MATHEMATICS

for the JEE Advanced

Manan Khurma
Prakash Rajpurohit

ALWAYS LEARNING

PEARSON

Mathematics for the JEE Advanced

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PEARSON

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To my parents
—Manan

To my parents
—Prakash

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Contents

Preface

This book is essentially a book of good Mathematics problems, designed for students taking the JEE Mains and (especially) the JEE Advanced Mathematics test. There are 16 chapters in the book which cover all the topics as per the prescribed syllabus. The structure of each chapter is as follows:

1. Part A (Theory): Summary material containing the main highlights for the chapter and important facts and formulae. This sections ends with a very useful feature called *Important Facts and Misconceptions*.
2. Part B (Illustrative Examples): A lot of problems with detailed solutions, and a special emphasis on *how to approach* a problem.
3. Part C (Advanced Problems): Unsolved problems of both objective and subjective types.
4. Part D (Solutions): Solutions to the problems in Part C. Most of the solutions are fairly detailed.

There is a *lot* of material in this book, designed to be covered over a period of two years. To get the most out of it, we recommend the following approach:

1. Once you have covered a chapter in your school (or elsewhere), read the corresponding Part A from this book. Clear any unresolved issues you might have at this stage.
2. Start Part B, but *try to solve the problems on your own*. In case you are able to solve a problem, compare your approach with the one in the book. In case a solution eludes you, analyze the provided solution to see where you faltered.
3. Each problem in Part C is very interesting and highlights the application of one or more concepts quite effectively. These problems are to be solved over an extended period of time, and not in one or two sittings. When you are unable to solve a problem, move to another and come back to the first problem later. Refer to the solutions only when you are convinced that you have made all the possible efforts.

This book is a must-have Mathematics resource for the IIT-JEE and NEET aspirants. We are hopeful that if used as suggested, it will enable its readers to master the art of problem solving in Mathematics.

Manan Khurma

Prakash Rajpurohit

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About the Authors

Manan Khurma graduated from IIT-Delhi in 2007 with a B.Tech in Electrical Engineering. An NTSE scholar and topper in multiple Math competitions, he has, over the last many years, taught Mathematics to IIT-JEE aspirants. That experience inspired the writing of this book. After graduation, Manan Khurma has worked on 'advanced solar research' in the Physics department of IIT-Delhi. Currently, he is working full-time in qLurn—his company which creates innovative learning solutions. His interests include cosmology, mathematics, travel, music and adventure sports.

Prakash Rajpurohit graduated from IIT-Delhi in 2007 with a B.Tech in Electrical Engineering. He and Manan Khurma are classmates from college and since then, they share an avid interest in mathematics. Prakash Rajpurohit secured All India Rank 4 in IIT-JEE 2003. In 2009, he secured All India Rank 2 in the Civil Services Examination conducted by the UPSC. He has also taught mathematics to UPSC aspirants. Currently, he is an IAS officer posted in Rajasthan. His interests include mathematics, movies, reading and travel.

CHAPTER 0: Mastering the Art of Problem Solving in Mathematics

Mathematics is one of the most important subjects for school students. Broadly speaking, the process of learning and mastering any topic in mathematics involves three steps or stages:

1. Learning the conceptual foundations of that topic
2. Understanding the applications of these concepts to problem solving
3. Solving 'new' problems

Let us take an example. Suppose that you have to master *Complex Numbers*. As described above, you will have to pass through three stages:

1. *Conceptual Foundations*. This will involving learning in-depth about the origin of complex numbers, their geometrical interpretation, how the various arithmetic operations apply to them, etc.
2. *Understanding Applications*. In this stage, you will have to understand how the various concepts you have studied in the previous stage apply when you have to deal with questions and problems. For example, you will have to understand how to calculate the principal argument of a complex number in different cases, or to plot the n-th roots of a complex number on the Argand plane.
3. *Problem Solving*. This is the stage where you have to apply all your knowledge about concepts and applications to solving new, unseen problems and challenges. This is the stage where you will encounter a problem which you might never have encountered earlier, and you have to marshal all your knowledge and decide what approach to take to be able to solve the problem with minimum effort.

   The most difficult stage is the third one, because of the fact that you are asked to apply your knowledge to new situations, to problems without precedent for you. How does one master this stage? How does one become an expert at the art of problem solving in Mathematics?

   We advocate something that we call the *PASPA* technique:

   PASPA = **P**re -**A**nalysis + **S**olution + **P**ost-**A**nalysis

   So what is this technique? Before we discuss this, we must emphasize on the following: If you *really* have to master the art of problem solving, then you must apply this technique to *every* Math problem you solve. You have to train your mind to build solutions according to the framework suggested by this technique. Only when you repeatedly work with and apply this technique will you start realizing its real power, its unlimited potential.

   The PASPA technique is simple. It says that whenever you have to solve a (seemingly difficult) problem:

   1. First do a thorough *Pre-Analysis*. This involves a lot of sub-steps, one of which will be to think of similar (not necessarily same) situations you have encountered in the past. For example, if you have to solve a problem on the roots of a complex number, your mind should automatically be activated to search through all the *files* it has on the topic of roots of complex numbers. Perhaps, if you are

lucky, your mind will come across a good match, and you can put your past experience to use. Other than this looking for known patterns, you must also work on ruling out certain approaches which you can be sure will definitely not work, or will lead to very long or inelegant solutions. You must consider the problem statement(s) from all possible aspects, and give importance to all parts of the problem. If the problem statement talks about a uni-modular complex number which satisfies a certain relation, then you must focus your mind on the adjective *uni-modular*, which might turn out to be the most important piece of information in the problem statement. You must also think 3–4 steps *ahead,* that is, you must try to intuitively feel what would happen if you were to take a certain approach. For example, in a problem on coordinate geometry, if you decide to use polar coordinates instead of Cartesian coordinates, you must try to intuitively feel what effect this choice will have on your solution, *without working out the solution itself.* This is important, you must *feel* or *anticipate* without putting pen to paper.

By the end of the *pre-analysis* stage, you must have decided exactly what approach you are going to take to solve the given problem. Also, most of the effort in this stage should be mental, rather than written. If you write and write and try out every approach, that will defeat the whole purpose of this stage. As we said earlier, you must *feel* what approach you are going to take towards a solution. It's similar to a trekking trip, you have multiple number of possible routes, and before starting the trip, you decide on one. Of course, that does not mean that you can *never* change routes mid-way, but changing routes should be rare (at least when it comes to mathematics problem solving).

2. Then work out the *solution.* The solution should be as brief and succinct as possible. This depends on how well you have done your pre-analysis. For many problems, especially those asked in examinations like the IIT-JEE, the overall solutions will generally not be very long or cumbersome. If you have done your pre-analysis properly, the solution to any such problem will only be a few lines long. An important consideration driving your solution should be *elegance*. If your calculations and algebraic manipulations are getting too long, you should be thinking: perhaps there is a more elegant way to do this. Once again, it is the role of the pre-analysis stage to tell you that elegant way. In this stage, you should only be implementing what you have already worked out in the decided in the pre-analysis stage.
3. Finally, do a thorough, meaningful *post-analysis*. Many good problem-solvers will work through the first two stages, but not attach much importance to this third, absolutely critical stage. Once you have solved a problem correctly, you must always reflect: *Could there have been a better way to do this*? *Could I have built a more elegant solution? Could I have taken a different route for this mental trek*? This stage may involve further calculations and manipulations, where you try to figure out alternate approaches and substitutions, etc. For example, once you have solved a coordinate geometry problem wherein you made use of Cartesian coordinates, you might think at this stage: *Could I have used polar coordinates to simplify my solution even more*? Then you might even rework your solution by using polar coordinates, and compare the two approaches.

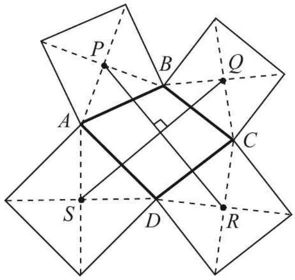
A good post-analysis will not only cement the solution in your mind, but it will also greatly help you in pre-analysis of problems which you solve subsequently.

The best problem solvers work through these three stages by default. Their mind is trained to do so without effort. If you also want to be a great problem solver, *force your mind* to work according to this structure. For each problem you solve, jot down your thoughts/workings from each of the three stages. Slowly, you will get better and better at doing this, and before long, you will not only become much better at solving problems, you will also start greatly enjoying the mental effort involved.

To make the PASPA idea more concrete, we are providing three examples in the following:

Example 1:

Q. 1 On each side of a quadrilateral $ABCD$, squares are drawn. The centers of the opposite squares are joined.



Show that PR and QS are equal in length and perpendicular to one another.

Pre-analysis: This problem involves one of the best applications of complex numbers we have seen, primarily because a pure geometry proof would be very lengthy, perhaps difficult. And since the quadrilateral is arbitrary, even the use of coordinate geometry would not be too helpful. With complex numbers, the proof can be complete in three lines!

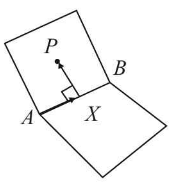
The motivation for the solution should come as follows:

- First of all, we need to select the best possible reference frame. Since there is no symmetry in the quadrilateral, a possible choice of origin could be a vertex of the quadrilateral say, the vertex A. Now we can assume complex representations for B, C and D. But wouldn't a simpler approach be to assume complex representations for the *sides* of the quadrilateral rather than its *vertices*? Thus, we could have

$$\overrightarrow{AB} = a, \overrightarrow{BC} = b, \overrightarrow{CD} = c, \overrightarrow{DA} = d.$$

Straightaway, we can see that $a+b+c+d=0$.

- Now, we need to find complex representations for P, Q, R, S. Lets first try to understand how we can reach P from the origin A. Each movement is represented by the appropriate vectors

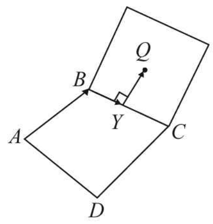


From A,

- go to $X\left(\dfrac{a}{2}\right)$
- go from X to $P\left(\dfrac{ia}{2}\right)$ (since $XP \perp AX$)

Thus, $P \equiv \dfrac{a}{2}+\dfrac{ia}{2}=\dfrac{a}{2}(1+i)$

Now let us see how we can reach Q from A:



From A,

- go to B (a)
- go from B to $Y\left(\frac{b}{2}\right)$
- go from Y to $Q\left(\frac{ib}{2}\right)$

Thus, $$Q \equiv a+\frac{b}{2}+\frac{ib}{2}=a+\frac{b}{2}(1+i)$$

By similar logic, we can simply write down the complex representations for R and S:

$$R \equiv a+b+\frac{c}{2}(1+i), \quad S \equiv a+b+c+\frac{d}{2}(1+i)$$

- Finally, we can write down the **vectors** PR and QS:

$$\overrightarrow{PR}=\vec{R}-\vec{P}=\frac{a}{2}(1-i)+b+\frac{c}{2}(1+i)$$

$$\overrightarrow{QS}=\vec{S}-\vec{Q}=\frac{b}{2}(1-i)+c+\frac{d}{2}(1+i)$$

- Now for the best part! The fact that PR and QS are perpendicular to each other and equal in length can be expressed succinctly as

$$\boxed{\overrightarrow{PR}=i\overrightarrow{QS}}$$

This is all we need to show!

Solution:

$$i\,\overrightarrow{QS}=\frac{i}{2}(b(1-i)+2c+d(1+i))$$

$$=\frac{i}{2}(b(1-i)+2c+(-a-b-c)(1+i)) \qquad \text{(since } a+b+c+d=0\text{)}$$

$$=\frac{i}{2}(a(-1-i)+2bi+c(1-i))$$

$$=\frac{a}{2}(-1-i)+b+\frac{c}{2}(1+i)=\overrightarrow{PR}$$

Post-analysis: Can you see the elegance of this complex numbers-based solution? Its amazing how a highly non-trivial result can be obtained in a few lines just by the appropriate choice of the axes and variables.

The question that may come to your mind is, “How can I think of an approach like this?” There is no single answer, and intuition too has a significant role to play (intuition can be developed with practice). Generally speaking, you should be guided by that vague quality called elegance. For example, a solution that requires too many variables or calculations at the JEE level will generally not be the best possible solution. You can try solving this problem using pure/co-ordinate geometry, and you’ll understand what we’re trying to say. For this problem, our approach of representing the directed sides using a, b, c, d (so that $a + b + c + d = 0$) makes all the difference.

Another Remark: How is it that complex numbers are able to easily accomplish what other techniques cannot? One way to think of this is as follows: a complex number, say z, contains a lot of ***information*** in its representation. For example, $z = 2 + 3i$, a single entity, has information about two parameters (x and y co-ordinates), whereas in co-ordinate geometry, you have to work with the coordinates separately. Another reason for the power of complex numbers is their similarity to vector behavior.

Example 2:

Q. 2 Let a complex number $\alpha, \alpha \neq 1$ be a root of $z^{p+q} - z^p - z^q + 1 = 0$ where p and q are distinct primes. Show that either

$$1+\alpha+\alpha^2+\cdots+\alpha^{p-1} = 0 \text{ or } 1+\alpha+\alpha^2+\cdots+\alpha^{q-1} = 0,$$

but not both together.

Pre-analysis: The given equation can be easily factored to $(z^p - 1)(z^q - 1) = 0$. If α is a root of this, then $\alpha^p = 1$ or $\alpha^q = 1$, so α is either a pth root of unity or a qth root of unity, which means that

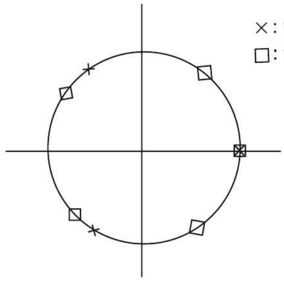
Either $\underbrace{1+\alpha+\alpha^2+\cdots\alpha^{p-1}}_{\text{sum of } p\text{th roots}} = 0$ (*)

or $\underbrace{1+\alpha+\alpha^2+\cdots\alpha^{p-1}}_{\text{sum of } q\text{th roots}} = 0$ (**)

This much was easy. However, the most important part of this problem is showing that (*) and (**) cannot hold true **simultaneously**, that is, α cannot be a pth and a qth root of unity at the same time. And why should that be? Because of the other piece of information given to us that p and q are distinct primes.

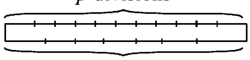
Let us discuss some concrete examples to illustrate this. Take i, for example. i is a 4th root of unity, because $i^4 = 1$. i is also on 8th root of unity, because $i^8 = 1$. Thus, the same number i is both a 4th and 8th root of unity. Take ω as another example: $\omega^3 = \omega^6 = \omega^9 = \omega^{12} = \cdots = 1$, so ω is a 3rd, a 6th, a 9th, a 12th,..., root of unity.

However, can you think of a complex number which is both a 3rd and a 5th root of unity? Lets plot the 3rd and 5th roots of unity on the same diagram, representing the 3rd roots by × and the 5th roots by □:



Note that no × and □ overlap (other than at 1). This means that a complex number $\alpha (\neq 1)$ cannot be a 3rd and a 5th root of unity simultaneously. The reason this is happening is because 3 and 5 are distinct primes, so they have no common factor.

In general, if p and q are distinct primes, and we plot the pth and qth roots of unity on the same diagram (representing the pth roots by × and the qth roots by □), we will find that no × and □ overlap. Many of you may be able to grasp this intuitively. Suppose I ask you to take a ruler of length 1 unit, and use pencil marks to divide the length of the ruler on one side into p equal units, and on the other side into q equal units.



If p and q are distinct primes, do you think any of the marks on the two sides can be at the same position? No! If this is not obvious to you intuitively, we may prove it rigorously.

Suppose that the mth mark on the upper side (spacing $\frac{1}{p}$) coincides with the nth mark on the lower side (spacing $\frac{1}{q}$). Then, we must have $\frac{m}{p} = \frac{n}{q}$, or $mq = np$, or $\frac{mq}{p} = n$. Thus, $\frac{mq}{p}$ must be an integer, but this is not possible since $m < p$, and p and q are prime. This implies that no marks will coincide.

In a similar manner, when the pth and qth roots of unity are plotted on the same unit circle, the × symbols corresponding to the pth roots divide the circumference into p divisions, while the □ symbols corresponding to the qth roots divide the circumference into q divisions. If p and q are prime, the two marks (× and □) cannot coincide in any position.

Solution: We proceed by contradiction, as in the case of the unit ruler. Let α be a pth and a qth root of unity simultaneously. Then, for some integers m, n, we must have

$$\alpha = e^{i\frac{2m\pi}{p}} = e^{i\frac{2n\pi}{q}} \Rightarrow \frac{m}{p} = \frac{n}{q}, \text{ where } 1 \le m < p, 1 \le n < q.$$

We have already see that this is not possible. That is all!

Post-analysis: This is one of the best conceptual questions from complex numbers in the JEE, since it combines the usage of the concept of the nth roots of unity from complex numbers elegantly with basic properties about prime numbers. Attentive readers might have noticed a further point. It is not necessary for p and q to be distinct primes for the result to hold.

As long as p and q are co-prime, that is, have no common factor, a complex number α can not simultaneously be a pth and qth root of unity. We would urge you to justify this along the lines of the solution I have presented.

One last comment: the form in which the problem was given involving two series, was not that illuminating. We had to look for an equivalent rephrasing. That rephrasing was:

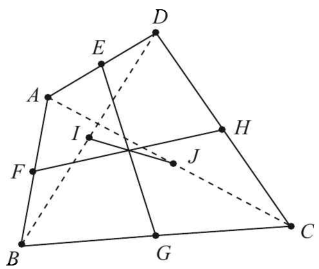
Show that either $1+\alpha+\alpha^2+\cdots\alpha^{p-1}=0$ or $1+\alpha+\alpha^2+\cdots\alpha^{q-1}=0$, but not both together	$\xrightarrow{\text{Rephrasing the question}}$	Show that either α is a pth root of unity or α is a qth root of unity, but not simultaneously

This skills of rephrasing a problem to a simpler form is very important for an exam like the JEE.

Example 3:

Q. 3 Prove that the line joining the midpoints of opposite sides of a quadrilateral and the line joining the midpoints of the diagonals are concurrent.

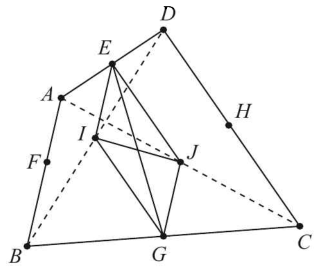
Pre-analysis We can try to solve this problem using coordinate geometry, but let us take a more elegant pure-geometry approach. The first step is to draw as accurate a figure as possible. In the figure below, $ABCD$ is a quadrilateral, and E, F, G and H are the midpoints of the four sides, while I and J are the midpoints of BD and AC respectively:



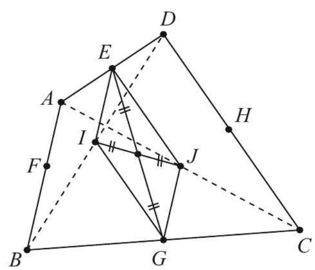
Clearly, EG, FH and IJ seem to be concurrent, but this is what we have to prove rigorously.

There are more than one ways in which the concurrency of three lines can be shown. For example, you can independently show that the three lines pass through the same point, or you can show that the point of intersection of one pair of lines is the same as the point of intersection of another pair. In our current example, we could show that each of EG, FH and IJ pass through some particular, fixed point. *Or*, we could show that the point of intersection of EG and IJ is the same as the point of intersection of FH and IJ. But how do we do that? What is the *key* to the solution?

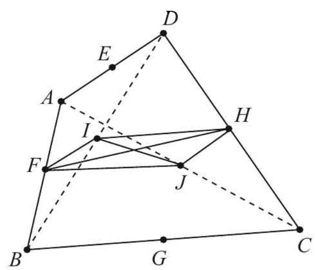
To discover the key, let us focus only on *EG* and *IJ*, and forget *FH*, for a moment. Let us complete the quadrilateral *EIGJ*:



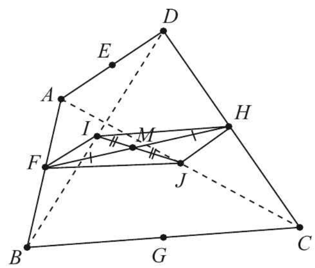
Does anything particular strike you about this quadrilateral? It should, if you observe it with a bit of attention: this seems to be a parallelogram! *Side Note*: without drawing an accurate diagram, it might be difficult to realize this fact. *EG* and *IJ* are the two diagonals of this parallelogram. What do we know about the diagonals of a parallelogram? We know that they bisect each other. Let us denote the point of intersection of *EG* and *IJ* by *K*. Thus, *EG* and *IJ* should get bisected at *K*:



Now, let us forget *EG* for a moment, and consider the segments *FH* and *IJ*. Let us complete the parallelogram *FIHJ*:



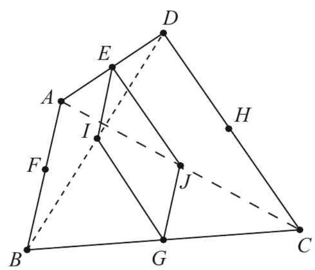
Once again, this seems to be a parallelogram, with *FH* and *IJ* as the two diagonals. If we denote their point of intersection by *M*, the two diagonals should get bisected at *M*:



But wait a minute! We conjectured earlier that *IJ* is bisected at *K* (in the supposed parallelogram *EIGJ*), and now we are talking about *IJ* getting bisected at *M* (in the supposed parallelogram *FIHJ*). If what we are saying is correct, then *K* and *M* are the same point, and each of *EG*, *IJ* and *FH* will pass through this point. On top of that, each of these three segments will get bisected at this point. Thus, if we are able to show that *EIGJ* and *FIHJ* are parallelograms, we will not only be able to show concurrency of the three segments, we will in addition be able to prove that the three segments are bisected at the point of concurrence. Really neat!

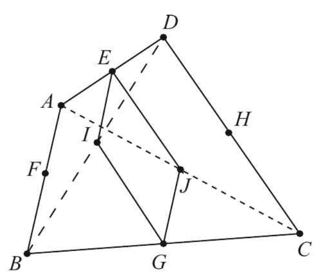
Let us now prove that *EIGJ* is a parallelogram. Then, the fact that *FIHJ* is a parallelogram will analogously be true, and the entire solution will automatically follow.

Solution: In the figure below, consider $\Delta ABCD$, the highlighted triangle:



E is the midpoint of *AD*, and *I* is the midpoint of *BD*. By the midpoint theorem, *EI* is parallel to *AB* and *EI* is equal to half of *AB*. Hold this fact in your mind for a moment.

In the figure below, consider ΔACB, the highlighted triangle:



G is the midpoint of *BC*, and *J* is the midpoint of *AC*. Again, by the midpoint theorem, *JG* is parallel to *AB* and *JG* is equal to half of *AB*.

Clearly, both *EI* and *JG* are parallel to *AB* and equal to half of *AB*. Thus, they are parallel to each other and equal to each other, which means that *EIGJ* is a parallelogram, and hence *EG* and *IJ* bisect each other at their point of intersection. The rest of the solution follows from the pre-analysis.

Post-analysis: Some key aspects of this problem (and the solution presented here) are as follows:

1. A significant step towards the correct solution was to draw an accurate diagram. Without that, the realization of certain quadrilaterals being parallelograms would have been difficult.
2. We had to show the concurrency of three line segments. We chose to show this by showing that the point of intersection of one pair of segments is the same as the choice of intersection of another pair. This choice played an important role.
3. Once we focused ourselves on a single pair of segments, say *EG* and *IJ*, our attention was caught by a realization that *EIGJ* is (seemed to be) a parallelogram, and thus *EG* and *IJ* are the diagonals of the parallelogram. This was the most crucial step in solving the problem. Once this is proven, we conclude that *EG* and *IJ* bisect each other. Similarly, *FH* and *IJ* bisect each other. Thus, both *EG* and *FH* bisect *IJ* (and in turn are bisected by *IJ*), which means that the three segments are concurrent.
4. The proof of *EIGJ* (and *FIHJ*) being parallelograms required a simple application of the midpoint theorem. The realization that the midpoint theorem will somehow be used should have come automatically to your mind when your first read the problem. Joining the midpoints of sides of triangles and quadrilaterals leads to interesting results, in which the midpoint theorem always has a key role to play.

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General Algebra

PART-A: Summary of Important Concepts

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1. Logarithms

Logarithms are a very convenient mathematical tool, used extensively to simplify calculations. In our experience, many students know how to use logarithms (logs for short), but do not know what they actually are. Here, we take a quick look at the concept of logs. Consider the following equation:

$$10^2 = 100$$

This would be read as 'Ten to the power two equals one hundred'. With logs, you can equivalently state this problem as:

$$\log_{10}(100) = 2,$$

which would be read as 'The logarithm of hundred to the base ten is two'. Consider some more examples:

$10^3 = 100 \quad \Rightarrow \quad \log_{10}(1000) = 3$

Read as: The log of 1000 to base 10 is 3'

$3^4 = 81 \quad \Rightarrow \quad \log_3(81) = 4$

Read as: 'The log of 81 to base 3 is 4'

$\sqrt{25} = 25^{\frac{1}{2}} = 5 \quad \Rightarrow \quad \log_{25}(5) = \frac{1}{2}$

Read as: 'The log of 5 to base 25 is $\frac{1}{2}$'

$27^{\frac{1}{3}} = 3 \quad \Rightarrow \quad \log_{27}(3) = \frac{1}{3}$

Read as: 'The log of 3 to base 27 is $\frac{1}{3}$'

Thus, we see that the log of any number to a given base is the power to which the base must be raised in order to equal the given number. In other words,

$$b^e = N \quad \Rightarrow \quad \log_b N = e, \text{ or}$$

$$(\text{Base})^{\text{Exponent}} = \text{Number} \quad \Rightarrow \quad \log_{\text{Base}}(\text{Number}) = \text{Exponent}$$

Thus, logs are just an alternate way of expressing exponential relations. As an exercise, show that $\log_{2\sqrt{2}} 32 \cdot (4^{\frac{1}{5}}) = 3 \cdot 6$.

Listed below are some properties that logs satisfy:

(1) $b^{\log_b N} = N$: This follows by definition.

(2) $\log_b N$ is defined only for $N > 0$, $b > 0$ and $b \neq 1$. We'll try to understand why b cannot equal 1:

1 raised to any power will always equal 1

$\Rightarrow$ $1^e = 1$ for all e

$\Rightarrow$ If we try to calculate $\log_1(N)$, we see that there can be no value possible for this log, since no matter what power you raise 1 to, you will never obtain N. Thus, 1 is not a valid base for logs.

(3) $\log_b 1 = 0$ for any base b. This is because any number raised to power 0 is 1, so that

$$b^0 = 1 \quad \Rightarrow \quad \log_b 1 = 0$$

(4) **Log of a product:** One of the most useful properties of logs is this:

$$\log_b(MN) = \log_b M + \log_b N$$

That is, the log of a product is equal to the sum of the individual logs. To justify this property, assume that $\log_b M = x, \log_b N = y$:

$$\Rightarrow \quad b^x = M,\ b^y = N$$

$$\Rightarrow \quad MN = b^x \times b^y = b^{x+y} \qquad (1)$$

$$\Rightarrow \quad \log_b(MN) = x + y = \log_b M + \log_b N$$

Thus, notice carefully that this property is nothing but a manifestation of the fact that when two exponential terms with the same base are multiplied, their powers add, as shown in (1). This is something you must always keep in mind.

(5) **Log of a fraction:** Analogous to the product property, we've the division property:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

As you might be expecting, this property is a result of the fact that when two exponential terms with the same base are divided, their powers subtract. Let us prove this property explicitly.

Assuming $\log_b M = x$ and $\log_b N = y$, we have

$$\Rightarrow \quad M = b^x,\ N = b^y \quad \Rightarrow \quad \frac{M}{N} = \frac{b^x}{b^y} = b^{x-y}$$

$$\Rightarrow \quad \log_b\left(\frac{M}{N}\right) = x - y = \log_b M - \log_b N.$$

(6) **Log of an exponential term:** This property states that

$$\log_b(N^e) = e\log_b N.$$

That is, the log of N^e to any base is e times the log of N to the same base. This is again straightforward to justify. Assume $\log_b N = x$:

$$\Rightarrow \quad N = b^x \quad \Rightarrow \quad N^e = (b^x)^e = b^{ex}$$

$$\Rightarrow \quad \log_b(N^e) = ex = e\log_b N$$

Thus, this property is a consequence of the fact that when an exponential term is raised to another power, the two powers multiply.

Illustration-1: **A calculation using logs.** Find the value of $\sqrt[5]{0.00000165}$

Working: Let $N = \sqrt[5]{0.00000165}$

$$\Rightarrow \quad \log N = \frac{1}{5}\log(0.00000165) = \frac{1}{5}(-6 + \log 1.65) \qquad \text{(why?)}$$

$$= \frac{1}{5}(-6 + .2175) = -1.2 + 0.0435$$

$$= -2 + 0.8435 \text{ (why did we do this?)}$$

$$= \bar{2}\cdot 8435$$

Thus,

$$N = \text{antilog }(0.8435) \times 10^{-2} = 6.974 \times 10^{-2} = 0.06974.$$

2. Progressions

2.1 Arithmetic Progressions

An *arithmetic progression* (AP for short) is a sequence with constant difference between successive terms. The *first term* in any AP is customarily denoted by a. Note that for an AP extending indefinitely into the left, *e.g.*

$$\ldots, -3, -1, 1, 3, \ldots$$

you can chose any of the AP's terms as your first term. The only purpose of the first term is to provide you with a reference point using which you can specify the other terms of the AP. It is similar to the '0' on a number line. The constant difference between successive terms is termed as the *common difference*, and is customarily denoted by d. Note that d can be any real number. Thus, in terms of a and d, an arbitrary AP can be written as

$$a, a+d, a+2d, a+3d, \ldots$$

or

$$\ldots, a-2d, a-d, a, a+d, a+2d, \ldots$$

etc.

For an AP that starts at a and has a common difference (abbreviated as CD from now on) d, the nth term t_n is given by

$$\boxed{t_n = a + (n-1)d}$$

To find the sum of n terms of the following AP:

$$S = a + (a+d) + (a+2d) + \cdots + (a + (n-1)d),$$

we write this series again, but this time in reverse:

$$S = (a + (n-1)d) + (a + (n-2)d) + \cdots + a.$$

Adding the two series (*i.e.*, adding term by term), we have

$$2S = \underbrace{(2a+(n-1)d)+(2a+(n-1)d)+\cdots+(2a+(n-1)d)}_{n \text{ times}}$$

$$= n\cdot\{2a+(n-1)d\}$$

$$\Rightarrow \qquad \boxed{S = \frac{n}{2}\cdot\{2a+(n-1)d}$$

Note that if we denote the last term by l, we have $l = a+(n-1)d$. so that

$$\boxed{S = \frac{n}{2}(a+l)}$$

This means, for example that $(1+2+\cdots+n) = \frac{n(n+1)}{2}$.

2.2 Arithmetic Means

If a and b are any two arbitrary real numbers, the *arithmetic mean* (abbreviated AM) c between a and b is a number c such that a, c and b are in AP. Thus, we should have

$$c-a=b-c \qquad \Rightarrow \qquad c = \frac{a+b}{2}$$

Thus, c is simply the average of the two numbers a and b. For example, the AM between 1 and 5 is $\frac{1+5}{2}=3$ because 1, 3 and 5 are in AP. We can insert more than one AMs between two numbers. If x and y are two arbitrary real numbers, then the n AMs between x and y are n real numbers $a_1, a_2, \ldots, a_n$, such that

$$x, a_1, a_2, \ldots, a_n, y$$

are in AP. Thus, the CD d is given by

$$d = \frac{y-x}{n+1} \text{ (verify)}$$

so that

$$a_1 = x+\frac{y-x}{n+1}, \quad a_2 = x+2\frac{(y-x)}{n+1}, \ldots, a_n = x+n\frac{(y-x)}{n+1}$$

The sum of these n AMs is

$$a_1+a_2+\cdots+a_n = nx+\frac{y-x}{n+1}(1+2+\cdots+n)$$

$$= nx+\frac{y-x}{n+1}\cdot\frac{n(n+1)}{2} = n\left(\frac{x+y}{2}\right)$$

$$= n\cdot\{\text{AM between } x \text{ and } y\}$$

Let us insert 5 AMs between 1 and 13; we have

$$d = \frac{13-1}{5+1} = 2 \qquad \Rightarrow \qquad a_1 = 3,\, a_2 = 5,\, a_3 = 7,\, a_4 = 9,\, a_5 = 11$$

Do not confuse n AMs between two numbers a and b, with the AM of n numbers. The AM (or average) of n numbers $a_1, a_2, \ldots, a_n$ is simply

$$\text{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Note also, however, that if we insert n AMs $a_1, a_2, \ldots, a_n$ between two numbers x and y, we have

$$\text{AM of } \{a_1, a_2, \ldots, a_n\} = \text{AM of } x \text{ and } y.$$

2.3 Geometric Progressions

A geometric progression (GP for short) is a sequence of numbers in which the *ratio* of successive terms is a constant. For example:

(a) $\{1, 2, 4, 8\ldots, 2^n, \ldots\}$ is a GP because each successive term is twice the preceeding term, *i.e.*, the constant ratio is 2. If we denote the general term by t_n, we have

$$\frac{t_n}{t_{n-1}} = 2 \text{ for all } n \in \mathbb{N}.$$

(b) $\{1, 0.1, 0.01, 0.001, \ldots\}$ is again a GP, with the ratio of successive terms being 0.1, *i.e.*,

$$\frac{t_n}{t_{n-1}} = 0.1 \text{ for all } n \in \mathbb{N}.$$

In a GP, the first term is again denoted by a, while the constant ratio, which is called the *common ratio* (CR), is customarily denoted by r. Thus, an arbitrary GP is of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots.$$

with the general, nth term being given by

$$t_n = ar^{n-1}.$$

Now, let us determine the sum of a general GP for n terms; this involves a small manipulation: We first write S_n, the series consisting of the n terms of the GP, and then we write the expression for rS_n:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \qquad (1)$$

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \qquad (2)$$

We now do (2) – (1); in this process, all terms on the RHS except a in (1) and ar^n in (2), vanish:

$$(r-1)S_n = ar^n - a = a(r^n - 1).$$

Thus, we have expression for S_n as

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right). \qquad (3)$$

Note that this holds only if $r \neq 1$. If $r = 1$, the GP is trivial and consists of repeated terms:

$$a, a, a, a \ldots.$$

In this case, the sum for n terms will simply be

$$S_n = na. \tag{4}$$

We can thus combine (3) and (4) and rewrite S_n as

$$S_n = \left\{ \begin{array}{cc} a\left(\dfrac{r^n - 1}{r-1}\right) & r \neq 1 \\ na & r = 1 \end{array} \right\}.$$

You may also notice that the sum for the case $r = 1$ can be obtained using the following limit:

$$\lim_{r \to 1} \frac{r^n - 1}{r - 1} = n.$$

A lot many times, we will encounter GPs with infinite terms. The sum of such a GP will be finite only if $|r| < 1$. For example, consider the following GP:

$$S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \infty.$$

This has an infinite number of terms, but S will still be finite. Intuitively, this is because as we progress along this series, the terms decrease *at a fast rate*. To understand this better, consider the following two series:

$$S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \infty,$$

$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \infty.$$

The first is a GP, which we have claimed has a finite sum, while in S_1, the terms to not *decrease fast enough*, so S_1 has an infinite sum. Coming back to evaluating S, we first consider the following GP:

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}.$$

This, as we have discussed, has the sum given by

$$S_n = a \cdot \left(\frac{r^n - 1}{r - 1} \right) = 1 \cdot \left\{ \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} \right\} = \frac{1 - (\frac{1}{2})^n}{(\frac{1}{2})} = 2 - \frac{1}{2^{n-1}}.$$

Now, if we let n approach ∞, we obtain S:

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(2 - \frac{1}{2^{n-1}} \right) = 2.$$

Thus, this proves that S is finite and its value is 2. The general case of the sum of an infinite GP with $|r|<1$ is dealt with similarly:

$$S = a + ar + ar^2 + ar^3 + \cdots \infty \text{ for } |r|<1 = \lim_{x\to\infty}\left\{a\left(\frac{r^n-1}{r-1}\right)\right\} = \lim_{x\to\infty}\left\{a\left(\frac{1-r^n}{1-r}\right)\right\}$$

$$= \frac{a}{1-r}, \text{ because } r^n \to 0 \text{ if } |r|<1 \text{ and } n\to\infty.$$

For example,

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \infty$$

is an infinite GP with $a = 1$ and $\text{CR} = r = \frac{1}{3} < 1$, so that

$$S = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2.4 Geometric Means

Similar to AM, we can define the *geometric mean* (GM) for two numbers a and b to be a number c such that

$$a, c, b \text{ are in G.P.} \quad \Rightarrow \quad \frac{c}{a} = \frac{b}{c} \quad \Rightarrow \quad c = \sqrt{ab}$$

For example, the GM between 1 and 9 will be $\sqrt{1\times 9} = 3$. We can also insert n GMs between two numbers x and y. This would be a set of numbers $\{a_1, a_2, \ldots, a_n\}$, such that

$$x, a_1, a_2, \ldots, a_n, y \text{ are in GP.}$$

If we denote the CR of this GP by r, we have

$$y = xr^{n+1} \text{ (why?)} \quad \Rightarrow \quad r = \left(\frac{y}{x}\right)^{\frac{1}{n+1}}.$$

Thus, the n GMs are given by

$$a_1 = xr, a_2 = xr^2, \ldots, a_n = xr^n.$$

As an exercise, insert 11 GMs between 1 and 2.

Illustration-2: The GM of n numbers $a_1, a_2, \ldots, a_n$ is defined as $(a_1 a_2 \ldots a_n)^{\frac{1}{n}}$. If we insert n GMs $\{a_1, a_2, \ldots, a_n\}$ between two real numbers x any y, show that GM of $\{a_1, a_2, \ldots, a_n\}$ = GM of $\{x, y\}$.

Working: The n GMs, as we've already seen, are given by

$$a_1 = xr, a_2 = xr^2, \ldots, a_n = xr^n, \text{ where } r = \left(\frac{y}{x}\right)^{\frac{1}{n+1}}$$

Thus,

$$a_1\, a_2 \ldots a_n = (xr)\cdot(xr^2)\cdot(xr^3)\cdot\cdots\cdot(xr^n)$$

$$= x^n\, r^{1+2+3+\cdots+n} = x^n\, r^{\frac{n(n+1)}{2}} = x^n \left\{\left(\frac{y}{x}\right)^{\frac{1}{n+1}}\right\}^{\frac{n(n+1)}{2}}$$

$$= x^n \cdot \frac{y^{\frac{n}{2}}}{x^{\frac{n}{2}}} = x^{\frac{n}{2}}\cdot y^{\frac{n}{2}} = \left(\sqrt{xy}\right)^n$$

$$\Rightarrow \quad (a_1\, a_2 \ldots a_n)^{\frac{1}{n}} = \sqrt{xy}$$

$$\Rightarrow \quad \text{GM of } \{a_1, a_2, \ldots, a_n\} = \text{ GM of } \{x, y\}$$

2.5 Harmonic Progressions

A sequence of numbers $a_1, a_2, \ldots, a_n$ is said to be in *Harmonic Progression* (HP) if

$$\frac{1}{a_1},\ \frac{1}{a_2}, \cdots, \frac{1}{a_n} \text{ are in AP}$$

This means that n numbers are in HP if their reciprocals are in AP. For example,

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}\right\} \text{ are in HP because } \{1, 2, 3, \ldots., n\} \text{ are in AP}$$

Thus, we see that the terms of an HP are of the form

$$\cdots\cdots \frac{1}{a},\quad \frac{1}{a+d},\quad \frac{1}{a+2d}, \cdots\cdots$$

Note that all terms in an HP must be non-zero. If the first term of an HP is denoted by h, we have the nth term given by

$$t_n = \frac{1}{\frac{1}{h} + (n-1)d} \tag{1}$$

where d is the CD of the corresponding AP formed by the reciprocal terms. (1) can also be written as

$$\frac{1}{t_n} = \frac{1}{h} + (n-1)\,d$$

As in the case of AM and GM, we can define the HM of two numbers a and b to be a number c such that

$$a, c, b \text{ are in HP} \Rightarrow \quad \frac{1}{a}, \frac{1}{c}, \frac{1}{b} \text{ are in AP} \Rightarrow \quad \frac{1}{c} = \frac{\frac{1}{a}+\frac{1}{b}}{2} \quad \Rightarrow \quad c = \frac{2ab}{a+b}$$

For example, the HM of 1 and 2 will be $\frac{4}{3}$. We can also insert n HMs between two numbers x and y: these will be n numbers $\{a_1, a_2, \ldots, a_n\}$, such that

$$x, a_1, a_2, \ldots, a_n, y \text{ are in HP}$$

$$\Rightarrow \quad \frac{1}{x}, \frac{1}{a}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}, \frac{1}{y} \text{ are in AP.}$$

If we let d denote the CD of this AP, we have

$$\frac{1}{y} = \frac{1}{x} + (n+1)d \text{ (why?)} \qquad \Rightarrow \qquad d = \frac{\frac{1}{y} - \frac{1}{x}}{n+1} = \frac{x-y}{(n+1)\,xy}$$

Thus, the n HMs are given by

$$\frac{1}{a_1} = \frac{1}{x} + d, \ \frac{1}{a_2} = \frac{1}{x} + 2d, \ldots, \frac{1}{a_n} = \frac{1}{x} + nd \tag{2}$$

As in the case of AMs and GMs, we can define the HM of n numbers using the AM of their reciprocals.

Illustration-3: If we insert n HMs $\{a_1, a_2, \ldots, a_n\}$ between two real numbers x and y, show that:

$$\text{HM of } \{a_1, a_2, \ldots, a_n\} = \text{HM of } \{x, y\}.$$

Working: The n HMs are given by (2), so that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \left(\frac{1}{x} + d\right) + \left(\frac{1}{x} + 2d\right) + \cdots + \left(\frac{1}{x} + nd\right) = \frac{n}{x} + d(1 + 2 + \cdots + n)$$

$$= \frac{n}{x} + \frac{x-y}{(n+1)xy} \cdot \frac{n(n+1)}{2} = \frac{n}{x} + \frac{n(x-y)}{2xy} = \frac{n}{x} + \frac{n}{2y} - \frac{n}{2x} = \frac{n}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\Rightarrow \quad \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n} = \frac{\frac{1}{x} + \frac{1}{y}}{2}$$

$$\Rightarrow \quad \text{HM of } \left\{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right\} = \text{HM of } \{x, y\}$$

2.6 AM-GM-HM Inequality

This inequality says that

$$\text{AM of } \{a_1, a_2, \ldots, a_n\} \geq \text{GM of } \{a_1, a_2, \ldots, a_n\} \geq \text{HM of } \{a_1, a_2, \ldots, a_n\}$$

$$\Rightarrow \quad \frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1\, a_2 \ldots a_n)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Mathematically inclined readers should try to justify this general inequality. Also, we note the following fact. Suppose that x and y are two numbers. We have

$$(\mathrm{AM})\times(\mathrm{HM})=\frac{x+y}{2}\times\frac{2xy}{x+y}=xy=(\mathrm{GM})^2.$$

Thus, the GM is also the geometric mean between the AM and the HM.

3. Quadratic Equations and Expressions

3.1 Basic Results

A general quadratic expression has the form

$$f(x)=ax^2+bx+c,\quad a\neq 0$$

A general quadratic equation has the form

$$f(x)=ax^2+bx+c=0,\quad a\neq 0$$

Note that at this level, we will always assume the constants a, b, c to be real numbers. We define the discriminant of this general quadratic expression to be

$$D=b^2-4ac$$

The sign of the determinant determines the nature of the roots, according to the following formula for the roots:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus, we see that

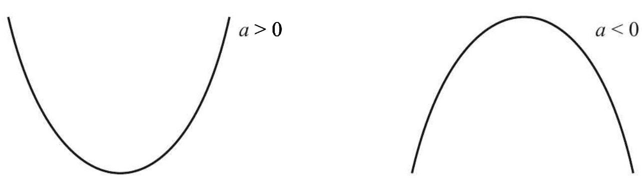
(i) $D>0$ implies that the roots are real and distinct.
(ii) $D=0$ implies that the roots are real and identical.
(iii) $D<0$ implies that the roots are non-real (and distinct).

3.2 The graph of $f(x)$

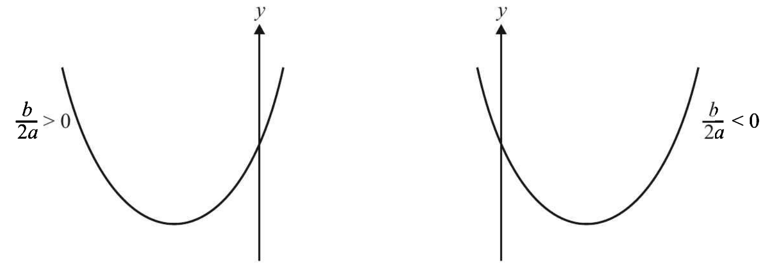
In terms of scaling and shifting operations, the graph of $f(x)$ can be obtained in the following steps:

$$x^2\xrightarrow{\text{Scaling}}ax^2\xrightarrow[\text{Shifting}]{\text{Horizontal}}a\left(x+\frac{b}{2a}\right)^2\xrightarrow[\text{Shifting}]{\text{Vertical}}a\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a}$$

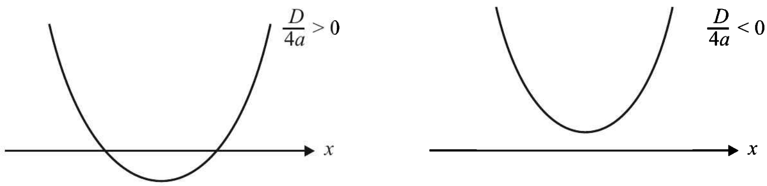
(i) If $a>0$, the graph is an upwards parabola, else it opens downwards:



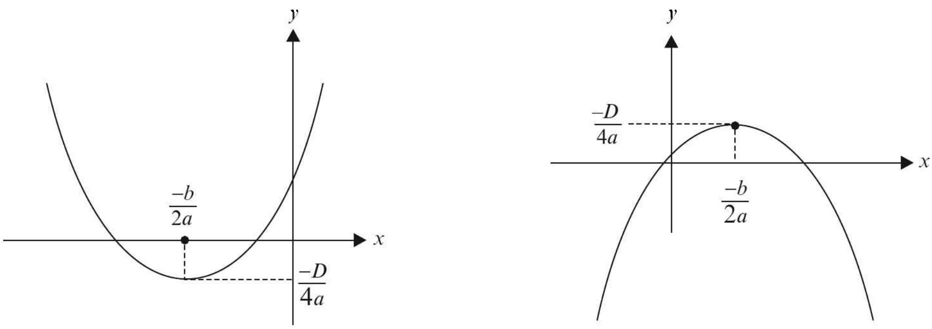
(ii) If $\frac{b}{2a} > 0$, the parabola is shifted left, else if $\frac{b}{2a} < 0$, it gets right shifted:



(iii) If $\frac{D}{4a} > 0$, the graph gets shifted downwards, else if $\frac{D}{4a} < 0$, it gets shifted upwards. The figure below is an example of this for $a > 0$:



(iv) The co-ordinates of the vertex of the parabola are given by $(\frac{-b}{2a}, \frac{-D}{4a})$, regardless of whether a is positive or negative:

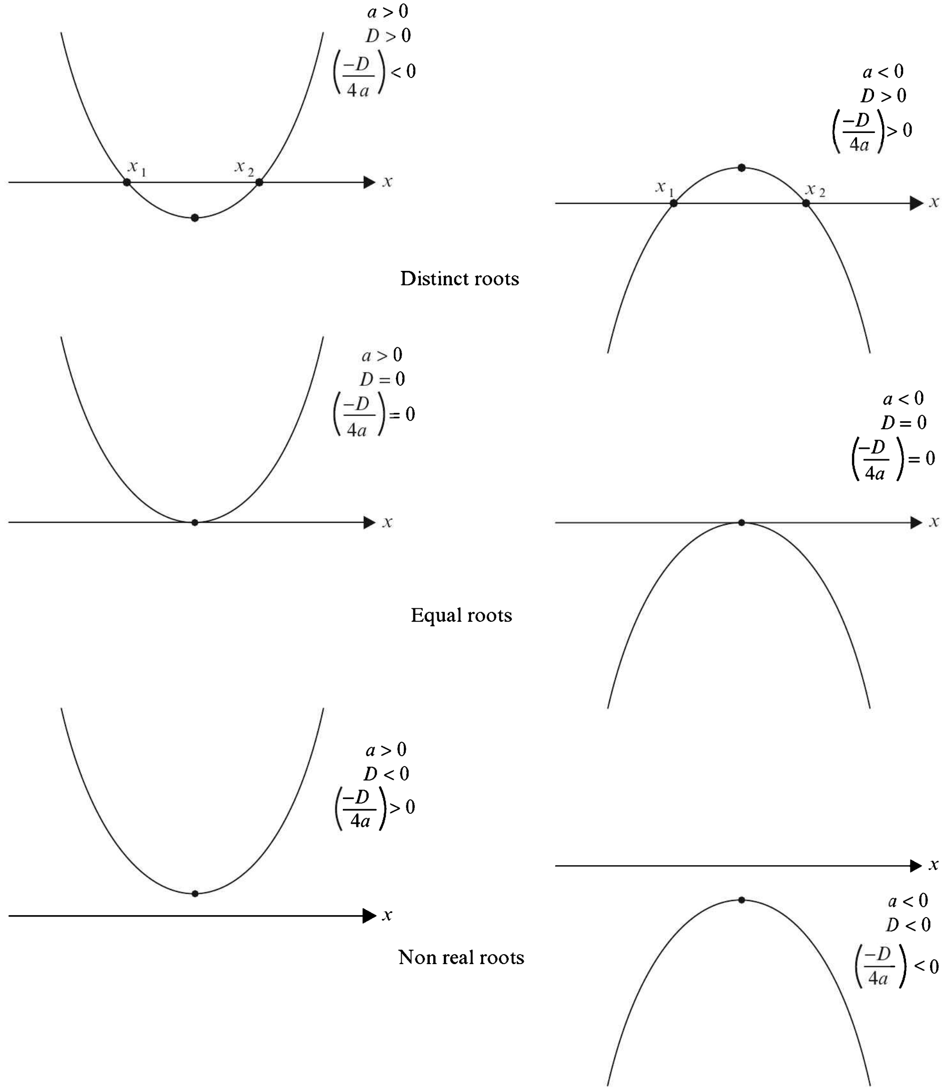


(v) If the parabola cuts the x-axis at two distinct points, we say that $f(x) = 0$ has two distinct roots, and the values of those roots are given by the quadratic formula. If the parabola just touches the x-axis (at a single point), or in other words the vertex of the parabola lies on the x-axis, $f(x) = 0$ has real and equal roots (both roots are the same). This is obvious from the quadratic formula, since if the vertex lies on the x-axis $D = 0$, so that:

$$x_1 = x_2 = \frac{-b}{2a}.$$

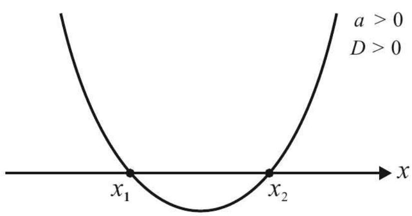
If the parabola does not cross the x-axis at all, we say that $f(x) = 0$ has non-real (or imaginary roots). Again, this is obvious from the quadratic formula. If the parabola lies entirely above the x-axis, the y-coordinate of the vertex $(-D/4a)$ is positive and a is also positive, implying D is negative. Similarly, if the parabola lies entirely below the x-axis, $(-D/4a)$ is negative and a is also negative, implying D is again negative. Thus, the formula for the roots gives

complex values of x. The graph in the figure below illustrate these three cases for $a > 0$ and $a < 0$ separately:



3.3 Quadratic Inequalities: $f(x) > 0$, $f(x) < 0$

Once you are able to draw the graph corresponding to $f(x)$, solving these inequalities should be easy. Consider an example of the graph of a quadratic function below:



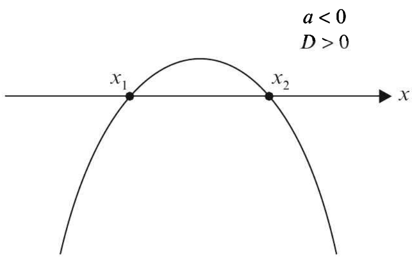
The roots are given by x_1 and x_2. For what values of x is $f(x) > 0$? Obviously, for those values for which the graph of $f(x)$ lies above the x-axis. *i.e.*,

$$f(x) > 0 \text{ for } x < x_1 \text{ and } x > x_2.$$

Similarly

$$f(x) < 0 \text{ for } x_1 < x < x_2.$$

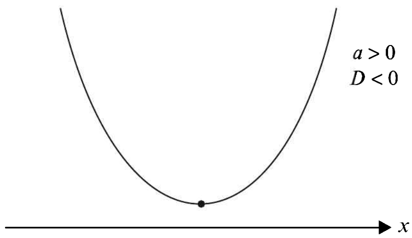
Consider another example:



We see from the graph that

$$f(x) > 0 \text{ for } x_1 < x < x_2 \text{ and } f(x) < 0 \text{ for } x < x_1 \text{ and } x > x_2.$$

Finally, consider a third example:



The graph for $f(x)$ lies entirely above the x-axis. Therefore,

$$f(x) > 0 \text{ for all values of } x \text{ and } f(x) < 0 \text{ for no value of } x.$$

Note how quadratic inequalities can easily be solved by considering the corresponding graphs.

3.4 Sum and Product of Roots

Suppose that the roots of a quadratic expression are denoted by α, β. We have

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Now,

$$\begin{aligned} f(x) = ax^2 + bx + c &= a(x-\alpha)(x-\beta) = a(x^2 - (\alpha+\beta)x + \alpha\beta) \\ &= ax^2 - a(\alpha+\beta)x + a\alpha\beta = ax^2 - aSx + aP \\ &= a(x^2 - Sx + P) \end{aligned}$$

where S denotes the sum and P the product of the roots. Comparing the coefficients of x on both sides gives:

$$-a(\alpha+\beta) = b$$
$$a\alpha\beta = c$$

$$\Rightarrow \qquad S = \text{Sum of roots} = \alpha+\beta = \frac{-b}{a}$$

$$\Rightarrow \qquad P = \text{Product of roots} = \alpha\beta = \frac{c}{a}$$

These are important results. They can of course be derived directly from the expression for the roots given by the quadratic formula. For example,

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a}.$$

3.5 Constraints on Roots

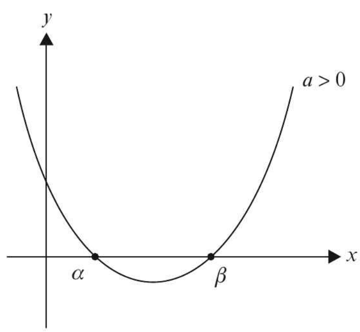
Let $f(x) = ax^2+bx+c$ and α, β be the zeroes of this quadratic expression. Consider the problem of placing constraints on a, b, c given some constraint(s) on α and β. We have already seen the constraint $D \geq 0$ given that α, β are real. Similarly, $D < 0$ if α, β are non-real. This is what we mean by saying that the nature of α, β places a constraint on a, b, c:

$$\alpha, \beta \text{ real} \quad \Rightarrow \quad b^2-4ac \geq 0$$
$$\alpha, \beta \text{ non-real} \quad \Rightarrow \quad b^2-4ac < 0$$

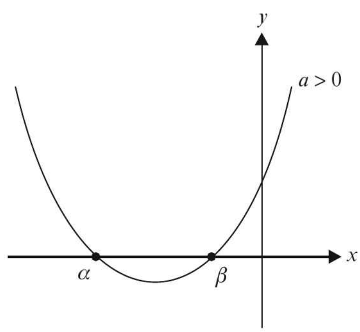
These are the most basic constraints possible. However, we can have more specific constraints, and some of the important ones are summarize below.

(A) $\boxed{\text{Both roots are of the same sign}}$

Since the roots are real, $D \geq 0$. Since they are of the same sign, $\alpha\beta > 0$. The graphs below illustrate examples for $a > 0$:



Both roots positive



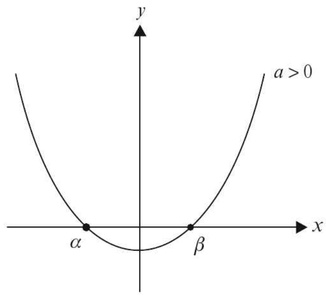
Both roots negative

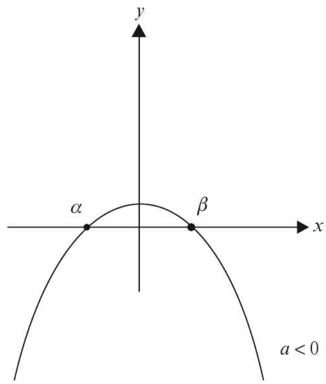
Hence, the constraints are:

$$b^2 - 4ac \geq 0, \ \frac{c}{a} > 0.$$

(B) $\boxed{\text{Roots are of opposite signs}}$

For real roots, $D \geq 0$. For roots of opposite sign, $\alpha\beta < 0$. However, notice that $\alpha\beta < 0$ ensures that $D > 0$ and hence writing the first constraint is unnecessary (can you see why?). Therefore, all we require is $\alpha\beta < 0$. The graphs below illustrate this case.





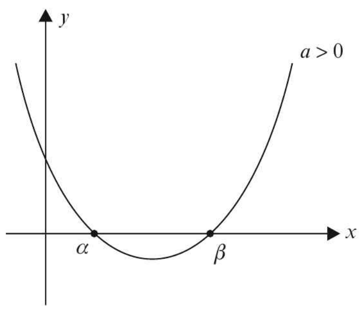
Roots are of opposite sign

The required constraint is:

$$\frac{c}{a} < 0.$$

(C) $\boxed{\text{Both roots are positive}}$

If you have followed the previous two cases properly, this case should be straight forward. We require $D \geq 0$, and in addition, since both the roots are positive, we must have $\alpha\beta > 0$ and $\alpha + \beta > 0$.

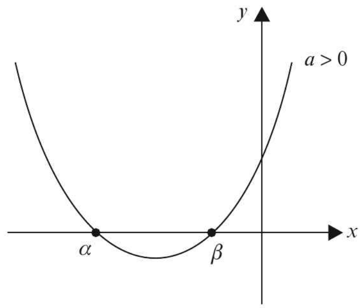


Both roots positive

$$b^2 - 4ac \geq 0, \ \frac{-b}{a} > 0, \ \frac{c}{a} > 0.$$

(D) $\boxed{\text{Both roots are negative}}$

This is similar to the previous case:



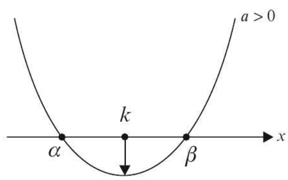
Both roots negative

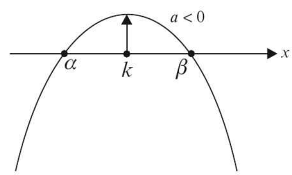
The required constraints are

$$b^2 - 4ac \geq 0,\ \frac{-b}{a} < 0,\ \frac{c}{a} > 0.$$

(E) $\boxed{\text{Roots lie on either side of } k}$

This means that $\alpha < k,\ \beta > k$:

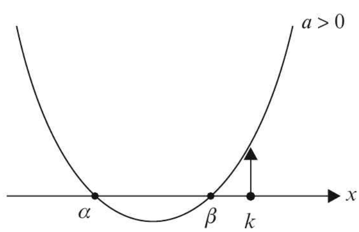


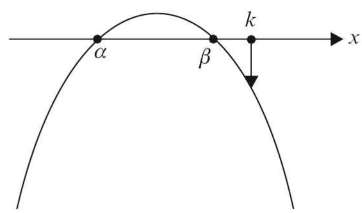


We see that if $a > 0, f(k) < 0$ and if $a < 0, f(k) > 0$. Combining these two observations, we can say concisely that we require $af(k) < 0$. This represents a necessary and also sufficient constraint for our requirement (verify that this condition alone is sufficient):

$$af(k) < 0$$

(F) $\boxed{\text{Both the roots are less than } k}$

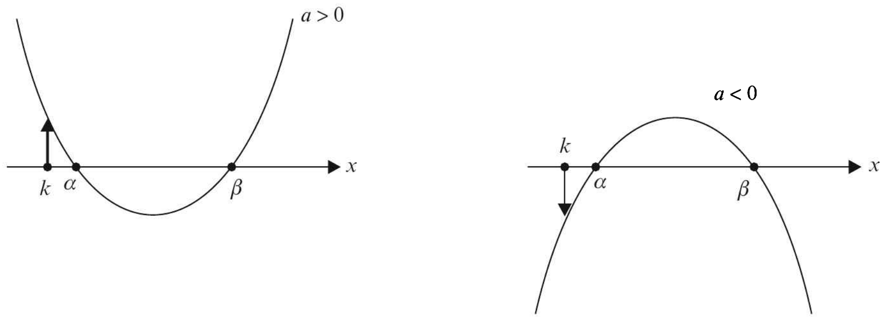




We see that for the left hand graph $(a > 0), f(k) > 0$ and for the right hand graph $(a < 0), f(k) < 0$. We can combine these two observations and say concisely that $af(k) > 0$. We also require $D \geq 0$, and since both the roots are less than k, $\alpha + \beta < 2k$. Hence, the constraints are:

$$D \geq 0,\ af(k) > 0,\ -\frac{b}{a} < 2k.$$

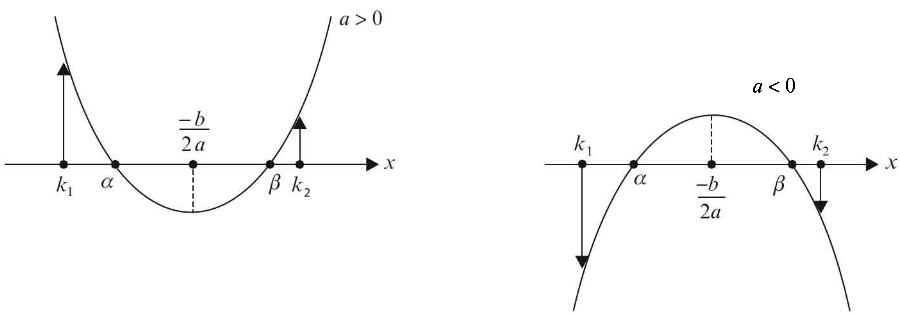
(G) $\boxed{\text{Both the roots are greater than } k}$



Here again, we see that $af(k) > 0$. Also, $\alpha + \beta > 2k$. The required constraints are

$$D \geq 0,\ af(k) > 0,\ -\frac{b}{a} > 2k.$$

(H) $\boxed{\text{Both the roots lie between } k_1 \text{ and } k_2}$

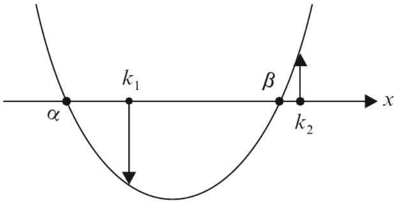


First of all, we require $D \geq 0$ for real roots. Now, notice that the x-coordinate of the vertex of the parabola lies between k_1 and k_2. Also, whether a is positive or negative, notice that $af(k_1)$ and $af(k_2)$ are positive. Therefore, the required constraints will be:

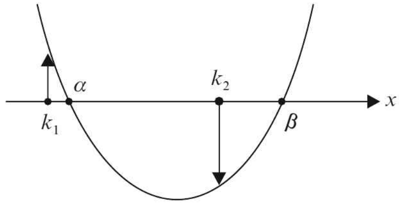
$$D \geq 0,\ af(k_1) > 0,\ af(k_2) > 0,\ k_1 < -\frac{b}{2a} < k_2.$$

(I) $\boxed{\text{Exactly one root lies between } k_1 \text{ and } k_2}$

$a > 0$

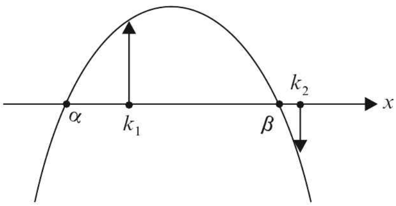


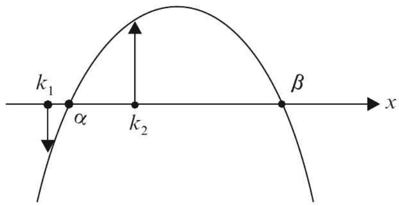
β lies between k_1 and k_2



α lies between k_1 and k_2

$a < 0$

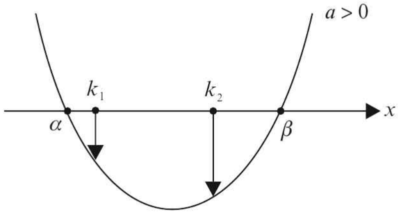


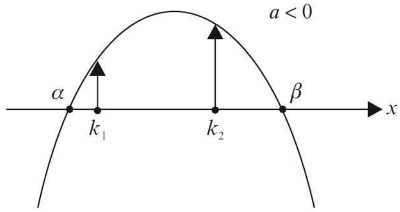


In all the four possible cases, $f(k_1)$ and $f(k_2)$ are of opposite sign. We can write this as $f(k_1)f(k_2) < 0$. Try to see that once we write this constraint, the basic constraint $D \geq 0$ becomes redundant (why? Because $f(k_1)$ and $f(k_2)$ can be of opposite sign only if the graph crosses the axis; this means that writing $f(k_1)\cdot f(k_2) < 0$ automatically implies that $f(x)$ will have real roots). The required constraint is:

$$f(k_1)\cdot f(k_2) < 0$$

(J) $\boxed{k_1 \text{ and } k_2 \text{ lie between the roots}}$





Here, we see that for both cases, $af(k_1)$ and $af(k_2)$ will be negative. Notice that this represents a sufficient condition. Thus, the required constraints are:

$$af(k_1) < 0, \;\; af(k_2) < 0.$$

3.6 Solving advanced inequalities

We know how to solve inequalities of the form $f(x) > 0$ or $f(x) < 0$, where $f(x)$ is a quadratic polynomial. We can extend this further to solve general rational inequalities. We note the following fundamental facts that hold for any inequality:

(a) If $a > b$, then $-a < -b$.

(b) If $a > b$, then $ac > bc$ only if $c > 0$. This means that you can multiply the two sides of an inequality by any factor *only if* that factor is positive. If that factor is negative, then the inequality gets reversed, *i.e.*, if $a > b$, and $c < 0$, then $ac < bc$. Part (a) above is a special case of this fact (multiplication by -1 on both sides).

(c) If $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$ (what if $0 > a > b$?).

(d) If $|a| < b$, then $-b < a < b$ (b is positive here).

(e) If $|a| > b$, then $a > b$ or $a < -b$.

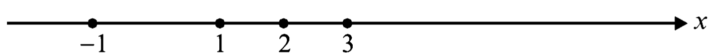
(f) If $b_1 < |a| < b_2$, then $b_1 < a < b_2$ or $-b_2 < a < -b_1$ (both b_1 and b_2 are positive).

(g) If $a^2 < b^2$, then $|a| < |b|$.

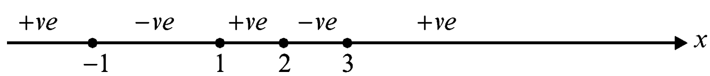
Suppose now that we need to solve the following inequality:

$$\frac{(x-1)(x-2)}{(x-3)(x+1)} > 0.$$

What we first do is multiply both sides of this inequality by $(x-3)^2(x+1)^2$ {since we know for sure that this term will be positive}. This multiplication reduces the expression above to $(x-1)(x-2)(x-3)(x+1) > 0$. The points where the left hand side can become 0 are $x = 1, 2, 3, -1$. We plot these points on a line:



If we take any value of x to the right 3, *i.e.*, $x > 3$, we see that all the four factors in the product above are positive, *i.e.*, for $x > 3$, the inequality above is satisfied. When $2 < x < 3$, $(x-3)$ becomes negative but the other factors still remain positive, i.e the product becomes negative and the inequality is not satisfied for this interval. Next, we see that when $1 < x < 2$, the product will be positive, when $-1 < x < 1$, the product is negative, while it is positive again for $x < -1$.



Hence, the requires values of x that satisfy the given inequality are:

$$x \in (-\infty, -1) \cup (1, 2) \cup (3, \infty).$$

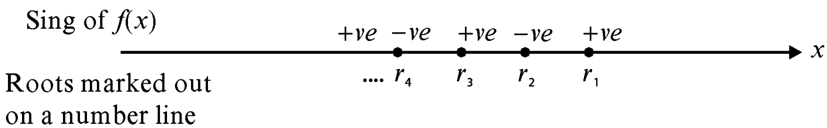
Generally speaking, let there be a rational function

$$f(x) = \frac{P(x)}{Q(x)}.$$

Suppose that this can be written, by factorizing the numerator and denominator, as follows:

$$f(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_n)}{(x-b_1)(x-b_2)\cdots(x-b_m)}.$$

To determine the sign of $f(x)$ in different intervals, we mark the roots of both the numerator and denominator on a number line (*i.e.*, all the $a_i's$ and $b_i's$). To the right of the largest root, *i.e.*, for $x >$ (the largest root), all the factors in both the numerator and denominator will be positive and $f(x) > 0$. $f(x)$ will then become negative in the preceding interval, then positive again in the next interval, and so on, *i.e.*, $f(x)$ will alternate in sign in adjacent intervals. Now, to solve $f(x) > 0$, we pick those values of x (those intervals) for which $f(x)$ is positive and to solve $f(x) < 0$, we pick those intervals of x for which $f(x)$ is negative.



IMPORTANT IDEAS AND TIPS

1. *Formulae Related to Progressions.* We have seen various formulae to evaluate the general term of a series, or the sum of the terms of a series. We also saw how to sum some special kinds of series, like AGPs. In all cases, it is imperative to understand the justification behind the formulae which you apply. This can help you in new scenarios where you can proceed by first principles if you know how to derive all these formulae.
2. *The Method of Differences.* One of the most versatile method of summing series is the method of differences. In fact, it is used not only to sum algebraic or number series, but also used extensively in the case of trigonometric or other kinds of progressions. The basic idea behind this method is simple: express each term of the series as a difference of two terms, so that there is cancellation in successive terms. For example,

$$\begin{aligned} S &= T_1 + T_2 + T_3 + \cdots + T_n \\ &= (R_1 - R_2) + (R_2 - R_3) + (R_3 - R_4) + \cdots + (R_n - R_{n+1}) \\ &= R_1 - R_{n+1} \end{aligned}$$

In the above example, after cancellation between successive terms, we are able to write the value of S in closed form from the original expanded form, *i.e.*, we are able to sum the series.

3. *Importance of Graphical Approach for Quadratics.* It can not be emphasized strongly enough how important graphical thinking is when it comes to solving problems related to quadratics. Many problems can be solved more quickly with a graphical rather than an algebraic approach. We will see examples of such problems in this chapter.

   (a) Given any quadratic expression, you should be able to draw its graph using shifting and scaling operations on the basic quadratic function $f(x) = x^2$. In particular, you should be clear about how the coordinates of the vertex of the parabola are obtained, and what those coordinates and the direction of the parabola (upwards or downwards opening) can tell us about the nature of the zeroes of the quadratic function. For example, if the y-coordinate of the vertex is positive, and the parabola opens upwards, this means that the parabola never intersects the x-axis, and so the roots are non-real.

(b) Solutions to inequalities are easy to picture using a graphical approach. For example, to solve the quadratic inequality $f(x) < 0$, we look for those values of x where the graph of f lies below the x-axis.

(c) In placing specific constraints on the roots of a quadratic (for example, the roots should lie between these particular values), a graphical approach would be of utmost necessity, because you will not be able to remember the conditions on coefficients for all the possible constraints. You will have to deduce those conditions from the constraints given in a particular problem—you will have to follow a first-principles graphical approach.

4. *Mistakes in Solving Inequalities.* The most common mistake in solving inequalities is to *cross-multiply.* For example, if a student is asked to solve the inequality

$$\frac{1}{x+1} > \frac{2}{(x+2)(x+3)},$$

she might be tempted to cross-multiply and write

$$(x+2)(x+3) > 2(x+2).$$

This is incorrect, because we are not sure of the signs of the factors which we are cross-multiplying. If they can be negative, then the direction of the inequality will not be preserved upon cross-multiplying. Never overlook this fact.

5. *Existence of Roots of a Quadratic.* When we say that the roots of a quadratic are non-real, we do *not* mean to say that the roots do not exist. The roots definitely exist, in *all* cases. However, when the roots are non-real, it means that they have complex values, so they do exist but not in the Real set.

General Algebra

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

How many digits will there be in 875^{16}?

(A) 47 (B) 48 (C) 49 (D) 50

Solution: We will follow an approach involving logarithms. The idea is to approximate the numerical value of the log of the given number to base 10. This will tell us the number of digits in the number. For example, consider $x = 300$.If we take the log of x to base 10, we have $\log_{10} x \approx 2.477$. That is, the logarithm of 10 has 2 in the integer part. This means that x should be larger than 10^2, but smaller than 10^3, or in other words, x should have 3 digits, which is correct. Applying this approach, we let $N = 875^{16}$:

$$\Rightarrow \quad \log N = 16\log(875) = 16\{\log(8.75\times 10^2)\} = 16(2+\log 8.75)$$

$$\Rightarrow \quad = 16 \times 2.9420 = 47.0721$$

$$\Rightarrow \quad N = \underbrace{\text{antilog}(0.0721)}_{\substack{\text{Contains 1 digit before}\\ \text{the decimal}}} \times \underbrace{10^{47}}_{\substack{\text{has 47}\\ \text{zeroes}}}$$

$$\Rightarrow \quad N \text{ contains 48 digits}$$

The correct option is (B). Note that we do not need to calculate the antilog of 0.0721 to arrive at the final answer. ■

Example 2

If the mth term of an AP is $\frac{1}{n}$ and the nth term is $\frac{1}{m}$, then the (mn)th term is

(A) 0 (B) 1 (C) $\dfrac{1}{mn}$ (D) $\dfrac{1}{m+n}$

Solution: From the relation for the general term t_r given by

$$t_r = a + (r-1)d,$$

we find that for this problem,

$$\Rightarrow \quad \frac{1}{n} = a + (m-1)d \tag{1}$$

$$\frac{1}{m} = a + (n-1)d$$

$$\Rightarrow \quad \frac{1}{n} - \frac{1}{m} = (m-n)d \quad \Rightarrow \quad \frac{m-n}{mn} = (m-n)d \quad \Rightarrow \quad mn = \frac{1}{d}$$

$$\Rightarrow \quad a = \frac{1}{mn} \text{ (from (1))}$$

Now, the (mn)th term is simply

$$t_{mn} = a + (mn-1)\ d = \frac{1}{mn} + (mn-1)\cdot\frac{1}{mn} = 1$$

The correct option is (B). ■

Example 3

If $b+c, c+a, a+b$ are in HP, then $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in

(A) AP (B) GP (C) HP (D) None of these

Solution: If the given numbers are in HP, then their reciprocals must be in AP:

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in AP}$$

$$\Rightarrow \quad \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b} \text{ are also in AP (why?)}$$

$$\Rightarrow \quad \frac{a+b+c}{b+c}-1, \frac{a+b+c}{c+a}-1, \frac{a+b+c}{a+b}-1 \text{ are also in AP}$$

$$\Rightarrow \quad \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in AP}$$

$$\Rightarrow \quad \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \text{ are in HP}$$

Therefore, the correct option is (C). ■

Example 4

How many real solutions exist for the equation $e^{\sin x} - e^{-\sin x} = 4$?

(A) 0 (B) 1 (C) 2 (D) More than 2

Solution: We observe that $(e^{\sin x})_{\text{min}} = e^{-1} = \frac{1}{e}$ and $(e^{\sin x})_{\text{max}} = e^{1} = e$. Denoting $e^{\sin x}$ by y, the given equation becomes

$$y - \frac{1}{y} = 4 \quad \Rightarrow \quad y^2 - 4y - 1 = 0$$

$$\Rightarrow \quad y = \frac{4 \pm \sqrt{20}}{2} = 2+\sqrt{5}, 2-\sqrt{5}$$

Since $2+\sqrt{5} > e$ and $2-\sqrt{5} < 0$, y can attain none of these values (in fact, the exact range of values that y can take is $[\frac{1}{e}, e]$). Thus, no real solutions exist for the given equation. The correct option is (A). ■

Example 5

If α, β be the roots of $m^2(x^2 - x) + 2mx + 3 = 0$ and m_1, m_2 be the two values of m for which α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{3}$, the value of $\frac{m_1^2}{m_2} + \frac{m_2^2}{m_1}$ is

(A) $\dfrac{-58}{3}$ (B) $\dfrac{-68}{3}$ (C) $\dfrac{-76}{3}$ (D) $\dfrac{-86}{3}$.

Solution: The statement of the problem might seem confusing at first, but the concept involved is simply the sum and product of roots of a quadratic equation. Since α, β are the roots of

$$f(x) = m^2x^2 + (2m - m^2)x + 3 = 0,$$

we have

$$\alpha + \beta = \frac{m^2 - 2m}{m^2} = 1 - \frac{2}{m} \text{ and } \alpha\beta = \frac{3}{m^2}.$$

Now,
$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} - 2 = \frac{4}{3}$$

$$\Rightarrow \quad (\alpha+\beta)^2 = \frac{10}{3}\alpha\beta \quad \Rightarrow \quad \left(1 - \frac{2}{m}\right)^2 = \frac{10}{m^2}$$

$$\Rightarrow \quad m^2 - 4m - 6 = 0 \quad \Rightarrow \quad m_1 + m_2 = 4, \quad m_1 m_2 = -6$$

$$\Rightarrow \quad \frac{m_1^2}{m_2} + \frac{m_2^2}{m_1} = \frac{m_1^3 + m_2^3}{m_1 m_2} = \frac{(m_1+m_2)^3 - 3m_1m_2(m_1+m_2)}{m_1m_2} = \frac{64 + 3\times 4\times 6}{-6} = \frac{-68}{3}.$$

Thus, the correct option is (B). ■

Example 6

In which of the following intervals can x lie so that it satisfies the inequality $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$?

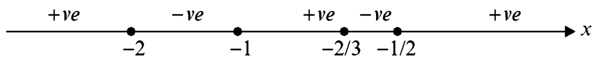
(A) $(-2, -1)$ (B) $\left(-\dfrac{2}{3}, \dfrac{1}{2}\right)$ (C) $\left(-\dfrac{1}{3}, \dfrac{1}{2}\right)$ (D) $\left(-\dfrac{3}{2}, -\dfrac{1}{2}\right)$

Solution: A common mistake in problems like these would be to cross-multiply. Since $(x+1)$ and $(2x^2+5x+2)$ are not necessarily positive, cross-multiplying would not be correct. Instead, we will have to proceed as follows:

$$\frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0 \quad \Rightarrow \quad \frac{2x(x+1) - (2x^2+5x+2)}{(x+1)(2x^2+5x+2)} > 0$$

$$\Rightarrow \quad \frac{-(3x+2)}{(x+1)(2x+1)(x+2)} > 0 \quad \Rightarrow \quad \frac{(3x+2)}{(x+1)(2x+1)(x+2)} < 0$$

We mark out the zeroes (of both the numerator and denominator) on a number line and pick the intervals where the expression is negative.



The required values of x are:

$$x \in (-2,-1) \cup \left(-\frac{2}{3},-\frac{1}{2}\right)$$

The correct options are (A) and (B). ■

Example 7

What is the range of values that $f(x)=\frac{x^2-3x+4}{x^2+3x+4}$ can assume, for real values of x?

(A) $\left[\frac{1}{3},3\right]$ (B) $\left[\frac{1}{4},4\right]$ (C) $\left[\frac{1}{7},7\right]$ (D) $\left[\frac{1}{4},1\right]$

Solution: The approach in such problems is as follows. We put $f(x)=y,$ find x in terms of y and find those values of y for which x is real. Imposing the constraint that x should be real restricts the values that y can take. These possible values of y form the range of $f(x)$ that we wish to determine.

$$f(x)=\frac{x^2-3x+4}{x^2+3x+4}=y$$

$$\Rightarrow \quad (1-y)x^2-3(1+y)x+4(1-y)=0 \qquad (1)$$

Since we want x to be real, the D of the quadratic in (1) above must be non-negative (this condition places a restriction on the values that y can take and hence gives us the range):

$$D \ge 0 \quad \Rightarrow \quad (3(1+y))^2 \ge 16(1-y)^2$$

$$\Rightarrow \quad 9y^2+18y+9 \ge 16y^2+16-32y \quad \Rightarrow \quad 7y^2-50y+7 \le 0$$

$$\Rightarrow \quad (7y-1)(y-7) \le 0 \quad \Rightarrow \quad \frac{1}{7} \le y \le 7$$

For these values of y, x is real in (1). Hence, $f(x)$ can take only these values. The range of $f(x)$ is

$$R=\left[\frac{1}{7},7\right]$$

Therefore, the correct option is (C). ■

SUBJECTIVE TYPE EXAMPLES

Example 8

Solve the following system of equations for x and y:

$$2^{x+y} = 6^y, \qquad 3^x = 3 \cdot 2^{y+1}$$

Solution: We take the logs of both sides of the two equations (using base 10):

$$(x+y)\log 2 = y\log 6 = y(\log 2 + \log 3)$$

$$\Rightarrow \quad x\log 2 = y\log 3 \tag{1}$$

And,

$$x\log 3 = \log 3 + (y+1)\log 2$$

$$\Rightarrow \quad (x-1)\log 3 = (y+1)\log 2 \tag{2}$$

Using (1) and (2), we have

$$x\log 2 = \left(\frac{(x-1)\log 3}{\log 2} - 1\right)\log 3$$

Using log 2 = a, log 3 = b for convenience, we have:

$$ax = \left(\frac{b(x-1)}{a} - 1\right)b \quad \Rightarrow \quad \frac{a}{b}x = \frac{b}{a}x - \frac{b}{a} - 1$$

$$\Rightarrow \quad \left(\frac{b}{a} - \frac{a}{b}\right)x = 1 + \frac{b}{a} = \frac{a+b}{a} \quad \Rightarrow \quad \left(\frac{b^2 - a^2}{ab}\right)x = \frac{a+b}{a}$$

$$\Rightarrow \quad x = \frac{b}{b-a} = \frac{\log 3}{\log 3 - \log 2} \approx \frac{0.4771}{0.1761} \approx 2.71$$

Similarly,

$$y = \frac{\log 2}{\log 3 - \log 2} \approx 1.71$$

■

Example 9

Can there be an AP with $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ as three of its terms?

Solution: Intuitively, the answer should be clear: there *cannot be* an AP with $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ as *any* of its three terms. Let us rigorously justify this; assume that there is an AP with first term a and CD d such that it has $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ has three of its terms. Let us go further and assume that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are the pth, qth and rth terms of this AP respectively, for some integers p, q, r. Thus,

$$\left.\begin{aligned}\sqrt{2} &= a + (p-1)d \\ \sqrt{3} &= a + (q-1)d \\ \sqrt{5} &= a + (r-1)d\end{aligned}\right\} \tag{1}$$

We need to show that the set of equations (1) cannot be solved for integer values of p, q, r. This can be done by contradiction:

$$\sqrt{3}-\sqrt{2}=(q-p)d \text{ and } \sqrt{5}-\sqrt{3}=(r-q)d$$

$$\Rightarrow \quad \frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{q-p}{r-q}=\lambda \text{ (assume)}$$

Note that λ must be some rational number since $\frac{q-p}{r-q}$ is rational. We now show that $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ is *not* rational (by contradiction). We have:

$$(\sqrt{3}-\sqrt{2})=\lambda(\sqrt{5}-\sqrt{3}).$$

Squaring gives

$$5-2\sqrt{6}=(8-2\sqrt{15})\lambda^2$$

$$\Rightarrow \quad \frac{8\lambda^2-5}{2}=\lambda^2\sqrt{15}-\sqrt{6}=\mu. \qquad \text{(assume)}$$

Again, μ should be some rational number. We now again square the last two expressions:

$$15\lambda^4+6-2\lambda^2\sqrt{90}=\mu^2$$

$$\Rightarrow \quad \sqrt{90}=\frac{15\lambda^4-\mu^2+6}{2\lambda^2} \quad \Rightarrow \quad \sqrt{10}=\frac{15\lambda^4-\mu^2+6}{6\lambda^2}.$$

This expression is contradictory, since $\sqrt{10}$ is irrational, while the right hand side is by assumption rational. Thus, the problem statement's assertion that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be three terms of the same AP, is true. ■

Example 10

If A_1 and A_2 be the AMs, G_1 and G_2 be the GMs, and H_1 and H_2 be the HMs, between two numbers x and y, show that:

$$\frac{G_1 G_2}{H_1 H_2}=\frac{A_1+A_2}{H_1+H_2}.$$

Solution: We have

$$\text{AM of } \{A_1, A_2\} = \text{AM of } \{x, y\}$$

$$\Rightarrow \quad \frac{A_1+A_2}{2}=\frac{x+y}{2} \quad \Rightarrow \quad A_1+A_2=x+y. \qquad (1)$$

Also,

$$\text{GM of } \{G_1, G_2\} = \text{GM of } \{x, y\}$$

$$\Rightarrow \quad \sqrt{G_1 G_2}=\sqrt{xy} \quad \Rightarrow \quad G_1 G_2 = xy \qquad (2)$$

Finally,

$$\text{HM of } \{H_1, H_2\} = \text{HM of } \{x, y\}$$

$$\Rightarrow \quad \frac{\frac{1}{H_1}+\frac{1}{H_2}}{2}=\frac{\frac{1}{x}+\frac{1}{y}}{2} \quad \Rightarrow \quad \frac{H_1 H_2}{H_1+H_2}=\frac{xy}{x+y} \qquad (3)$$

Thus,

$$\frac{G_1\,G_2}{H_1\,H_2} = \frac{xy}{\frac{xy}{x+y}\cdot(H_1+H_2)} \qquad \text{(Using (2) and (3))}$$

$$= \frac{x+y}{H_1+H_2} = \frac{A_1+A_2}{H_1+H_2} \qquad \text{(Using (1))}$$

■

Example 11

For three positive real numbers x, y, z, show that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9.$$

Solution: We simply use the AM–HM inequality on x, y, z:

$$\text{AM of } \{x, y, z\} \ge \text{HM of } \{x, y, z\} \quad \Rightarrow \quad \frac{x+y+z}{3} \ge \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$\Rightarrow \quad (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9.$$

Can you think of any alternative way of proving this inequality? When does the equality hold?

■

Example 12

The fourth power of the CD (common difference) of an AP with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.

Solution: We can assume values for the first of the four consecutive terms, and for the CD, say a and d respectively. The four terms will then be $a, a+d,\ a+2d$ and $a+3d,$ and our problem boils down to proving that

$$a(a+d)(a+2d)(a+3d)+d^4 \qquad (1)$$

is a perfect square. This will obviously work, but the form of the expression in (1) is such that algebraic manipulations to prove (1) to be a perfect square will be lengthy.

Instead, what we do is the following: since the choice of variables lies with us, we assume the four consecutive terms to be $a-3d,\ a-d,\ a+d$ and $a+3d,$ *i.e.*, we assume symmetric expressions for the four consecutive terms, with the CD being equal to $2d$. This helps because now the product of the four consecutive terms has a very simple expression. Also, note that there is no loss of generality in doing this, *i.e.*, this assumption is correct for every case. Thus,

$$(a-3d)(a-d)(a+d)\,(a+3d)+(2d)^4 = (a^2-9d^2)(a^2-d^2)+16d^4$$

$$= a^4-10a^2d^2+25d^4 = (a^2-5d^2)^2, \text{ which is obviously a perfect square}$$

Thus, you should appreciate how easy the algebraic manipulations have been rendered by an appropriate choice of variables for the four consecutive terms. This techniques holds in general: whenever you need to assume some terms for any AP, assume them symmetrically. ■

Example 13

Find the sum of the following series:

(a) $5 + 55 + 555 + \cdots$ n terms **(b)** $0.7 + 0.77 + 0.777 + \cdots$ n terms

Solution: Both these series are neither APs nor GPs. However, a little manipulation will convert both of them into a form we can sum:

(a) $S_1 = 5 + 55 + 555 + \cdots n$ terms $= 5\,(1 + 11 + 111 + \cdots n$ terms)

$$= \frac{5}{9}(9 + 99 + 999 + \cdots n \text{ terms}) \quad \{\text{Why we did this becomes clear in the next two steps}\}$$

$$= \frac{5}{9}\{(10-1) + (100-1) + (1000-1) + \cdots + (10^n - 1)\}$$

$$= \frac{5}{9}\left\{\underbrace{10 + 100 + 1000 + \cdots + 10^n}_{\text{A Geometric Progression!}} - n\right\} = \frac{5}{9}\left\{\frac{10(10^n - 1)}{10 - 1} - n\right\}$$

$$= \frac{5}{81}(10^{n+1} - 9n - 10)$$

(b) $S_2 = 0.7 + 0.77 + 0.777 + \cdots n$ terms

$$= 7\,(0.1 + 0.11 + 0.111 + \cdots n \text{ terms}) = \frac{7}{9}\,(0.9 + 0.99 + 0.999 + \cdots n \text{ terms})$$

$$= \frac{7}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \cdots \left(1 - \frac{1}{10^n}\right)\right\}$$

$$= \frac{7}{9}\left\{n - \underbrace{\left(\frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{10^n}\right)}_{\text{A Geometric Progression!}}\right\} = \frac{7}{9}\left\{n - \frac{1}{10}\left(\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}\right)\right\}$$

$$= \frac{7}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\} = \frac{7}{81}\left(9n - 1 + \frac{1}{10^n}\right)$$ ■

Example 14

Evaluate the following sums:

(a) $S_1 = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots n$ terms

(b) $S_2 = 1 - 3x + 5x^2 - 7x^3 + \cdots \infty$ for $|x| < 1$

Solution: These interesting series are examples of Arithmetico-Geometric Progressions (AGPs). The general term of an AGP is the product of the corresponding terms of an AP and a GP. For example, in S_1,

$$1, 4, 7, 10 \ldots \text{ are in AP and } 1, \frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3} \cdots \text{ are in GP}$$

An AGP has the general form

$$a, (a+d)r, (a+2d)r^2, \dots, \underbrace{(a+(n-1)d)r^{n-1}}_{\text{the } n\text{th term}}, \dots$$

Let us evaluate the sum S_n of an AGP for the general case, for n terms:

$$S_n = a + (a+d)r + (a+2d)r^2 + \cdots + (a+(n-1)d)r^{n-1} \tag{1}$$

The trick is to multiply S_n with the common ratio r of the GP:

$$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + \cdots + (a+(n-1)d)r^n \tag{2}$$

Doing (1)–(2) yields

$$\begin{aligned}(1-r)S_n &= a + dr + dr^2 + \cdots + dr^{n-1} - (a+(n-1)d)r^n \\ &= a + d(r + r^2 + \cdots + r^{n-1}) - (a+(n-1)d)r^n \\ &= a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - (a+(n-1)d)r^n\end{aligned}$$

$$\Rightarrow \qquad S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r}.$$

This is the general technique to sum an AGP. You need not memorize the general expression for S_n; its easier to apply this technique everytime we're required to sum an AGP. Coming back to the problems of this example:

(a) $S_1 = 1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \cdots + \dfrac{(3n-2)}{5^{n-1}}$ {Note carefully how the n^{th} term was written}

$$\Rightarrow \quad \frac{S_1}{5} = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \frac{10}{5^4} + \cdots + \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad S_1\left(1 - \frac{1}{5}\right) = 1 + \frac{3}{5} + \frac{3}{5^2} + \cdots + \frac{3}{5^{n-1}} - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad \frac{4}{5}S_1 = 1 + \frac{3}{5}\underbrace{\left(1 + \frac{1}{5} + \cdots + \frac{1}{5^{n-2}}\right)}_{\text{A Geometric Progression!}} - \frac{3n-2}{5^n} = 1 + \frac{3}{4}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{3n-2}{5^n}$$

$$\Rightarrow \quad S_1 = \frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{3n-2}{4.5^{n-1}}$$

(b) $S_2 = 1 - 3x + 5x^2 - 7x^3 + \cdots \infty$ For $|x| < 1$

Note that the CR of the GP is $-x$, so we have

$$-xS_2 = -x + 3x^2 - 5x^3 + 7x^4 - \cdots \infty$$

$$\Rightarrow \quad (1+x)S_2 = 1 - 2x + 2x^2 - 2x^3 + 2x^4 - \cdots \infty$$

$$= 1 - 2x(1 - x + x^2 - x^3 + \cdots \infty) = 1 - \frac{2x}{1+x} \quad \text{(how?)}$$

$$= \frac{1-x}{1+x}$$

$$\Rightarrow \quad S_2 = \frac{1-x}{(1+x)^2}$$

■

Example 15

Sum the following series to n terms:

(a) 1. 2. 3. + 2. 3. 4 + 3. 4. 5 + ⋯

(b) $3.8+6.11+9.14+\cdots$

(c) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

(d) $1^3+3.2^2+3^3+3.4^2+5^3+3.6^2+\cdots$ for even n (= $2m$)

(e) 5 + 7 + 13 + 31 + 85 + ⋯

(f) $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\cdots$

(g) $\dfrac{1}{1.2.3.4}+\dfrac{1}{2.3.4.5}+\dfrac{1}{3.4.5.6}+\cdots$

(h) $\dfrac{1}{1+1^2+1^4}+\dfrac{2}{1+2^2+2^4}+\dfrac{3}{1+3^2+2^4}+\cdots$

Solution: These eight series have been grouped together into one question since the approach in summing any such series is similar: identify the general term and write it using an index r. Then, split the general term into simpler terms, and sum over r. In each of the following parts, we denote the general, rth term by T_r.

(a)

$$T_r = r(r+1)(r+2) = r^3+3r^2+2r$$

$$\Rightarrow \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}(r^3+3r^2+2r) = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= \frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2}+(2n+1)+2\right\} = \frac{n(n+1)}{2}\cdot\left(\frac{n^2+5n+6}{2}\right)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

This is the required sum. Notice how we split the general term into simpler terms like $\Sigma r^3, \Sigma r^2$ and Σr, which we could sum easily.

(b)

$$T_r = 3r(3r+5) \quad \{\text{Understand this carefully}\}$$

$$= 9r^2+15r$$

$$\Rightarrow \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}(9r^2+15r)$$

$$= 9\sum_{r=1}^{n} r^2 + 15\sum_{r=1}^{n} r = \frac{9n(n+1)(2n+1)}{6} + \frac{15n(n+1)}{2}$$

$$= 3n(n+1)(n+3)$$

(c)

$$T_r = 1^2+2^2+3^2+\cdots+r^2 = \frac{r(r+1)(2r+1)}{6}$$

$$= \frac{1}{6}(2r^3+3r^2+r)$$

$$\Rightarrow \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\frac{1}{6}(2r^3+3r^2+r) = \frac{1}{6}\left\{2\sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right\}$$

$$= \frac{1}{6}\left\{\frac{2n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{12}\,n(n+1)\,\{n(n+1)+(2n+1)+1\} = \frac{n(n+1)^2\,(n+2)}{12}.$$

(d) This series can be split up into two separate series, both containing m terms.

$$S_1 = 1^3 + 3^3 + 5^3 + \cdots m \text{ terms},$$

$$S_2 = 3.2^2 + 3.4^2 + 3.6^2 + \cdots m \text{ terms}.$$

We evaluate both these series separately.

$$S_1 = \sum_{r=1}^{m} (2r-1)^3 = \sum_{r=1}^{m} (8r^3 - 12r^2 + 6r - 1)$$

$$= 8\sum_{r=1}^{m} r^3 - 12\sum_{r=1}^{m} r^2 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= \frac{8m^2(m+1)^2}{4} - \frac{12m(m+1)\,(2m+1)}{6} + \frac{6m(m+1)}{2} - m$$

$$= 2m^2(m+1)^2 - 2m(m+1)\,(2m+1) + 3m(m+1) - m \tag{1}$$

Now we evaluate the sum of the second series:

$$S_2 = \sum_{r=1}^{m} 3.(2r)^2 = 12\sum_{r=1}^{m} r^2 = \frac{12m\,(m+1)\,(2m+1)}{6}$$

$$= 2m(m+1)\,(2m+1) \tag{2}$$

Adding (1) and (2) gives us the actual sum S.

$$S = S_1 + S_2 = 2m^2(m+1)^2 + 3m(m+1) - m$$

$$= \frac{2n^2(n+2)^2}{16} + \frac{3n(n+2)}{4} - \frac{n}{2} \qquad \left(\text{Putting } m = \frac{n}{2}\right)$$

$$= \frac{n}{8}\,(n^3 + 4n^2 + 10n + 8)$$

(e) Note that the differences between successive terms form the sequence

2, 6, 18, 54, ...

which is a GP with CR $r = 3$. The kth term of this GP is $2 \cdot r^{k-1} = 2 \cdot 3^{k-1}$. Thus, if we denote the general term of the original sequence by T_k, we have

$$T_{k+1} - T_k = 2.3^{k-1}, \text{ with } T_1 = 5$$

Writing this repeatedly for $k = 1$ to $k = n - 1$, we have

$$T_2 - T_1 = 2.3^0$$

$$T_3 - T_2 = 2.3^1$$

$$T_4 - T_3 = 2.3^2$$

$$\vdots$$

$$T_n - T_{n-1} = 2.3^{n-2}$$

Adding these $(n-1)$ relations gives

$$T_n - T_1 = 2\cdot(3^0 + 3^1 + \cdots + 3^{n-2}) = 2\cdot\left(\frac{3^{n-1}-1}{3-1}\right) = 3^{n-1}-1$$

$$\Rightarrow \quad T_n = 3^{n-1} + T_1 = 3^{n-1} + 4$$

Thus, we've succeeded in writing the expression for the general term of the series. Note that this could have been written by (keen!) observation also, but we did it the way we did to outline a rigorous technique using which you can write the general term. Finally, we can evaluate the sum S for n terms now:

$$S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (3^{r-1}+4) = \frac{3^n-1}{3-1} + 4n = \frac{3^n + 8n - 1}{2}$$

(f) We have

$$T_r = \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\left\{\frac{1}{2r-1} - \frac{1}{2r+1}\right\}$$

$$\Rightarrow \quad \sum_{r=1}^{n} T_r = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\left\{\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$$

(g)
$$T_r = \frac{1}{r(r+1)(r+2)(r+3)} \tag{1}$$

Note that we cannot express T_r as simply as in part (*f*). Instead, we'll have to resort to a slightly more complex manipulation: specifically, we write a relation between T_r and T_{r+1}. We have,

$$T_{r+1} = \frac{1}{(r+1)(r+2)(r+3)(r+4)} \tag{2}$$

Using (1) and (2) gives

$$rT_r = (r+4)T_{r+1}$$

$$\Rightarrow \quad rT_r - (r+1)T_{r+1} = 3T_{r+1} \tag{3}$$

We did this because now (3) can be repeatedly applied for $r = 1$ to $r = n-1$.

$$1.T_1 - 2.T_2 = 3.T_2$$
$$2.T_2 - 3.T_3 = 3.T_3$$
$$3.T_3 - 4.T_4 = 3.T_4$$
$$\vdots$$
$$(n-1).T_{n-1} - n.T_n = 3.T_n$$

Adding these gives

$$T_1 - nT_n = 3(T_2 + T_3 + \cdots + T_n)$$

Adding $3T_1$ on both sides gives

$$4T_1 - nT_n = 3(T_1 + T_2 + \cdots T_n)$$

Thus, the sum $\sum_{r=1}^{n} T_r$ is simply

$$\sum_{r=1}^{n} T_r = \frac{4T_1 - nT_n}{3} = \frac{4\cdot\frac{1}{1.2.3.4} - n\cdot\frac{1}{n.(n+1)(n+2)(n+3)}}{3}$$

$$= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

(h)

$$T_r = \frac{r}{1+r^2+r^4} = \frac{r}{(1-r+r^2)(1+r+r^2)}$$

$$= \frac{1}{2}\left\{\frac{1}{1-r+r^2} - \frac{1}{1+r+r^2}\right\} \quad \text{(Understand this carefully)}$$

Thus,

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\sum_{r=1}^{n}\left\{\frac{1}{1-r+r^2} - \frac{1}{1+r+r^2}\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{1-n+n^2}-\frac{1}{1+n+n^2}\right)\right\}$$

$$= \frac{1}{2}\left(1-\frac{1}{1+n+n^2}\right) = \frac{n(n+1)}{2(n^2+n+1)}.$$

■

Example 16

Let a, b, c be real numbers. If $ax^2 + bx + c = 0$ has two real roots α, β where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Solution: We see that –1 and 1 lie between the roots of the equation. Therefore, the constraints that need to be satisfied are:

(i) $af(k_1) < 0 \;\Rightarrow\; a\cdot\{a(1)^2 + b(1) + c\} < 0$

$\Rightarrow\; a\{a+b+c\} < 0$

Now, we divide both sides of the inequality by a^2 (this is allowed since $a^2 > 0$):

$$\Rightarrow\; 1 + \frac{b}{a} + \frac{c}{a} < 0 \qquad (1)$$

(ii) $af(k_2) < 0 \;\Rightarrow\; a\cdot\{a(-1)^2 + b(-1) + c\} < 0$

$\Rightarrow\; a\{a-b+c\} < 0$

Again, we divide both sides by a^2:

$$\Rightarrow \quad 1-\frac{b}{a}+\frac{c}{a}<0 \tag{2}$$

(1) and (2) can be combined into

$$1+\left|\frac{b}{a}\right|+\frac{c}{a}<0.$$

■

Example 17

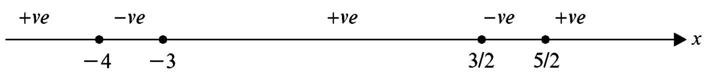
Find the values of x which satisfy $\dfrac{8x^2+16x-51}{(2x-3)(x+4)}<3$

Solution: Once again, do not be tempted to cross multiply by the denominator on both sides since the denominator is not necessarily positive. The correct way to solve this inequality is as follows:

$$\frac{8x^2+16x-51}{(2x-3)(x+4)}-3<0 \quad \Rightarrow \quad \frac{(8x^2+16x-51)-3(2x^2+5x-12)}{(2x-3)(x+4)}<0$$

$$\Rightarrow \quad \frac{2x^2+x-15}{(2x-3)(x+4)}<0 \quad \Rightarrow \quad \frac{(2x-5)(x+3)}{(2x-3)(x+4)}<0$$

We mark out the zeroes (of the numerator and denominator) on a number line and pick the intervals where the expression is negative.



The required values of x are

$$x\in(-4,-3)\cup\left(\frac{3}{2},\frac{5}{2}\right)$$

■

Example 18

Solve the following inequalities for x:

(a) $\dfrac{10x}{x^2+9}\le 1$ (b) $\left|\dfrac{x^2-5x+4}{x^2-4}\right|\le 11$ (c) $\left|\dfrac{x^2-1}{x^2+x+1}\right|<1$ (d) $\log_{10}(x^2-3x+3)\ge 0$

Solution: **(a)** For this case, notice that the denominator x^2+9 is always positive; this means that we can directly cross-multiply the denominator to get

$$x^2+9\ge 10x \quad \Rightarrow \quad x^2-10x+9\ge 0$$

$$\Rightarrow \quad (x-1)(x-9)\ge 0 \quad \Rightarrow \quad x\le 1 \text{ or } x\ge 9$$

(b) This inequality is equivalent to

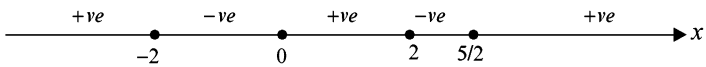
$$\overbrace{-1 \le}^{A} \frac{x^2 - 5x + 4}{x^2 - 4} \underbrace{\le 1}_{B}$$

We need to consider inequalities A and B separately and then find those values of x that satisfy both these inequalities simultaneously.

Inequality A:

$$-1 \le \frac{x^2 - 5x + 4}{x^2 - 4} \quad \Rightarrow \quad \frac{x^2 - 5x + 4}{x^2 - 4} + 1 \ge 0 \quad \Rightarrow \quad \frac{2x^2 - 5x}{x^2 - 4} \ge 0$$

$$\Rightarrow \quad \frac{x(2x - 5)}{(x - 2)(x + 2)} \ge 0$$



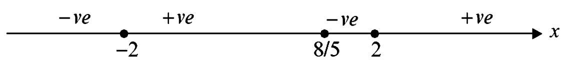
$$\Rightarrow \quad x < -2 \text{ or } 0 \le x < 2 \text{ or } x \ge \frac{5}{2}$$

Note above that x cannot take the values -2 or 2 since the denominator would then become 0.

Inequality B:

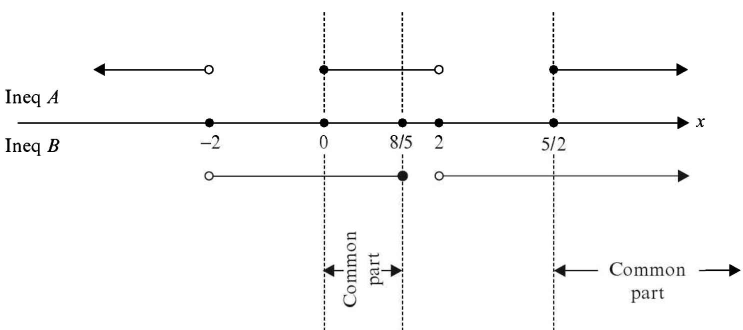
$$\frac{x^2 - 5x + 4}{x^2 - 4} \le 1 \quad \Rightarrow \quad \frac{x^2 - 5x + 4}{x^2 - 4} - 1 \le 0 \quad \Rightarrow \quad \frac{8 - 5x}{x^2 - 4} \le 0$$

$$\Rightarrow \quad \frac{5x - 8}{(x + 2)(x - 2)} \ge 0$$



$$\Rightarrow \quad -2 < x \le \frac{8}{5} \text{ or } x > 2$$

We now find the intersection of the solutions to these two inequalities:



The solution for x is therefore:

$$x \in \left[0, \frac{8}{5}\right] \cup \left[\frac{5}{2}, \infty\right)$$

(c) This inequality can be written equivalently as

$$-1 < \frac{x^2 - 1}{x^2 + x + 1} < 1$$

Notice that the denominator of the given rational expression is always positive and hence it can directly be cross-multiplied across both the inequalities to get

$$-x^2 - x - 1 < x^2 - 1 < x^2 + x + 1$$

The two inequalities considered separately are:

$$\begin{array}{lll} \text{(i)}\ \ x^2 - 1 < x^2 + x + 1 & \Rightarrow & x > -2 \\ \text{(ii)}\ \ -x^2 - x - 1 < x^2 - 1 & \Rightarrow & 2x^2 + x > 0 \\ & \Rightarrow & x(2x+1) > 0 \\ & \Rightarrow & x < -1/2 \text{ or } x > 0 \end{array}$$

The common part of the two solutions can be seen to be

$$x \in \left(-2, -\frac{1}{2}\right) \cup (0, \infty)$$

(d) Since $\log_{10}(x^2 - 3x + 3) \geq 0$

$$\begin{array}{llll} \Rightarrow & x^2 - 3x + 3 \geq 10^0 = 1 & \Rightarrow & x^2 - 3x + 2 \geq 0 \\ \Rightarrow & (x-1)(x-2) \geq 0 & \Rightarrow & x \leq 1 \quad \text{or} \quad x \geq 2 \end{array}$$

■

Example 19

If $a < b < c < d$, then prove that for any real λ, $f(x) = (x-a)(x-c) + \lambda(x-b)(x-d) = 0$ will always have real roots.

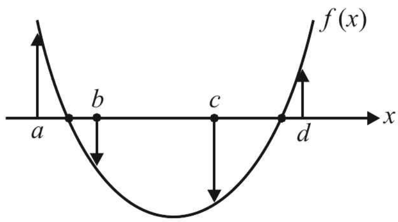
Solution: We can of course follow the approach of writing $f(x)$ in the form of a standard quadratic equation, evaluating its discriminant and proving it to be non-negative for all real values of λ. However, such an approach would become unnecessarily lengthy. Instead, we will use a graphical approach that is quicker and also gives us more insight into the nature of the roots than the discriminant approach. We consider three cases, $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$ and show that in all cases, the roots are real. In graphical terms, this means that in all cases, the graph should intersect the x-axis.

$$\boxed{\lambda = 0} \quad \Rightarrow \quad f(x) = (x-a)(x-c) = 0.$$

This obviously has two real roots, namely $x = a, c$

$$\boxed{\lambda > 0} \Rightarrow \begin{aligned} f(a) &= \lambda(a-b)(a-d) > 0 \\ f(b) &= (b-a)(b-c) < 0 \\ f(c) &= \lambda(c-b)(c-d) < 0 \\ f(d) &= (d-a)(d-c) > 0 \end{aligned}$$

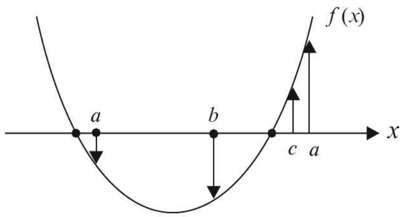
Notice that $f(a)$ and $f(b)$ are of opposite signs. This means that the graph has to necessarily cross the x-axis between a and b, or in other words, there is a zero of $f(x)$ between a and b. Similarly, there lies another zero of $f(x)$ between c and d. The graph for $f(x)$ is approximately sketched below:



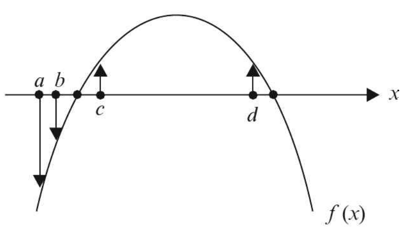
We see that the graphical approach not only tells us that two real and distinct roots exist, but also gives us the location of the roots.

$$\boxed{\lambda < 0} \Rightarrow \begin{aligned} f(a) &= \lambda(a-b)(a-d) < 0 \\ f(b) &= (b-a)(b-c) < 0 \\ f(c) &= \lambda(c-b)(c-d) > 0 \\ f(d) &= (d-a)(d-c) > 0 \end{aligned}$$

Since $f(b)$ and $f(c)$ are of the opposite signs, there must be a root of $f(x)$ between b and c. $f(x)$ can have the following configurations:



Since the parabola opens upwards here, the coefficient of x^2 in $f(x)$ *i.e.*, $(1+\lambda)$ should be positive, so that $\lambda > -1$.



Since the parabola opens downwards, $1+\lambda < 0$ or $\lambda < -1$.

Here also, since the graph of $f(x)$ intersects the x-axis, $f(x)$ has real roots. In addition, notice that the graph also tells us that if $\lambda > -1$, the second root lies to the left of a, while if $\lambda < -1$, the second root lies to the right of d. (What if $\lambda = -1$?)

In all cases, $f(x)$ has real roots. ■

Example 20

If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \delta$, $\beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$ for some constant δ, then prove that

$$\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$$

Solution: This result can be achieved by some straight-forward algebraic manipulation. We have:

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

$$\alpha + \beta + 2\delta = -\frac{B}{A} \quad \alpha \quad (\alpha + \delta)(\beta + \delta) = \frac{C}{A}$$

$$\text{LHS } = \frac{b^2 - 4ac}{a^2} = \left(\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) = (\alpha + \beta)^2 - 4\alpha\beta \;= (\alpha - \beta)^2$$

$$\text{RHS } = \left(\frac{B}{A}\right)^2 - 4\frac{C}{A} \;= (\alpha + \beta + 2\delta)^2 - 4(\alpha + \delta)(\beta + \delta)$$

$$= \{(\alpha + \delta) + (\beta + \delta)\}^2 - 4(\alpha + \delta)(\beta + \delta) = \{(\alpha + \delta) - (\beta + \delta)\}^2 = (\alpha - \beta)^2$$

■

Example 21

Prove that the equation

$$\frac{1}{x} + \frac{1}{x+a} + \frac{1}{x-b} = 0 \qquad (a, b > 0)$$

has two roots, one each in the intervals $(-\frac{2a}{3}, -\frac{a}{3})$ and $(\frac{b}{3}, \frac{2b}{3})$.

Solution: The given equation, when written in the form of a standard quadratic equation, becomes $3x^2 + 2(a-b)x - ab = 0$. Let the left hand expression in the problem be denoted by $f(x)$. To prove that a zero of $f(x)$ lies between, for example, $-\frac{2a}{3}$ and $-\frac{a}{3}$, the obvious thing to do is evaluate $f(-\frac{2a}{3})$ and $f(-\frac{a}{3})$ and show that these are of opposite signs, or, $f(-\frac{2a}{3}) \cdot f(-\frac{a}{3}) < 0$. Following this approach, we have

$$f\left(-\frac{2a}{3}\right) \cdot f\left(-\frac{a}{3}\right) = \left(\frac{4a^2}{3} - \frac{4a(a-b)}{3} - ab\right) \times \left(\frac{a^2}{3} - \frac{2a(a-b)}{3} - ab\right)$$

$$= -\frac{a^2 b}{9}(a+b) < 0$$

Similarly

$$f\left(\frac{b}{3}\right)\cdot f\left(\frac{2b}{3}\right)=\left(\frac{b^2}{3}+\frac{2b(a-b)}{3}-ab\right)\times\left(\frac{4b^2}{3}+\frac{4b(a-b)}{3}-ab\right)=-\frac{-ab^2}{9}(a+b)<0.$$

Hence, the assertion in the problem is true. ■

Example 22

If a, b, c, d be real numbers in GP and u, v, w satisfy the equations

$$u+2v+3w=6 \qquad 4u+5v+6w=12 \qquad 6u+9v=4,$$

then show that the roots of the following two quadratic equations are reciprocal to each other.

$$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+\{(b-c)^2+(c-a)^2+(d-b)^2\}x+(u+v+w)=0$$

$$20x^2+10(a-d)^2x-9=0.$$

Solution: We first solve the three given equation for u, v, w to get:

$$u=-\frac{1}{3} \qquad v=\frac{2}{3} \qquad w=\frac{5}{3}.$$

Now, since a, b, c, d are in GP, we let $b=ar$, $c=ar^2$, $d=ar^3$. The coefficient of x in the first given equation is:

$$(b-c)^2+(c-a)^2+(d-b)^2=a^2\{(r-r^2)^2+(r^2-1)^2+(r^3-r)^2\}$$

$$=a^2(r-1)^2(r^2+r+1)^2=a^2(r^3-1)^2=(d-a)^2.$$

Therefore, the given equation reduces to (verify)

$$-9x^2+10(d-a)^2x+20=0.$$

If α, β, are its roots, then:

$$\alpha+\beta=\frac{10}{9}(d-a)^2,\ \alpha\beta=-\frac{20}{9}.$$

Now let us find the equation whose roots are $\frac{1}{\alpha},\frac{1}{\beta}$:

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{1}{2}(d-a)^2$$

$$\frac{1}{\alpha}\cdot\frac{1}{\beta}=\frac{1}{\alpha\beta}=-\frac{9}{20}$$

Therefore, the required equation is

$$x^2 + \frac{1}{2}(d-a)^2 - \frac{9}{20} = 0$$

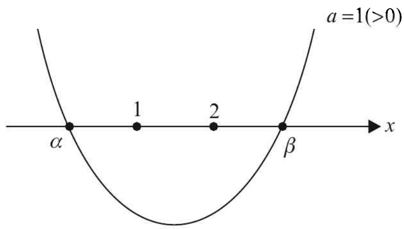
or $$20x^2 + 10(a-d)^2 - 9 = 0.$$

This is the second given equation. Hence the roots of the two given equations are reciprocals of each other. ■

Example 23

Find the values of a for which the inequality $x^2 + ax + a^2 + 6a < 0$ is satisfied for all $x \in [1,2]$.

Solution: We will first try to visualize (graphically) what the problem statement means. Since $f(x) = x^2 + ax + a^2 + 6a < 0$ for all $x \in [1,2]$, this means that the parabola for $f(x)$ should remain below the x-axis for the entire interval [1, 2]; this can only happen if the interval [1, 2] falls between the zeroes of $f(x)$:



We see that $f(x) < 0$ for all $x \in [1, 2]$

We can now impose the following constraints:

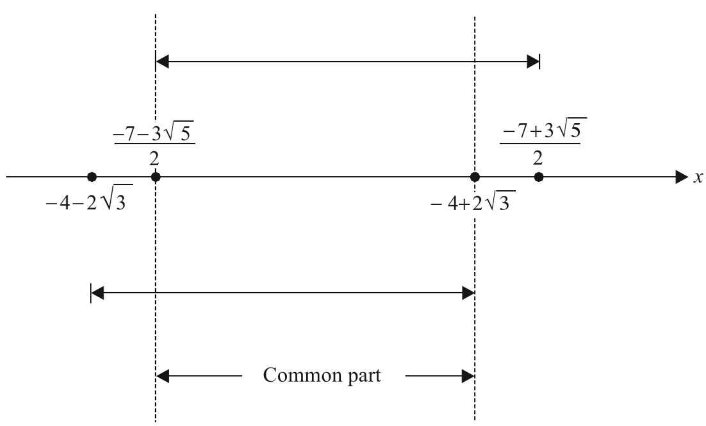
$$1 \cdot f(1) < 0 \Rightarrow 1 \cdot \{(1)^2 + a \cdot (1) + a^2 + 6a\} < 0$$

$$\Rightarrow a^2 + 7a + 1 < 0 \quad \Rightarrow \quad \frac{-7-3\sqrt{5}}{2} < a < \frac{-7+3\sqrt{5}}{2}$$

$$1 \cdot f(2) < 0 \Rightarrow 1 \cdot \{\}(2)^2 + a \cdot (2) + a^2 + 6a | < 0$$

$$\Rightarrow a^2 + 8a + 4 < 0 \quad \Rightarrow \quad -4 - 2\sqrt{3} < a < -4 + 2\sqrt{3}$$

We now find the values of a satisfying both the constraints above:



We see that:

$$\frac{-7-3\sqrt{5}}{2} < a < -4+2\sqrt{3}.$$ ■

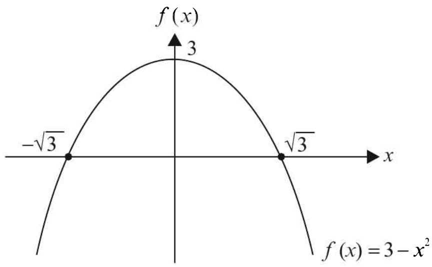
Example 24

Find the values of a for which the inequality $x^2 + |x-a| - 3 < 0$ is satisfied by at least one negative x.

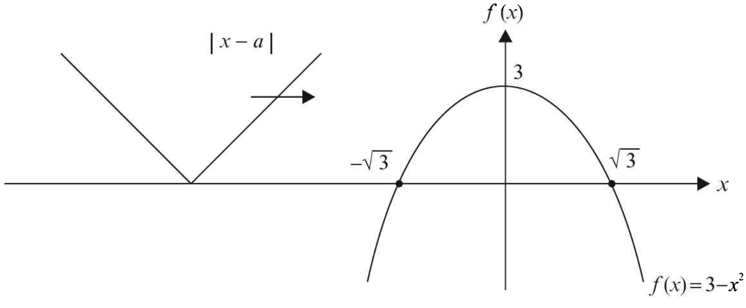
Solution: Although this problem can be solved analytically, we will follow a graphical approach here and you will see just how powerful thinking in terms of graphs is. The given inequality can be written as:

$$|x-a| < 3 - x^2$$

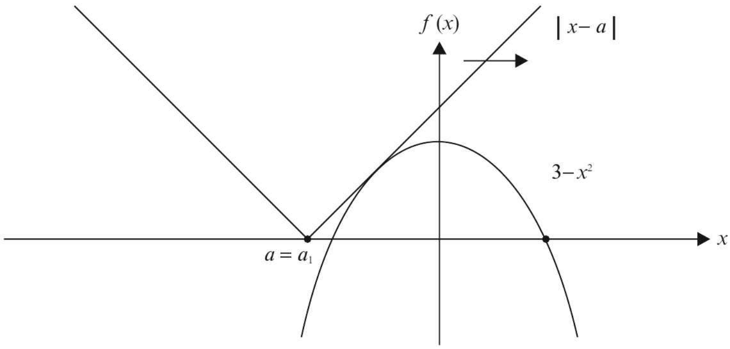
This should be satisfied for at least one negative value of x. Equivalently stated, the graph of the function $|x-a|$ should lie *beneath* the graph of the function $(3-x^2)$ for at least one negative value of x. The graph of $3-x^2$ is fixed:



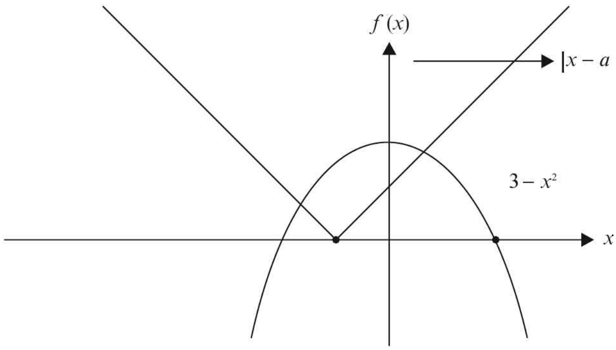
The graph of $|x-a|$ is a left or right-shifted version of the graph of $|x|$, depending on the value of a. We need a to be such a value so that the graph of $|x-a|$ goes beneath that of $(3-x^2)$ for some negative x. Imagine that we start with large negative values of a so that the graph is left shifted to a large extent.



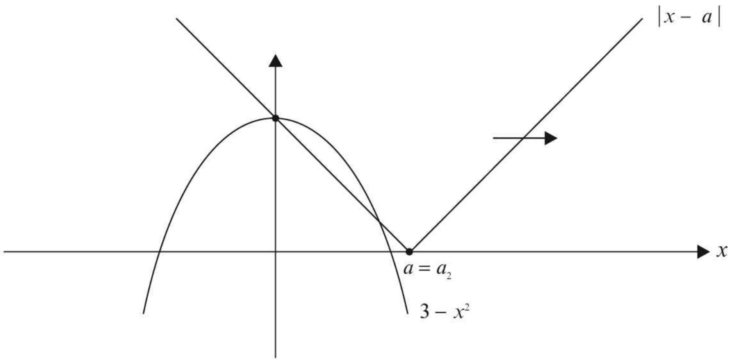
As we decrease the magnitude of a, the mod-graph shifts to the right. At some point (some particular value of a, say $a = a_1$), the right arm of the mod-graph just touches the parabola (becomes a tangent to it):



As we further right-shift the mod-graph, the right arm of the mod graph goes below the parabola for some negative values of x.



In fact, after a point, the left arm also goes below the parabola for certain negative values of x (you are urged to visualise all this). The left arm remains below the parabola for certain negative x values until the other extreme is achieved (at another value of a, say $a = a_2$), when the left arm passes through the vertex of the parabola):



Hence, when $a_1 < a < a_2$, at least one of the arms of the mod-graph will be below the parabola for some negative values of x. We can, using coordinate geometry, easily evaluate a_1 and a_2 to be $-\frac{13}{4}$ and 3 respectively. This is left to the reader as an exercise. Therefore,

$$a \in \left(-\frac{13}{4}, 3\right). \qquad \blacksquare$$

General Algebra

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. What is the value of the sum S given by

$$S = \frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \cdots + \frac{1}{\log_{100} N}$$

(A) $\dfrac{1}{\log_N 100!}$ (B) $\log_N 100!$ (C) $\dfrac{N}{\log_N 100!}$ (D) $\dfrac{1}{N}\log_N 100!$

P2. Consider the value of x which satisfies the following relation:

$$\frac{6}{5} a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} = 3^{\log_{10} \frac{x}{10}} + 9^{\log_{100} x + \log_4 2}.$$

This value of x lies between:

(A) 10 and 20 (B) 30 and 40 (C) 75 and 85 (D) 95 and 105.

P3. Let $k = \log_2 \log_3 5^{\lambda}$. What set of values can λ take such that the following inequality is satisfied?

$$1 < (25)^{2^{-k}} < 3$$

(A) $(-\tfrac{1}{2}, \infty)$ (B) $(\tfrac{1}{2}, \infty)$ (C) $(1, \infty)$ (D) $(2, \infty)$

P4. If $6x^5 + 5x^3 + x^2 + 4x - 1 = a(x-1)^5 + b(x-1)^4 + c(x-1)^3 + d(x-1)^2 + e(x-1) + f$, what is the value of $a + 2b + 3c + 4d + 5e + 6f$?

(A) 780 (B) 820 (C) 870 (D) 910

P5. If $a_1, a_2, \ldots, a_n$ is an AP of non-zero terms, what is the sum of the following series?

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}$$

(A) $\frac{(n-1)^2}{a_1a_2a_{n-1}a_n}$ (B) $\frac{n^2}{a_1a_2a_{n-1}a_n}$ (C) $\frac{n-1}{a_1a_n}$ (D) $\frac{n}{a_1a_n}$

P6. Let α, β be the roots of $x^2+px+1=0$, and γ, δ be the roots of $x^2+qx+1=0$. What is the value of the following expression?

$$P=(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$$

(A) q^2-p^2 (B) p^2-q^2 (C) $(p+q)^2$ (D) $(p-q)^2$

P7. What is the set of values of a for which the inequality

$$4^x-a(2^x)-a+3\le 0$$

is satisfied by at least one real x?

(A) $[-1,\infty)$ (B) $[0,\infty)$ (C) $[1,\infty)$ (D) $[2,\infty)$

P8. If $x^2+(a-b)x+(1-a-b)=0$ where $a, b\in R$, then what is the complete set of values of a for which this equation has unequal real roots for all values of b?

(A) $(1,\infty)$ (B) $\left(\frac{3}{2},\infty\right)$ (C) $(2,\infty)$ (D) $\left(\frac{5}{2},\infty\right)$

P9. The sum of all possible integers λ such that $(x-\lambda)(x-10)+1=0$ has integer roots is

(A) 12 (B) 15 (C) 20 (D) 24

P10. If $x^2-10ax-11b=0$ has roots c and d, and $x^2-10cx-11d=0$ has roots a and b, then the value of $a+b+c+d$ lies between

(A) 1150 and 1170 (B) 1200 and 1220 (C) 1240 and 1260 (D) 1280 and 1300

P11. What is the set of real values of c so that $\frac{x^2+2x+c}{x^2+4x+3c}$ can take all real values for $x\in\mathbb{R}$?

(A) [0, 1] (B) [0, 2] (C) [1, 2] (D) [0, 3]

P12. What is the minimum value of $f(x)=\frac{(x+a)(x+b)}{(x+c)}$, where $x>-c,\ a>c,\ b>c$?

(A) $\sqrt{a-c}+\sqrt{b-c}$ (B) $\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$

(C) $2\left(\sqrt{a-c}+\sqrt{b-c}\right)$ (D) $2\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$

P13. For what values of the parameter a does the equation $x^4+2ax^3+x^2+2ax+1=0$ have at least two distinct negative roots?

(A) $(0,\infty)$ (B) $(\frac{1}{4},\infty)$ (C) $(\frac{1}{2},\infty)$ (D) $(\frac{3}{4},\infty)$

P14. If β is such that $\sin 2\beta \neq 0$, then the value of the expression $\frac{x^2+2x\cos 2\alpha+1}{x^2+2x\cos 2\beta+1}(x \in \mathbb{R})$ will lie between which of the following two values?

(A) $\tan^2\alpha,\ \tan^2\beta$ (B) $\dfrac{\cos^2\alpha}{\cos^2\beta}, \dfrac{\sin^2\alpha}{\sin^2\beta}$ (C) $\dfrac{\cos^2\alpha}{\sin^2\beta}, \dfrac{\sin^2\alpha}{\cos^2\beta}$ (D) $\dfrac{\cos^2\beta}{\cos^2\alpha}, \dfrac{\sin^2\beta}{\sin^2\alpha}$

P15. For any real x, what is the maximum value of the expression $y = 2(k-x)(x+\sqrt{x^2+k^2})$?

(A) $\dfrac{k^2}{4}$ (B) $\dfrac{k^2}{2}$ (C) k^2 (D) $2k^2$

P16. For what real values of a do the roots of the equation $x^2-2x-(a^2-1)=0$ lie between the roots of the equation $x^2-2(a+1)x+a(a-1)=0$?

(A) $\left(-\dfrac{1}{2},\dfrac{3}{2}\right)$ (B) $\left(-\dfrac{1}{2},2\right)$ (C) $\left(-\dfrac{1}{4},\dfrac{6}{5}\right)$ (D) $\left(-\dfrac{1}{4},1\right)$

P17. Let $f(x)=ax^2+bx+c$, where $a,b,c \in \mathbb{R}$. Suppose that $|f(x)| \leq 1\ \forall\ x \in [0,1]$. Which of the following statements are true?

(A) $|a| \leq 8$ (B) $|b| \leq 8$ (C) $|c| \leq 1$ (D) $|a|+|b|+|c| \leq 17$

SUBJECTIVE TYPE EXAMPLES

P18. If $\log_4(x+2y)+\log_4(x-2y)=1$, find the minimum value of $|x|-|y|$.

P19. If a, b, c are positive real numbers, then is the following inequality true?

$$\{(1+a)(1+b)(1+c)\}^7 > 7^7\, a^4\, b^4\, c^4$$

P20. Prove that $\underbrace{1111......1}_{91\text{ times}}$ is not a prime number.

P21. If $a_1, a_2, ..., a_{2n}$ be in AP, prove that

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}\left(a_1^2 - a_{2n}^2\right)$$

P22. Find the sum S given by

$$S = 1^2 + 2^2 x + 3^2 x^2 + 4^2\, x^3 + \cdots\infty \text{ for } |x| < 1$$

P23. Let

$$x = 1 + 3a + 6a^2 + 10a^3 + \cdots\infty, \quad |a| < 1$$

$$y = 1 + 4b + 10b^2 + 20b^3 + \cdots\infty, \quad |b| < 1$$

Find

$$S = 1 + 3(ab) + 5(ab)^2 + \cdots\infty$$

in terms of x and y.

P24. Let α, β be the roots of the equation $x^2 - x + b = 0$. Let $S_k = \alpha^k + \beta^k$. Find b such that S_2, S_3 and S_5 are in AP.

P25. The sum of a certain number of consecutive positive integers is 1000. Find these integers.

P26. Let $a_1 = 1$ and $a_n + a_{n-1} = n^2 + 2n, n \geq 2$. Evaluate $S_1 = \sum_{k=1}^{2m} a_k$ and $S_2 = \sum_{k=1}^{2m+1} a_k$.

P27. Consider a sequence with $a_1 = 1$, $a_2 = 1$ and satisfying the property $P: a_n = a_{n-1} + a_{n-2}, n > 2$. If the general solution for the nth term of this sequence is

$$a_n = C_1(r_1)^n + C_2(r_2)^n$$

where r_1, r_2 are the roots of the equation formed by substituting $a_n = r^n$ in P, find

(a) a_n (b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

P28. If $\sum_{r=1}^{n} t_r = \frac{1}{8}n(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{t_r}$.

P29. Let $a_1, a_2, a_3, \ldots,$ be positive real numbers in geometric progression. For each *n*, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of $a_1, a_2, \ldots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$.

P30. A sequence $a_1, a_2, a_3, \ldots, a_n, a_{n+1}$ of real numbers is such that

$$a_1 = 0, |a_2| = |a_1+1|, |a_3| = |a_2+1|, \ldots, |a_n| = |a_{n-1}+1|,\ |a_{n+1}| = |a_n+1|$$

Can the arithmetic mean of $\{a_1, a_2, \ldots, a_n\}$ be less than $-1/2$?

P31. If *a*, *b*, *c* are in GP and the equation $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root, then is it true that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in GP?

P32. The sum to *n* terms of two series in AP are in the ratio $(7n+1):(4n+17)$, where *n* is any positive integer. Find the ratio of their *n*th terms.

P33. Find the sum of the following series:

$$\frac{1^4}{1.3}+\frac{2^4}{3\cdot 5}+\frac{3^4}{5\cdot 7}+\cdots+\frac{n^4}{(2n-1)(2n+1)}$$

P34. Let *a*, *b* be positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, then

(a) Show that $\dfrac{G_1G_2}{H_1H_2} = \dfrac{A_1+A_2}{H_1+H_2}$

(b) Find the value of these two expressions.

P35. Consider the sum *S* given by

$$S = \frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001}$$

(a) Is it true that $S > 1$?

(b) Is it true that $S < \frac{3}{2}$?

(c) Is it true that $S < \frac{4}{3}$?

P36. Evaluate $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m+m3^n)}$

P37. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n+1}$ respectively, then find the value of $S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$.

P38. Sum the series

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\ldots(1+a_n)}$$

P39. Let $a_1, a_2, \ldots, a_n$ be the terms of a GP whose common ratio is r. Let S_k denote the sum of the first k terms of the GP Find the value of $\sum_{1 \le i < j \le m} a_i\, a_j$ in terms of S_{m-1} and S_m.

P40. Determine all values of a such that the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have at least one common root.

P41. Let α, β be the roots of the equation $ax^2 + 2bx + c = 0$ and γ, δ be the roots of the equation $px^2 + 2qx + r = 0$. If $\alpha, \beta, \gamma, \delta$ are in GP, then show that

$$\frac{ac}{b^2} = \frac{pr}{q^2}.$$

P42. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Find the value of r in terms of p and q.

P43. Let $f(x) = Ax^2 + Bx + C$ $(A, B, C \in \mathbb{R})$. Prove that if $f(x)$ is an integer whenever x is an integer, then $2A$, $A + B$ and C are all integers. Prove the converse also.

P44. Solve the following equation for x, assuming that the value of the parameter a is such that a solution exists.

$$\sqrt{a + \sqrt{a + x}} = x.$$

P45. If α is a real root of the quadratic equation $ax^2 + bx + c = 0$ and β is a real root of $-ax^2 + bx + c = 0$, show that there is a root γ of the equation $\frac{a}{2}x^2 + bx + c = 0$ which lies between α and β.

P46. Find the real values of a for which the equation

$$(\tan^2\theta + 1)^2 + 4a\tan\theta(\tan^2\theta + 1) + 16\tan^2\theta = 0$$

has four distinct real roots in $(0, \frac{\pi}{2})$.

P47. For $a \le 0$, determine all real roots of the equation

$$x^2 - 2a\,|x - a| - 3a^2 = 0.$$

P48. Find all the solutions of the equation

$$|x+1|-|x|+3|x-1|-2|x-2|=x+2.$$

P49. If $\frac{1}{a+b+c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then show that for odd n,

$$\frac{1}{a^n+b^n+c^n}=\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}.$$

P50. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that

$$[nx]=[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right].$$

P51. If $a \in \mathbb{R}$, solve the following equation:

$$x+a^3=\sqrt[3]{a-x}.$$

P52. Consider a $S=\{x \mid x^n=1\}$ and a subset of S given by $S_1=\{(x^k)^p \mid x^k \in S, p \in \mathbb{Z}\}$. For what values of k is S_1 equals to S?

General Algebra

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. Use the following two facts about logs:

(i) $\dfrac{1}{\log_k N} = \log_N k$

(ii) $\log_N k_1 + \log_N k_2 = \log_N (k_1 k_2)$

The answer can be obtained as $\log_N 100!$. The correct option is thus (B).

S2. We use the fact that $\log_b a = \frac{\log_c a}{\log_c b}$ to simply both the sides.

$$\frac{6}{5} a^{\frac{\log x}{\log a} \cdot \frac{\log a}{\log 10} \cdot \frac{\log 5}{\log a}} = 3^{\log_{10} x - 1} + 9^{\frac{1}{2}\log_{10} x + \frac{1}{2}} \tag{1}$$

Consider the term on the left side:

$$a^{\frac{\log x}{\log a} \cdot \frac{\log 5}{\log 10}} = a^{\log_a x \cdot \log_{10} 5} = x^{\log_{10} 5} = 5^{\log_{10} x} \qquad \text{(how?)}$$

Using this in (1), along with the substitution $\log_{10} x = t$, we have

$$\frac{6}{5} 5^t = 3^{t-1} + 9^{\frac{t+1}{2}} = 3^{t-1} + 3^{t+1} = 3^t \left(3 + \frac{1}{3}\right) = 10.3^{t-1}$$

$$\Rightarrow \quad 5^{t-2} = 3^{t-2} \quad \Rightarrow \quad t = 2 \quad \Rightarrow \quad x = 100$$

Thus, the correct option is (D).

S3. $k = \log_2 \log_3 5^\lambda \quad \Rightarrow \quad \lambda \log_3 5 = 2^k \quad \Rightarrow \quad 2^{-k} = \log_5 3^{1/\lambda}$

$$\Rightarrow \quad 1 < 25^{\log_5 3^{1/\lambda}} < 3 \Rightarrow \quad 1 < 3^{2/\lambda} < 3$$

$$\Rightarrow \quad 0 < \frac{2}{\lambda} < 1 \quad \Rightarrow \quad \lambda > 2$$

The correct option is therefore (D).

S4. We express x in terms of a new variable y as follows:

$$x-1=\frac{1}{y} \quad \Rightarrow \quad x=\frac{y+1}{y}$$

Why we did this will become clear only once you understand the entire solution. Substituting this into the given relation, we have

$$\frac{6(y+1)^5}{y^5}+\frac{5(y+1)^3}{y^3}+\frac{(y+1)^2}{y^2}+\frac{4(y+1)}{y}-1=\frac{a}{y^5}+\frac{b}{y^4}+\frac{c}{y^3}+\frac{d}{y^2}+\frac{e}{y}+f$$

Multiplying by y^6 across both sides, we have

$$6y(y+1)^5+5y^3(y+1)^3+y^4(y+1)^2+4y^5(y+1)-y^6=ay+by^2+cy^3+dy^4+ey^5+fy^6$$

Differentiating both sides and then substituting $y = 1$ gives us the value we are seeking:

$$a+2b+3c+4d+5e+6f=910$$

The straightforward calculation of this value is left to the reader as an exercise. The correct option is (D).

S5. We note that

$$\frac{1}{a_r a_{r+1}}=\left(\frac{1}{a_r}-\frac{1}{a_{r+1}}\right)\times\frac{1}{(a_{r+1}-a_r)}$$

Now, since the $a_i's$ are in AP, $a_{r+1}-a_r$ will be constant for all values of r, and will equal the CD of the AP, say d. Thus,

$$\frac{1}{a_r a_{r+1}}=\frac{1}{d}\left(\frac{1}{a_r}-\frac{1}{a_{r+1}}\right)$$

so that

$$\frac{1}{a_1a_1}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}=\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\frac{1}{d}\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\cdots+\frac{1}{d}\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)$$

$$=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots+\frac{1}{a_{n-1}}-\frac{1}{a_n}\right\}=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_n}\right\}$$

$$=\frac{1}{d}\left\{\frac{a_n-a_1}{a_1a_n}\right\}=\frac{1}{d}\left\{\frac{(n-1)d}{a_1a_n}\right\}=\frac{n-1}{a_1a_n}$$

Thus, we've been able to express the sum in terms of the first and last terms of the AP. The correct option is (C).

S6. We have

$$P = (\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta) = (\alpha\beta-\gamma(\beta+\alpha)+\gamma^2)\,(\alpha\beta+\delta(\alpha+\beta)+\delta^2)$$

$$= (\gamma^2+\gamma p+1)(\delta^2-\delta p+1)$$

Now, $\gamma^2+q\gamma+1=0 \;\Rightarrow\; \gamma^2+1=-q\gamma$

Similarly, $\delta^2+q\delta+1=0 \;\Rightarrow\; \delta^2+1=-q\delta$

$$\Rightarrow\; P=q^2-p^2$$

The correct option is (A).

S7. Note that since $2^x > 0$, if we consider the quadratic inequality as a quadratic in $y = 2^x$:

$$y^2-ay-a+3\le 0,$$

this must be satisfied for at least one *positive* y, which means that both the roots of the equation

$$y^2-ay-a+3=0 \tag{1}$$

cannot be non-positive. Now, (1) will have real roots if

$$a^2+4(a-3)\ge 0 \;\Rightarrow\; a\in(-\infty,-6]\cup[2,\infty)$$

Also, both roots are non-positive if

$$a\le 0, -a+3\ge 0 \;\Rightarrow\; a\in(-\infty,-6]$$

Thus, for at least one positive root, we have $a\in[2,\infty)$.

S8. The important part of the question is the phrase 'for *all* values of b, *i.e.*, $\forall\, b\in\mathbb{R}$, the equation should have unequal real roots. Imposing the constraint $D > 0$, we have

$$(a-b)^2-4(1-a-b)>0$$

$$\Rightarrow\; b^2+(4-2a)b+a^2+4a-4>0 \tag{1}$$

(1) should hold true $\forall b\in\mathbb{R}$, which means that the discriminant of the quadratic expression in (1) must be *negative*:

$$(4-2a)^2-4(a^2+4a-4)<0$$

$$\Rightarrow\; -32a+32<0 \;\Rightarrow\; a>1.$$

The correct option is (A).

S9. Let $(x-\lambda)(x-10)+1=(x+\alpha)(x+\beta), \alpha, \beta \in \mathbb{Z}$. Substitute $x=-\alpha$:

$$(\alpha+\lambda)(\alpha+10)=-1 \quad \Rightarrow \quad \alpha+\lambda=1 \quad \text{and} \quad \alpha+10=-1$$

$$\text{or}$$

$$\alpha+\lambda=-1 \quad \text{and} \quad \alpha+10=1$$

$$\Rightarrow \quad \alpha=-11 \text{ so } \lambda=12, \text{ or } \alpha=-9, \text{ so } \lambda=8$$

The two possible values of λ are 8 and 12. The correct option is (C), as 8 + 12 = 20.

S10. We have $10a=c+d$ and $10c=a+b;$ adding these, we obtain:

$$a+b+c+d=10(a+c), \tag{1}$$

while subtracting these, we obtain

$$b-d=11(c-a) \tag{2}$$

Now, since c is a root of the first equation, we have $a^2-10ac-11b=0$. Similarly, since a is a root of the second equation, we have $a^2-10ac-11b=0$. Subtracting these, we obtain

$$c^2-a^2-11(b-d)=0 \tag{3}$$

Using $b-d=11(c-a)$ in(1), we have

$$c^2-a^2=121(c-a)$$

If $c=a$, then the two equations become identical (why?); we assume that $c \neq a$, and thus $a+c=121$. Using this in (1), we have $a+b+c+d=1210$. Thus, the correct option is (B).

S11. We want the given expression to assume all real values (for appropriate x), *i.e*, we want the range of the given expression to be $\mathbb{R}$.

$$\frac{x^2+2x+c}{x^2+4x+3c}=y$$

$$\Rightarrow \quad (1-y)x^2+(2-4y)x+c(1-3y)=0\cdot$$

For x to be real, the D for this equation should be non-negative:

$$\Rightarrow \quad 4(1-2y)^2 \geq 4c(1-y)(1-3y)$$

$$\Rightarrow \quad 4y^2-4y+1 \geq c(3y^2-4y+1)$$

$$\Rightarrow \quad (4-3c)y^2+(4c-4)y+(1-c) \geq 0 \tag{1}$$

Now comes the crucial step. Since we want the range of y to be $\mathbb{R}$, the constraint (1) should be satisfied by each real value of y. This means that the parabola for the left hand side of (1) should not go below the axis for any value of y.

$\Rightarrow$ The discriminant for the LHS of (1) cannot be positive

$\Rightarrow$ D of (1) ≤ 0 $\quad\Rightarrow\quad 16(1-c)^2 \leq 4(1-c)(4-3c)$

$\Rightarrow$ $4c^2 - 8c + 4 \leq 3c^2 - 7c + 4 \quad\Rightarrow\quad c^2 - c \leq 0$

$\Rightarrow$ $c(c-1) \leq 0 \quad\Rightarrow\quad 0 \leq c \leq 1$

The correct option is (A).

S12. We let

$$y = \frac{(x+a)(x+b)}{(x+c)} \quad\Rightarrow x^2 + (a+b-y)x + (ab-cy) = 0$$

Since x is real, $D \geq 0$:

$$\Rightarrow (a+b-y)^2 - 4(ab-cy) \geq 0$$

$$\Rightarrow y^2 - 2(a+b-2c)y + (a+b)^2 - 4ab \geq 0$$

$$\Rightarrow y^2 - 2(a+b-2c)y + (a-b)^2 \geq 0 \qquad (1)$$

The roots of this quadratic expression in y are

$$y_1, y_2 = \frac{2(a+b-2c) \pm \sqrt{4(a+b-2c)^2 - 4(a-b)^2}}{2}$$

$$= (a+b-2c) \pm \sqrt{4ab + 4c^2 - 4c(a+b)} = a+b-2c \pm 2\sqrt{ab + c^2 - c(a+b)}$$

$$= a+b-2c \pm 2\sqrt{(a-c)(b-c)} = (a-c) + (b-c) \pm 2\sqrt{(a-c)(b-c)}$$

$$= \left(\sqrt{a-c} \pm \sqrt{b-c}\right)^2$$

Let $\quad y_1 = \left(\sqrt{a-c} - \sqrt{b-c}\right)^2$ and $y_2 = \left(\sqrt{a-c} + \sqrt{b-c}\right)^2$

The solution for y in (1) is

$$y \leq y_1 \quad \text{and} \quad y \geq y_2$$

We can discard the solution $y \leq y_1$ (Why? Make sure you understand this step!) so that $y \geq y_2$. The minimum value of y is y_2, or

$$y_{\min} = \left(\sqrt{a-c} + \sqrt{b-c}\right)^2$$

The correct option is (B).

S13. This is a fourth degree polynomial. Notice that the coefficients are 'symmetric about the middle', so we can proceed as follows. We divide the equation by x^2 to get

$$x^2+2ax+1+\frac{2a}{x}+\frac{1}{x^2}=0 \quad \Rightarrow \quad x^2+\frac{1}{x^2}+2a\left(x+\frac{1}{x}\right)+1=0$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2-2+2a\left(x+\frac{1}{x}\right)+1=0$$

$$\Rightarrow \quad y^2+2y-1=0 \quad \text{where } y=x+\frac{1}{x}.$$

The roots for this equation are:

$$y=-a\pm\sqrt{1+a^2}$$

We let

$$x+\frac{1}{x}=-a+\sqrt{1+a^2} \tag{1}$$

$$\Rightarrow \quad x^2+\left(a-\sqrt{1+a^2}\right)x+1=0$$

$$\Rightarrow \quad S=\sqrt{1+a^2}-a>0$$

$$P=1>0.$$

This means that both the roots of (1) are positive. Therefore, for at least two negative roots, both roots of the second quadratic equation, which we can form (using the second value of y), that is,

$$x+\frac{1}{x}=-a-\sqrt{1+a^2} \tag{2}$$

should be negative.

$$\Rightarrow \quad x^2+\left(a+\sqrt{1+a^2}\right)x+1=0 \tag{3}$$

$$S=-\left(a+\sqrt{1+a^2}\right)<0$$

$$P=1>0$$

Hence, we see that both the roots will be negative. But we have to still ensure that a satisfies the basic constraint $D>0$ for equation (3):

$$D>0 \quad \Rightarrow \quad \left(a+\sqrt{1+a^2}\right)^2>4$$

$$\Rightarrow \quad \left(a+\sqrt{1+a^2}+2\right)\left(a+\sqrt{1+a^2}-2\right)>0$$

Since the first term is always positive, the inequality above reduces to

$$\sqrt{1+a^2}>2-a$$

This can easily be solved graphically (or analytically) to get $a>\frac{3}{4}$. The correct option is (D).

S14. Essentially, we have to find the range of the given expression. We let

$$y = \frac{x^2 + 2x\cos 2\alpha + 1}{x^2 + 2x\cos 2\beta + 1}$$

$$\Rightarrow \quad (1-y)x^2 + 2(\cos 2\alpha - y\cos 2\beta)x + (1-y) = 0$$

Since x is real, the D for this equation is non-negative:

$$D \ge 0 \quad \Rightarrow \quad 4(\cos 2\alpha - y\cos 2\beta)^2 \ge 4(1-y)^2$$

$$\Rightarrow \quad \cos^2 2\alpha + y^2\cos^2 2\beta - 2y\cos 2\alpha\cos 2\beta \ge y^2 + 1 - 2y$$

$$\Rightarrow \quad (\cos^2 2\beta - 1)y^2 + 2y(1 - \cos 2\alpha\cos 2\beta) + (\cos^2 2\alpha - 1) \ge 0 \qquad (1)$$

The roots of this expression are

$$y_1, y_1 = \frac{(\cos 2\alpha\cos 2\beta - 1) \pm \sqrt{(1-\cos 2\alpha\cos 2\beta)^2 - (\cos^2 2\alpha - 1)(\cos^2 2\beta - 1)}}{(\cos^2 2\beta - 1)}$$

$$= \frac{(\cos 2\alpha\cos 2\beta - 1) \pm \sqrt{\cos^2 2\alpha + \cos^2 2\beta - 2\cos 2\alpha\cos 2\beta}}{\cos^2 2\beta - 1}$$

$$= \frac{(\cos 2\alpha\cos 2\beta - 1) \pm (\cos 2\alpha - \cos 2\beta)}{\cos^2 2\beta - 1}$$

$$= \frac{(\cos 2\beta + 1)(\cos 2\alpha - 1)}{\cos^2 2\beta - 1}, \frac{(\cos 2\beta - 1)(\cos 2\alpha + 1)}{\cos^2 2\beta - 1}$$

$$= \frac{\cos 2\alpha - 1}{\cos 2\beta - 1}, \frac{\cos 2\alpha + 1}{\cos 2\beta + 1} = \frac{\sin^2\alpha}{\sin^2\beta}, \frac{\cos^2\alpha}{\cos^2\beta}$$

Notice that in (1), the coefficient of y^2 is negative so its solution will be all values of y lying between the roots. Hence, y lies between $\frac{\sin^2\alpha}{\sin^2\beta}$ and $\frac{\cos^2\alpha}{\cos^2\beta}$.

The correct option is (B).

S15. We have upon rearranging:

$$x + \sqrt{x^2 + k^2} = \frac{y}{2(k-x)} \qquad (1)$$

$$\Rightarrow \quad \frac{k^2}{\sqrt{x^2+k^2} - x} = \frac{y}{2(k-x)}$$

$$\Rightarrow \quad \sqrt{x^2+k^2} - x = \frac{2k^2(k-x)}{y} \qquad (2)$$

By (1) – (2) and subsequent rearrangement, we can form a quadratic in x:

$$4(k^2-y)x^2-4k(2k^2-y)x+(4k^4-y^2)=0 \tag{3}$$

Now, we impose the condition that the roots of (3) must be real:

$$16k^2(2k^2-y)^2 \geq 16(k^2-y)(4k^4-y^2)$$

$$\Rightarrow \quad y^2(y-2k^2)\leq 0 \quad \Rightarrow \quad y\leq 2k^2$$

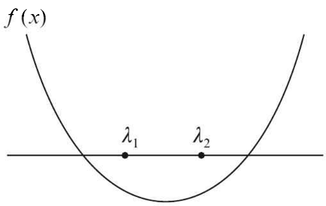
Therefore, the maximum value of y is $2k^2$. The correct option is (D).

S16. The first equation is

$$x^2-2x+1=a^2 \;\Rightarrow\; (x-1)=\pm a \;\Rightarrow\; x=1\pm a=\lambda_1,\lambda_2\,(\text{say})$$

These two values should lie between the roots of

$$f(x)\equiv x^2-2(a+1)x+a(a-1)=0$$



Thus,

$$\begin{matrix} f(\lambda_1)=f(1-a)<0 \\ f(\lambda_2)=f(1+a)<0 \end{matrix} \;\Rightarrow\; \left\{\begin{matrix} (1-a)^2+2(a^2-1)+a(a-1)<0 \;\Rightarrow\; -\dfrac{1}{4}<a<1 \\ (1+a)^2-2(1+a)^2+a(a-1)<0 \;\Rightarrow\; a>-\dfrac{1}{3} \end{matrix}\right\}$$

Taking the intersection of these two solution sets, the required answer is $a\in(-\frac{1}{4},1)$. The correct option is (D).

S17. We substitute $x=0,1$ and $\frac{1}{2}$ in the inequality $|f(x)|\leq 1$:

$$|c|\leq 1 \;\Rightarrow\; -1\leq c\leq 1 \tag{1}$$

$$|a+b+c|\leq 1 \;\Rightarrow\; -1\leq a+b+c\leq 1 \tag{2}$$

$$\left|\frac{a}{4}+\frac{b}{2}+c\right|\leq 1 \;\Rightarrow\; -4\leq a+2b+4c\leq 4 \tag{3}$$

From Eq. (3), since $c\in[-1,1]$, we have

$$-8\leq a+2b\leq 8 \tag{4}$$

Modifying Eqs. (2) and (3) slightly, we have

$$-4\leq 4a+4b+4c\leq 4$$

$$-4\leq -a-2b-4c\leq 4$$

Adding, we have

$$-8 \le 3a + 2b \le 8 \tag{5}$$

From Eqs (4) and (5), we have

$$-16 \le 2a \le 16 \quad \text{(How?)}$$

$$\Rightarrow \quad -8 \le a \le 8 \tag{6}$$

Multiplying Eq. (4) by 3, we have

$$-24 \le 3a + 6b \le 24 \tag{7}$$

From Eqs. (5) and (7)

$$-32 \le 4b \le 32$$

$$\Rightarrow \quad -8 \le b \le 8 \tag{8}$$

Thus, $|a| \le 8, |b| \le 8, |c| \le 1$ and consequently, $|a| + |b| + |c| \le 17$. All the statements given in the problem are true.

SUBJECTIVE TYPE EXAMPLES

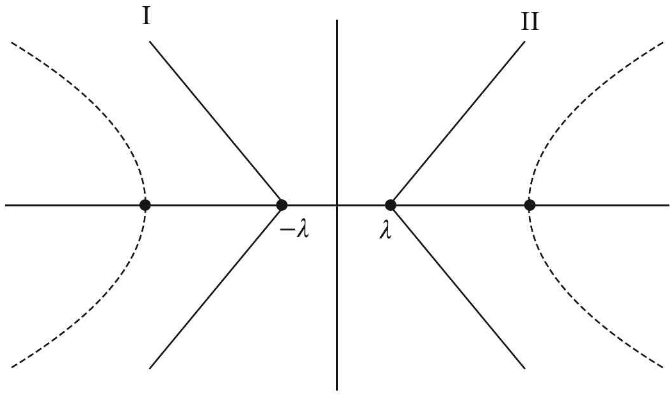
S18. We will develop a solution using coordinate geometry. We have

$$\log_4[(x+2y)(x-2y)]=1 \quad \Rightarrow \quad \log_4(x^2-4y^2)=1$$

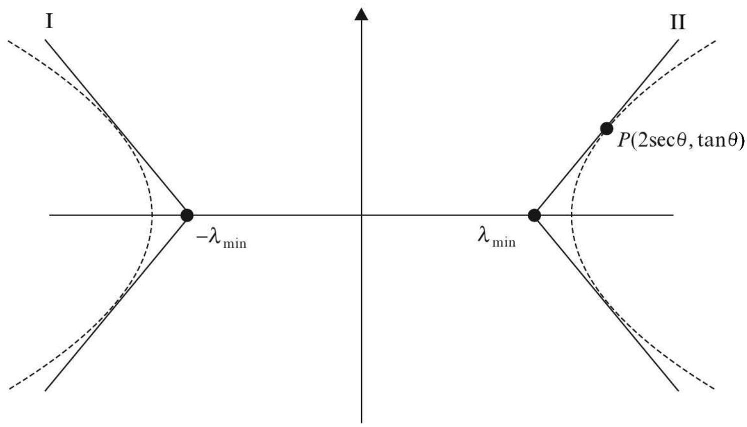
$$\Rightarrow \quad x^2-4y^2=4$$

$$\Rightarrow \quad \frac{x^2}{4}-y^2=1 \tag{1}$$

This represents a hyperbola, any point on which can be assumed to be $P(2\sec\theta, \tan\theta)$. Let us denote the value of $|x|-|y|$ by λ, and its minimum possible value by $\lambda_{\min}$. We note that $|x|-|y|=\lambda$ represents a set of lines (actually, rays) as shown below (along with the hyperbola in Eq. (1)):



As λ is increased, the segments I and II move 'outwards', and at $\lambda_{\min}$, these segments will be tangent to the two hyperbolic segments:



At P, the slope of the tangent is $\frac{\operatorname{cosec}\theta}{2}$, which must be equal to 1:

$$\frac{\operatorname{cosec}\theta}{2}=1 \quad \Rightarrow \quad \operatorname{cosec}\theta=2 \quad \Rightarrow \quad \theta=\frac{\pi}{6}$$

The point P is thus $(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and the equation of the upper part of segment II is:

$$y - \frac{1}{\sqrt{3}} = x - \frac{4}{\sqrt{3}} \quad \Rightarrow \quad x - y = \sqrt{3}$$

The intersection of this line with the x-axis is the point $(\sqrt{3}, 0)$. Thus,

$$\lambda_{\min} = \sqrt{3}$$

S19. Yes. We have

$$(1+a)(1+b)(1+c) = 1 + a + b + c + ab + bc + ca + abc$$

$$\geq 1 + 7(a^4b^4c^4)^{1/7} \qquad \text{(AM – GM inequality)}$$

$$> 7(a^4b^4c^4)^{1/7}$$

$$\Rightarrow \quad \{(1+a)(1+b)(1+c)\}^7 > 7^7 a^4b^4c^4$$

S20. To prove so, we need to show that this number (call it N) can be factorized. We have,

$$N = \underbrace{1111...1}_{91 \text{ times}} = 10^{90} + 10^{89} + 10^{88} + \cdots + 1 = \frac{10^{91} - 1}{10 - 1}$$

Now, since 91 can be factorized as 7×13, we have

$$N = \frac{10^{91}-1}{10-1} = \frac{10^{91}-1}{10^7-1} \times \frac{10^7-1}{10-1} = \frac{(10^7)^{13}-1}{10^7-1} \times \frac{10^7-1}{10-1}$$

$$= \{(10^7)^{12} + (10^7)^{11} + \cdots 1\} \times \{10^6 + 10^5 + \cdots 1\} \qquad \text{(how?)}$$

$$= N_1 \times N_2$$

Thus, N is not a prime number.

S21. Assuming d to be the CD of the AP, we have

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \cdots + (a_{2n-1} - a_{2n})(a_{2n-1} + a_{2n})$$

$$= -d(a_1 + a_2) - d(a_3 + a_4) - \cdots - d(a_{2n-1} + a_{2n})$$

$$= -d\{a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n}\}$$

$$= -dn\{a_1 + a_{2n}\} \qquad \text{(why?)}$$

$$= \frac{-dn\{a_1^2 - a_{2n}^2\}}{a_1 - a_{2n}} = \frac{-dn\{a_1^2 - a_{2n}^2\}}{-(2n-1)d} = \frac{n}{2n-1}\left(a_1^2 - a_{2n}^2\right)$$

S22. Note that the x-terms are in GP, but the numeric terms are not in AP, so this not an AGP. However, as the alert reader will realise, this can be reduced to an AGP as follows:

$$S = 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots \infty$$

$$xS = 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots \infty$$

$$\Rightarrow (1-x)S = 1^2 + (2^2 - 1^2)x + (3^2 - 2^2)x^2 + (4^2 - 3^2)x^3 + \cdots \infty$$

$$= \underbrace{1 + 3x + 5x^2 + 7x^3 + \cdots \infty}_{\text{An AGP now!}} \qquad (1)$$

Now we apply the conventional method of summing an AGP. We have

$$x(1-x)S = x + 3x^2 + 5x^3 + 7x^4 + \cdots \infty \qquad (2)$$

Doing (1)–(2) yields

$$(1-x)S - x(1-x)S = 1 + 2x + 2x^2 + 2x^3 + \cdots \infty$$

$$\Rightarrow (1-x)^2 S = 1 + 2x(1 + x + x^2 + \cdots \infty) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

$$\Rightarrow S = \frac{1+x}{(1-x)^3}$$

S23. The series for both x and y are not AGPs. However, they can be reduced to AGPs. We do this for both x and y.

For x

$$x = 1 + 3a + 6a^2 + 10a^3 + \cdots \infty$$

$$\Rightarrow \quad ax = a + 3a^2 + 6a^3 + \cdots \infty$$

$$\Rightarrow \quad (1-a)x = 1 + 2a + 3a^2 + 4a^3 + \cdots \infty \qquad \text{(An AGP!)}$$

$$\Rightarrow \quad a(1-a)x = a + 2a^2 + 3a^3 + \cdots \infty$$

$$\Rightarrow \quad (1-a)x - a(1-a)x = 1 + a + a^2 + a^3 + \cdots \infty$$

$$\Rightarrow \quad (1-a)^2 x = \frac{1}{1-a}$$

$$\Rightarrow \quad x = \frac{1}{(1-a)^3}$$

$$\Rightarrow \quad a = 1 - x^{-1/3} \qquad (1)$$

For y

$$y = 1 + 4b + 10b^2 + 20b^3 + \cdots \infty$$

$$\Rightarrow \quad by = b + 4b^2 + 10b^3 + \cdots \infty$$

$$\Rightarrow \quad (1-b)y = 1 + 3b + 6b^2 + 10b^3 + \cdots \infty$$

$$= \frac{1}{(1-b)^3} \qquad \{\text{Using the sum of the series for } x\text{, with } b \text{ in place of } a\}$$

$$\Rightarrow \quad y = \frac{1}{(1-b)^4}$$

$$\Rightarrow \quad b = 1 - y^{-1/4} \tag{2}$$

Finally, we have

$$S = 1 + 3(ab) + 5(ab)^2 + \cdots \infty \qquad \text{(An AGP)}$$

$$(ab)S = (ab) + 3(ab)^2 + \cdots \infty$$

$$\Rightarrow \quad (1-ab)S = 1 + 2(ab) + 2(ab)^2 + \cdots \infty$$

$$= 1 + 2ab(1 + ab + (ab)^2 + \cdots \infty)$$

$$= 1 + \frac{2ab}{1-ab} = \frac{1+ab}{1-ab}$$

$$\Rightarrow \quad S = \frac{1+ab}{(1-ab)^2} = \frac{1 + (1-x^{-1/3})(1-y^{-1/4})}{\{1 - (1-x^{-1/3})(1-y^{-1/4})\}^2} \qquad \text{(Using (1) and (2))}$$

S24. The roots of the given equation are $\frac{1 \pm \sqrt{1-4b}}{2}$:

$$\Rightarrow \quad S_k = \alpha^k + \beta^k = \left(\frac{1+\sqrt{1-4b}}{2}\right)^k + \left(\frac{1-\sqrt{1-4b}}{2}\right)^k$$

$$= \frac{1}{2^{k-1}}(1 + {}^kC_2(1-4b) + {}^kC_4(1-4b)^2 + \cdots) \qquad \text{(how?)}$$

$$\Rightarrow \quad S_2 = 1 - 2b, \; S_3 = 1 - 3b, \; S_5 = 1 - 5b + 5b^2$$

$$\Rightarrow \quad 2S_3 = S_2 + S_5 \quad \Rightarrow \quad 2 - 6b = 2 - 7b + 5b^2 \quad \Rightarrow \quad b = 0, \frac{1}{5}$$

S25. Let the first of these integers be m, and there be $k + 1$ consecutive integers in all. Thus,

$$m + (m+1) + (m+2) + \cdots + (m+k) = 1000$$

$$\Rightarrow \quad (k+1)(k+2m) = 2000 = 2^4 5^3$$

Note that $(k+2m)-(k+1)=2m-1$, an odd number. Thus, the following cases arise:

(a) k+1 is odd, 2m + k is even

(i) $k+1=1 \Rightarrow k=0, 2m+k=2000 \Rightarrow m=1000$

(ii) $k+1=5 \Rightarrow k=4, 2m+k=400 \Rightarrow m=198$

(iii) $k+1=25 \Rightarrow k=24, 2m+k=80 \Rightarrow m=28$

(iv) $k+1=125 \Rightarrow m<0$ (Not feasible)

(b) 2m+k is odd, k + 1 is even

The only possibility is $2m+k=125$, $k+1=16 \Rightarrow m=55, k=15$

Thus, all possible sequences become known:

$$\{1000\},\ \{198, 199, ..., 202\},\ \{28, 29, ..., 52\}, \{55, 56, ..., 70\}$$

S26. We have $a_1=1, a_n+a_{n-1}=n^2+2n, n\geq 2$. Putting $n=2$, we obtain $a_2=7$. Now,

$$a_n+a_{n-1}=n^2+2n$$

$$a_{n+1}+a_n=(n+1)^2+2(n+1) \Rightarrow a_{n+1}-a_{n-1}=2n+3$$

Thus,

$$\begin{array}{ll}
n=2: a_3-a_1=2(2)+3 & n=3: a_4-a_2=2(3)+3 \\
n=4: a_5-a_3=2(4)+3 & n=5: a_6-a_4=2(5)+3 \\
n=6: a_7-a_5=2(6)+3 & n=7: a_8-a_6=2(7)+3 \\
\vdots & \vdots \\
\underline{n=2m: a_{2m+1}-a_{2m-1}=2(2m)+3} & \underline{n=2m-1: a_{2m}-a_{2m-2}=2(2m-1)+3} \\
\sum: a_{2m+1}-a_1=2(2+4+...+2m) & \sum: a_{2m}-a_2=2(3+5+\cdots+(2m-1)) \\
\qquad + & \qquad + \\
\qquad 3m & \qquad 3(m+1) \\
\qquad =2m^2+5m & \qquad =2m^2+3m-5 \\
\Rightarrow \quad a_{2m+1}=2m^2+5m+1 & \Rightarrow \quad a_{2m}=2m^2+3m+2
\end{array}$$

Now,

$$S_1=\sum_{k=1}^{2m} a_k=\sum_{k=1,\,k\text{ odd}}^{2m-1} a_k+\sum_{k=2,\,k\text{ even}}^{2m} a_k$$

$$=\sum_{\substack{r=0\\k=2r+1}}^{m-1}(2r^2+5r+1)+\sum_{\substack{r=1\\k=2r}}^{m}(2r^2+3r+2)$$

$$=\left(4\sum_{r=1}^{m} r^2-2m^2\right)+\left(\sum_{r=0}^{m-1}(5r+1)\right)+\left(\sum_{r=1}^{m}(3r+2)\right)$$

$$=\frac{2m(2m^2+1)}{3}+5\frac{m(m-1)}{2}+m+3\frac{m(m+1)}{2}+2m$$

$$=\frac{4m(m+1)(m+2)}{3}$$

$$
\begin{aligned}
S_2 &= S_1 + a_{2m+1} \\
&= \frac{4m(m+1)(m+2)}{3} + 2m^2 + 5m + 1 \\
&= \frac{4m^3 + 18m^2 + 23m + 3}{3}
\end{aligned}
$$

S27. Substituting $a_n = r^n$ generates the quadratic $r^2 = r + 1$, whose roots are $r_1, r_2 = \frac{1 \pm \sqrt{5}}{2}$. Thus,

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(a) $\quad a_1 = 1 \Rightarrow C_1\left(1+\sqrt{5}\right) + C_2\left(1-\sqrt{5}\right) = 2$

(b) $\quad a_2 = 1 \Rightarrow C_1\left(6+2\sqrt{5}\right) + C_2\left(6-2\sqrt{5}\right) = 4$

$$\Rightarrow C_1 = \frac{1}{\sqrt{5}},\ C_2 = -\frac{1}{\sqrt{5}}$$

Therefore,

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\sqrt{5}+1}{2}$$

S28.

$$\sum_{r=1}^{n} t_r = \frac{1}{8} n(n+1)(n+2)(n+3)$$

$$\Rightarrow \sum_{r=1}^{n-1} t_r = \frac{1}{8} n(n-1)(n+1)(n+2)$$

Subtracting the two, we obtain t_n:

$$t_n = \frac{n(n+1)(n+2)}{2}$$

Thus,

$$
\begin{aligned}
\sum_{r=1}^{n} \frac{1}{t_r} = \sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} &= 2\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)\left(\frac{1}{r+2}\right) \\
&= 2\sum_{r=1}^{n}\left(\frac{1}{r(r+2)} - \frac{1}{(r+1)(r+2)}\right) \\
&= \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right) - 2\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)
\end{aligned}
$$

$$= \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$= \left(1-\frac{1}{n+1}\right)-\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$= \frac{1}{2}-\frac{1}{(n+1)(n+2)}$$

S29. Let r be the common ratio of the G.P $a_1, a_2, a_3, ...$

$$\Rightarrow \quad A_n = \frac{a_1+a_2+\cdots+a_n}{n} = \frac{a_1(1+r+r^2+\cdots+r^{n-1})}{n}$$

$$G_n = (a_1 a_2 \ldots a_n)^{1/n} = (a_1 \cdot a_1 r \cdot a_1 r^2 \ldots a_1 r^{n-1})^{1/n} = a_1 r^{(n-1)/2}$$

$$H_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} = \frac{n}{\frac{1}{a_1}+\frac{1}{a_1 r}+\frac{1}{a_1 r^2}+\cdots+\frac{1}{a_1 r^{n-1}}}$$

$$= \frac{n a_1 r^{n-1}}{1+r+r^2+\cdots+r^{n-1}} = \frac{G_n^2}{A_n}$$

$$\Rightarrow \quad G_n = \sqrt{A_n H_n}$$

The geometric mean G of $G_1, G_2, ..., G_n$ will be given by

$$G = (G_1 G_2 ... G_n)^{1/n}$$

$$= \left(\sqrt{A_1 H_1}\,\sqrt{A_2 H_2}\,...\,\sqrt{A_n H_n}\right)^{1/n}$$

$$= (A_1 A_2 ... A_n H_1 H_2 ... H_n)^{\frac{1}{2n}}$$

S30. No. If we square the given relations, we have

$$a_1^2 = 0$$
$$a_2^2 = a_1^2 + 2a_1 + 1$$
$$a_3^2 = a_2^2 + 2a_2 + 1$$
$$\vdots$$
$$a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1$$
$$a_{n+1}^2 = a_n^2 + 2a_n + 1$$

Adding all these relations, we have

$$a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) + 2(a_1 + a_2 + \cdots + a_n) + n$$

$$\Rightarrow \quad 2(a_1 + a_2 + \cdots + a_n) = -n + a_{n+1}^2 \geq -n$$

$$\Rightarrow \quad \frac{a_1 + a_2 + \cdots + a_n}{n} \geq -\frac{1}{2}$$

S31. No. In fact, they are in AP. If λ is the common root, then

$$\begin{aligned} a\lambda^2 + 2b\lambda &= -c \\ d\lambda^2 + 2e\lambda &= -f \end{aligned} \quad \Rightarrow \quad \lambda^2 = \frac{\begin{vmatrix} -c & 2b \\ -f & 2e \end{vmatrix}}{\begin{vmatrix} a & 2b \\ d & 2e \end{vmatrix}}, \ \lambda = \frac{\begin{vmatrix} a & -c \\ d & -f \end{vmatrix}}{\begin{vmatrix} a & 2b \\ d & 2e \end{vmatrix}}$$

$$\Rightarrow \quad \lambda^2 = \frac{bf - ce}{ae - bd}, \ \lambda = \frac{cd - af}{2(ae - bd)}$$

$$\Rightarrow \quad 4(ae - bd)(bf - ce) = (cd - af)^2$$

Dividing both sides by (ab) (bc), and noting that $b^2 = ac$, we have

$$4\left(\frac{e}{b} - \frac{d}{a}\right)\left(\frac{f}{c} - \frac{e}{b}\right) = \frac{(cd - af)^2}{ab^2c} = \left(\frac{d}{a} - \frac{f}{c}\right)^2$$

From this, it is straightforward to deduce that

$$\frac{e}{b} - \frac{d}{a} = \frac{f}{c} - \frac{e}{b},$$

which means that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP. The last step is left to the reader as an exercise.

S32. Let the first terms and common differences of the two series be a_1, d_1 and a_2, d_2 respectively. Then,

$$\frac{\frac{n}{2}(2a_1 + (n-1)d_1)}{\frac{n}{2}(2a_2 + (n-1)d_2)} = \frac{7n+1}{4n+17}$$

The important point to note is that this is an identity in n, *i.e.*, it holds for any n. Substituting $n \to 2n-1$ gives:

$$\frac{a_1 + (n-1)d_1}{a_2 + (n-1)d_2} = \frac{7(2n-1)+1}{4(2n-1)+17} = \frac{14n-6}{8n+13}$$

As is evident, this is the required ratio of the nth terms.

S33.
$$\text{LHS} = \sum_{r=1}^{n} \frac{r^4}{(2r-1)(2r+1)} = \frac{1}{16}\sum_{r=1}^{n}\left\{\left(\frac{16r^4-1}{4r^2-1}\right) + \frac{1}{(2r-1)(2r+1)}\right\}$$

$$= \frac{1}{16}\sum_{r=1}^{n}\left\{\left(4r^2+1\right) + \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)\right\}$$

$$= \frac{1}{16}\left\{\left(\frac{4n(n+1)(2n+1)}{6} + n\right) + \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{n(4n^2+6n+5)}{48} + \frac{n}{16(2n+1)}$$

S34. We have

$$A_1 = a + \left(\frac{b-a}{3}\right) = \frac{b+2a}{3}, \qquad A_2 = a + 2\left(\frac{b-a}{3}\right) = \frac{a+2b}{3}$$

$$G_1 = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}b^{1/3}, \qquad G_2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}b^{2/3}$$

$$H_1 = \left(\frac{1}{a} + \frac{\left(\frac{1}{b}-\frac{1}{a}\right)}{3}\right)^{-1} = \frac{3ab}{a+2b}, \qquad H_2 = \left(\frac{1}{a} + \frac{2\left(\frac{1}{b}-\frac{1}{a}\right)}{3}\right)^{-1} = \frac{3ab}{2a+b}$$

(a) and **(b)**
$$\frac{G_1G_2}{H_1H_2} = \frac{ab}{\frac{9a^2b^2}{(a+2b)(2a+b)}} = \frac{(2a+b)(a+2b)}{9ab}$$

$$\frac{A_1+A_2}{H_1+H_2} = \frac{\left(\frac{2a+b}{3}\right) + \left(\frac{a+2b}{3}\right)}{\left(\frac{3ab}{a+2b}\right) + \left(\frac{3ab}{2a+b}\right)} = \frac{(2a+b)(a+2b)}{9ab}$$

S35. We use the AM–HM inequality as follows:

$$\left(\frac{\sum_{r=1001}^{3001} r}{2001}\right) > \left(\frac{2001}{\sum_{r=1001}^{3001} \frac{1}{r}}\right)$$

Since, $\sum_{r=1001}^{3001} r = (2001)^2$, we obtain

$$\sum_{r=1001}^{3001} \frac{1}{r} > 1$$

Thus, (a) is true. Now, there are 2001 terms in the given sequence. We divide the series into groups of 500 and proceed accordingly:

$$S = \sum_{r=1001}^{1500} \frac{1}{r} + \sum_{r=1501}^{2000} \frac{1}{r} + \sum_{r=2001}^{2500} \frac{1}{r} + \sum_{r=2501}^{3000} \frac{1}{r} + \frac{1}{3001}$$

$$< \sum_{r=1001}^{1500} \left(\frac{1}{1000}\right) + \sum_{r=1501}^{2000} \left(\frac{1}{1500}\right) + \sum_{r=2001}^{2500} \left(\frac{1}{2000}\right) + \sum_{r=2501}^{3000} \left(\frac{1}{2500}\right) + \frac{1}{3001}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3001} < \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000} = \frac{3851}{3000} < \frac{4}{3}.$$

Thus, $S < \frac{4}{3}$, and hence also $S < \frac{3}{2}$.

$\Rightarrow$ Both (b) and (c) are true.

S36.

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)} \tag{1}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^2 m}{3^n (m3^n + n3^m)} \quad \{\text{Interchanging the variables } m \text{ and } n\}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 m}{3^n (m3^n + n3^m)} \quad \{\text{Interchanging the order of summation}\} \tag{2}$$

We add the expressions for S in (1) and (2):

$$2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{nm}{3^n\, 3^m} = \left(\sum_{n=1}^{\infty} \frac{n}{3^n}\right) = \left(\frac{3}{4}\right)^2 = \frac{9}{16} \quad \Rightarrow \quad S = \frac{9}{32}$$

S37.

$$S_k = k + kr + kr^2 + \cdots \infty, \left(r = \frac{1}{k+1}\right)$$

$$= k\left(\frac{1}{1-r}\right) = k+1$$

$$\Rightarrow \quad S_1^2 + S_2^2 + \cdots + S_{2n-1}^2 = \sum_{k=1}^{2n-1} (k+1)^2.$$

$$= \sum_{r=2}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - 1 = \frac{2n(2n+1)(4n+1)}{6} - 1 = \frac{n(2n+1)(4n+1)}{3} - 1$$

S38. If we denote the rth term by T_r, then

$$T_1 = \frac{a_1}{1+a_1} = 1 - \frac{1}{(1+a_1)}.$$

$$T_2 = \frac{a_2}{(1+a_1)(1+a_2)} = \left(\frac{1}{1+a_1}\right)\left(1 - \frac{1}{1+a_2}\right) = \frac{1}{(1+a_1)} - \frac{1}{(1+a_1)(1+a_2)}$$

$$T_3 = \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} = \frac{1}{(1+a_1)(1+a_2)} - \frac{1}{(1+a_1)(1+a_2)(1+a_3)}$$

$\vdots$

Adding these, we obtain the sum S as

$$S = 1 - \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)}$$

S39. We have

$$\sum_{1\le i<j\le m} a_i a_j = \frac{1}{2}\left\{\left(\sum_{i=1}^{m} a_i\right)^2 - \sum_{i=1}^{m} a_i^2\right\} \quad \text{(how?)}$$

$$= \frac{1}{2}\left\{\left(\frac{a_1(1-r^m)}{1-r}\right)^2 - \frac{a_1^2(1-r^{2m})}{1-r^2}\right\}$$

$$= \frac{a_1^2}{2(1-r)}\left\{\frac{(1-r^m)^2}{1-r} - \frac{1-r^{2m}}{1+r}\right\}.$$

$$= \frac{a_1^2}{2(1-r)^2(1+r)}\{(1+r)(1-2r^m+r^{2m})-(1-r)(1-r^{2m})\}$$

$$= \frac{a_1^2(r-r^m-r^{m+1}+r^{2m})}{(1-r)^2(1+r)}$$

$$= \frac{r}{1+r}\left\{\frac{a_1(1-r^{m-1})}{1-r}\right\}\left\{\frac{a_1(1-r^m)}{1-r}\right\}$$

$$= \frac{r}{1+r}S_{m-1}S_m$$

S40. Substitute $a = -(x^2+x)$ into the first equation to obtain $x^3 = 1$. This equation has its roots as the cube roots of unity.

$$\Rightarrow \quad x = 1, \omega, \omega^2$$

$$\Rightarrow \quad a = -2 \text{ or } 1 \quad \text{(how?)}$$

S41. We have

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \quad \Rightarrow \quad \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 = \left(\frac{\gamma+\delta}{\gamma-\delta}\right)^2 \quad \Rightarrow \quad \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2-4\alpha\beta} = \frac{(\gamma+\delta)^2}{(\gamma+\delta)^2-4\gamma\delta}$$

$$\Rightarrow \quad \frac{b^2}{b^2-ac} = \frac{q^2}{q^2-pr} \quad \Rightarrow \quad \frac{1}{1-\frac{ac}{b^2}} = \frac{1}{1-\frac{pr}{q^2}}$$

$$\Rightarrow \quad \frac{ac}{b^2} = \frac{pr}{q^2}$$

S42. $$\alpha+\beta = p, \quad \frac{\alpha}{2}+2\beta = q \quad \Rightarrow \quad \alpha+4\beta = 2q$$

$$\Rightarrow \quad \beta = \frac{1}{3}(2q-p), \alpha = \frac{2}{3}(2p-q)$$

$$\Rightarrow \quad r = \alpha\beta = \frac{2}{9}(2p-q)(2q-p)$$

S43. First, we assume that $f(x)$ is an integer whenever x is an integer. Thus,

$f(0)$ is an integer $\Rightarrow$ C is an integer

$f(1)$ is an integer $\Rightarrow$ $A + B + C$, and hence $A + B$ are integers

$f(-1)$ is an integer $\Rightarrow$ $A - B + C$ is an integer

$\Rightarrow\ (A+B+C)+(A-B+C)=2A+2C$ is an integer, and hence $2A$ is an integer

We have thus proved that $2A$, $A + B$ and C are integers. Now, we prove the converse. We assume that $2A$, $A + B$ and C are integers, and we have to prove that if x is an integer, then $f(x)$ is an integer. For that, we assume two cases:

x* is even, say 2*n

$$f(x)=Ax^2+Bx+C$$

$$=(2A)(2n^2)+(2B)n+C,$$

which is an integer

***x* is odd, say 2*n* + 1**

$$f(x)=Ax^2+Bx+C$$

$$=A(4n^2+4n+1)+B(2n+1)+C$$

$$=(2A)(2n^2+2n)+(2B)n+(A+B)+C,$$

which is an integer

Thus, $f(x)$ is an integer in both cases.

S44. Step-1: Square, rearrange, square again:

$$a+\sqrt{a+x}=x^2 \quad\Rightarrow\quad \sqrt{a+x}=x^2-a$$

$$\Rightarrow\quad a+x=x^4+a^2-2ax^2$$

Step-2: Now, treat this as a (quadratic) equation in a rather than as an equation in x:

$$a^2-a(1+2x^2)+x^4-x=0$$

$$\Rightarrow\quad a=\frac{(1+2x^2)\pm\sqrt{(1+2x^2)^2-4(x^4-x)}}{2}=\frac{(1+2x^2)\pm\sqrt{4x^2+4x+1}}{2}$$

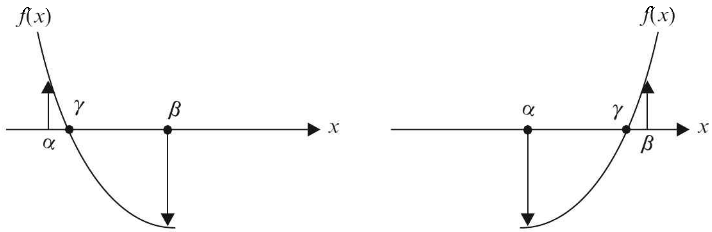
$$=\frac{(1+2x^2)\pm(1+2x)}{2}=x^2-x,\ x^2+x+1$$

Step-3: We now have two quadratics:

$$x^2-x=a \quad\Rightarrow\quad x=\frac{1\pm\sqrt{1+4a}}{2}$$

$$x^2+x+1=a \quad\Rightarrow\quad x=\frac{-1\pm\sqrt{4a-3}}{2}$$

S45. Let us denote $\frac{a}{2}x^2+bx+c$ by $f(x)$. Since we want γ (a zero of $f(x)$) to lie between α and β, our approach should be to somehow show that $f(\alpha)$ and $f(\beta)$ are of opposite signs, so that the graph of $f(x)$ crosses the x-axis between α and β. The figure below illustrates this point:



Now, since α is a root of $ax^2 + bx + c = 0$, we have

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\Rightarrow \quad \frac{a}{2}\alpha^2 + b\alpha + c = -\frac{a}{2}\alpha^2$$

$$\Rightarrow \quad f(\alpha) = -\frac{a}{2}\alpha^2 \tag{1}$$

Since β is a root of $-ax^2 + bx + c = 0$,

$$\Rightarrow \quad -a\beta^2 + b\beta + c = 0$$

$$\Rightarrow \quad \frac{a}{2}\beta^2 + b\beta + c = \frac{3a}{2}\beta^2$$

$$\Rightarrow \quad f(\beta) = \frac{3a}{2}\beta^2 \tag{2}$$

From (1) and (2), notice that $f(\alpha)$ and $f(\beta)$ are of opposite signs, and hence a root γ of $f(x)$ lies between α and β.

S46. Dividing throughout by $\tan^2\theta$, we have

$$(\tan\theta + \cot\theta)^2 + 4a(\tan\theta + \cot\theta) + 16 = 0$$

Using $\tan\theta + \cot\theta = y$,

$$y^2 + 4ay + 16 = 0 \quad \Rightarrow \quad y = \tan\theta + \cot\theta = -2\left(a \pm \sqrt{a^2 - 4}\right) \tag{1}$$

Now, let $a \pm \sqrt{a^2 - 4}$ be represented as b. Thus,

$$y = \tan\theta + \frac{1}{\tan\theta} = -2b \quad \Rightarrow \quad \tan\theta = -b \pm \sqrt{b^2 - 1} \tag{2}$$

For real roots, we have

From (1): $a^2 > 4 \quad \Rightarrow \quad a < -2$ or $a > 2$

From (2): $b^2 > 1 \quad \Rightarrow \quad \left(a \pm \sqrt{a^2 - 4}\right)^2 > 1$

$$\Rightarrow \quad -\frac{5}{2} < a < \frac{5}{2} \quad \text{(Verify!)}$$

Thus, $a \in (-\frac{5}{2}, -2) \cup (2, \frac{5}{2})$. However, we have not paid attention to one particular fact: that we need four distinct roots in $(0, \frac{\pi}{2})$. For that, we need to take only negative values of a, that is, $a \in (-\frac{5}{2}, -2)$. The justification of why this should be so is left to the reader as an exercise.

S47. Case-I: $x < a$

$$\begin{aligned} & x^2 - 2a(a-x) - 3a^2 = 0 \\ \Rightarrow \quad & x^2 + 2ax - 5a^2 = 0 \\ \Rightarrow \quad & x = a\left(-1 \pm \sqrt{6}\right) \end{aligned}$$

Since $a < 0$ and $x < a$, we must have

$$x = a\left(-1 + \sqrt{6}\right)$$

Case-II: $x > a$

$$\begin{aligned} & x^2 - 2a(x-a) - 3a^2 = 0 \\ \Rightarrow \quad & x^2 - 2ax - a^2 = 0 \\ \Rightarrow \quad & x = a\left(1 \pm \sqrt{2}\right) \end{aligned}$$

Since $a < 0$ and $x > a$, we must have

$$x = a\left(1 - \sqrt{2}\right)$$

Thus, the two solutions are

$$x = a\left(-1 + \sqrt{6}\right), a\left(1 - \sqrt{2}\right)$$

S48. Assume $f(x) = |x+1| - |x| + 3|x-1| - 2|x-2| - (x+2)$. We now split $f(x)$ according to the appropriate intervals:

$$f(x) = \begin{Bmatrix} -2x-4 & -\infty < x \le -1 \\ -2 & -1 \le x \le 0 \\ -2x-2 & 0 \le x \le 1 \\ 4x-8 & 1 \le x < 2 \\ 0 & 2 \le x < \infty \end{Bmatrix}$$

From this definition, $f(x) = 0$ when $x \in [2, \infty) \cup \{-2\}$.

S49. To prove the assertion, we'll have to take a rather indirect approach. We will construct a polynomial with roots a, b and c. From the relation given to us, we have:

$$(a+b+c)(ab+bc+ca) = abc$$

If we let $a+b+c = -p$ and $ab+bc+ca = q$, the required polynomial is:

$$\begin{aligned} f(x) &= x^3 + px^2 + qx + pq \qquad \text{(why?)} \\ &= (x+p)(x^2+q) \end{aligned}$$

One of the roots of $f(x)$, say a, is equal to $-p$:

$$a = -p = a + b + c$$
$$\Rightarrow \quad b + c = 0$$
$$\Rightarrow \quad c = -b$$

Thus, for odd n,

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n} + \frac{1}{b^n} - \frac{1}{b^n} = \frac{1}{a^n} = \frac{1}{a^n + b^n - b^n} = \frac{1}{a^n + b^n + c^n}$$

S50. Assume that the fractional part of x, that is $\{x\} = x - [x]$, falls in the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$, where $0 \le k < n$ and $k \in \mathbb{Z}$. Thus,

$$k \le nx - n[x] < k + 1 \quad \Rightarrow \quad 0 \le nx - n[x] - k < 1$$
$$\Rightarrow \quad [nx] = n[x] + k$$

On the other hand, the right side is

$$[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [x] + [x] + \cdots + \underbrace{([x]+1) + ([x]+1) + \cdots}_{k \text{ terms}} \quad \text{(how?)}$$
$$= n[x] + k$$

The two sides are thus equal.

S51. Rather than forming a cubic equation, we take a more elegant approach. We rewrite the equation as follows:

$$\sqrt[3]{\sqrt[3]{a-x} - x} = a \tag{1}$$

We now define a function of the 'variable' a as follows:

$$f(a) = \sqrt[3]{a-x} \quad \text{(\textit{Note:} the independent variable is } a\text{, not } x)$$

In terms of f, (1) can be re-written as

$$f(f(a)) = a \tag{2}$$

Now, we note that $f(a)$ is an increasing function of a. We claim that $f(a)$ must be equal to a. To see why, consider the other two cases:

If $f(a) < a$, then $a = f(f(a)) < f(a) < a$ and If $f(a) > a$, then $a = f(f(a)) > f(a) > a$

Both cases cannot hold; the only possibility is that

$$\begin{aligned} & f(a) = a \\ \Rightarrow\ & \sqrt[3]{a-x} = a \\ \Rightarrow\ & a - x = a^3 \\ \Rightarrow\ & x = a - a^3 \end{aligned}$$

S52. This is a very deep problem and its basis lies in a fascinating area of mathematics knows as Group Theory. Without going into the details of this theory, we will try to understand the solution to this problem from the perspective of our current knowledge. We start by taking a concrete example from Complex Numbers, which the reader can relate to. Consider the set of the eighth roots of unity:

$$S = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}, \quad \text{where } \alpha = e^{i\frac{2\pi}{8}}$$

For any $x \in S$, we have $x^8 = 1$. This serves as a good test example for the present problem, with $n = 8$.

Now, if we consider a specific element of S, say $x = \alpha$, and consider the sequence x^r where $r \in \mathbb{Z}$, we have

$r =$	0	1	2	3	4	5	6	7
$x^r =$	1	α	α^2	α^3	α^4	α^5	α^6	α^7

We see that as r is varied from 0 to 7, we are able to *generate* each element in S. We will formally say that the element α is a *generator* for S. The question we ask the reader to consider is: Is every element of S a generator for S? Obviously, the element 1 is *not*, since $1^r = 1\ \forall\ r \in \mathbb{Z}$. We consider $x = \alpha^2$, and see what set x^r generates as r varies over $\mathbb{Z}$:

$r =$	0	1	2	3	4	$\cdots$	6
$x^r =$	1	α^2	α^4	α^6	$\alpha^8 = 1$	$\cdots$	α

We see that only four elements of S are generated, namely $\{1, \alpha^2, \alpha^4, \alpha^6\}$ and giving r other values will generate these elements again (try this). Thus, $x = \alpha^2$ is *not* a generator for S. Now, lets try $x = \alpha^3$:

$r =$	0	1	2	3	4	5	6	7
$x^r =$	1	α^3	α^6	$\alpha^9 = \alpha$	$\alpha^{12} = \alpha^4$	$\alpha^{15} = \alpha^2$	$\alpha^{18} = \alpha^2$	$\alpha^{21} = \alpha^5$

Carefully note how all the elements in S are generated, implying that α^3 is a generator for S. Continuining this exercise, you will find that α^4 and α^6 will *not* be generators for S, while α^5 and α^7 ***will be***. You are urged to verify this.

Now, we return to our orignial problem. We have to essentially find those values of k for which x^k is a generator for S. This means that every element in S, say x^m (where $0 \le m < n$, $m \in \mathbb{Z}$) should be expressible as $(x^k)^p$ for some $p \in \mathbb{Z}$, *i.e.*,

$$x^m = (x^k)^p \text{ for some } p \in \mathbb{Z}.$$

In particular, this must be satisfied for $m = 1$:

$$x^{kp} = x \text{ for some } p \in \mathbb{Z}$$

$$\Rightarrow \quad x^{kp} = x^{nl+1} \text{ for some } p \in \mathbb{Z}, l \in \mathbb{Z} \quad \text{(why?)}$$

$$\Rightarrow \quad kp + n(-\ell) = 1 \text{ for some } p, l \in \mathbb{Z}$$

This implies (using a well known result from basic number theory, which you are urged to recall/review) that k and n are co-prime.

$$\Rightarrow \quad k \text{ is co-prime to } n.$$

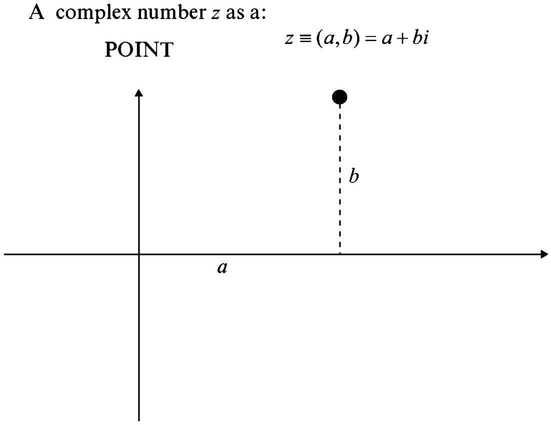
Complex Numbers

PART-A : Summary of Important Concepts

1. Fundamental Facts

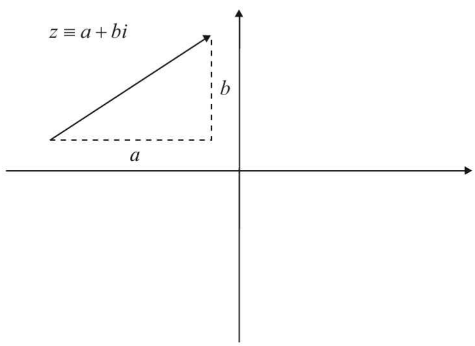
Most of you would have been introduced to complex numbers in the following manner: a complex number z is a number of the form $a+bi$ where $a, b \in \mathbb{R}$ while i is a non-real number given by $i^2 = -1$. This is a perfectly correct statement, and that is what complex numbers actually are. However, what we would like to tell you here is that if you really intend to attain a deep mastery of this topic, you should always think of a complex number $z = a + bi$ as a point in the plane with the coordinates (a, b), or as a vector, whose x-component is a and y-component is b:

A complex number z as a:



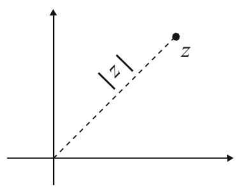
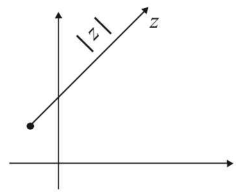
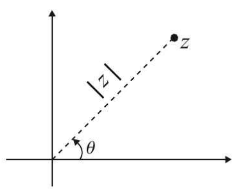
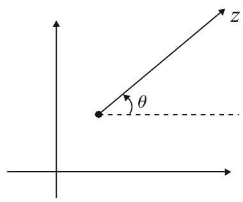
A complex number z as a:

VECTOR



Note that when we think of z as a point, then it has a fixed location in the plane, whereas if we think of z as a vector, then it is a free entity; we can move it around the plane as we wish, as long as we don't change its magnitude and direction. Whether we should think of z as a point or as a vector depends on context, and once you practice a lot of problems, you will attain an intuitive understanding necessary to make that choice.

Now, let us see that meaning we should attach to the modulus and argument of a complex number $z = a + bi$:

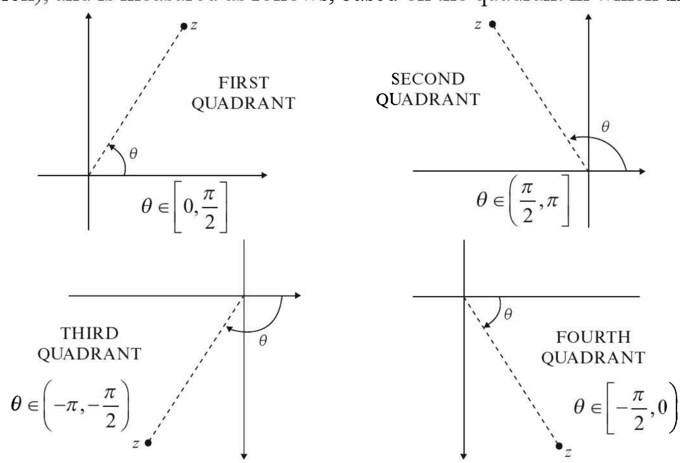
	Thinking of z as a POINT	Thinking of z as a VECTOR
Modulus $\lvert z \rvert$	The distance of the point z from the origin: 	The length of the vector z 
Argument arg (z)	The angle made by the line joining the origin to the point z with the positive direction of the x-axis 	The angle made by the vector z with the positive x-direction 

Using these interpretations, observe the following examples:

$|z| = 1$: This would mean that z is a point whose distance from the origin is 1, or, a vector whose magnitude is 1.

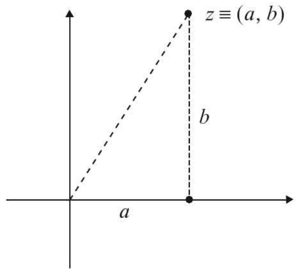
$\arg(z) = 90°$: This would mean that z is a point on the positive y-axis, or z is a vector pointing in the positive y-direction.

At this point, we must make note of the distinction between *an* argument and *the* principal argument of a complex number. Observe carefully that we've used the expressions '*an* argument' and '*the* principal argument' implying that there are multiple values of the argument of a complex number, but only one value of the principal argument. In fact, the principal argument (call it θ) lies in the range $(-\pi, \pi)$ (that is the generally accepted convention), and is measured as follows, based on the quadrant in which the point z lies:

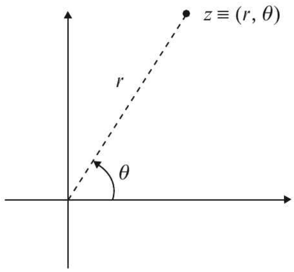


We make the important observation that if θ is the value of the principal argument of z, then for any integer k, $\theta + 2k\pi$ is also a valid value for the argument of z.

We now note that there are two interchangeable representations of any complex number z. If we think of z as a point in the plane, then we can uniquely determine the location of z either by specifying its co-ordinates (*Cartesian* form) or by specifying its modulus and argument (*Polar* form):



Representing z using coordinates.
This is the CARTESIAN from.

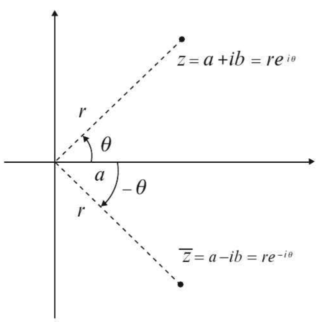


Representing z using its modulus and argument.
This is the POLAR from.

The polar form $z \equiv (r, \theta)$ leads to the Euler form $z = re^{i\theta}$, which is a very powerful representation of any complex number, as it leads to a great amount of simplification in multiplication and division operations, by virtue of its exponential form. Note carefully the following examples where we've written a few complex numbers in all the three forms:

Complex Number	Cartesian From	Polar From	Euler From
z = 1	$1 + i0$ or $(1, 0)$	$(1, 0)$	$1e^{i0}$
z = 1+i	$1 + i1$ or $(1, 1)$	$\left(\sqrt{2}, \frac{\pi}{4}\right)$	$\sqrt{2}e^{i\pi/4}$
z = –2	$-2 + i0$ or $(-2, 0)$	$(2, \pi)$	$2e^{i\pi}$
z = –1 – i	$-1 + i(-1)$ or $(-1, -1)$	$\left(\sqrt{2}, \frac{3\pi}{4}\right)$	$\sqrt{2}e^{-i3\pi/4}$

To conclude this section, we note that the conjugate of a complex number $z = a + ib = re^{i\theta}$ will be the mirror reflection of the point z in the x-axis, and will be given by $\bar{z} = a - ib = re^{-i\theta}$:



Note that:

- $|z| = |\bar{z}| = r$
- $z + \bar{z} = 2a = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = 2ib = 2i\,\mathrm{Im}(z)$
- $\arg(\bar{z}) = -\theta = -\arg(z)$
- $z\bar{z} = r^2 = |z|^2$

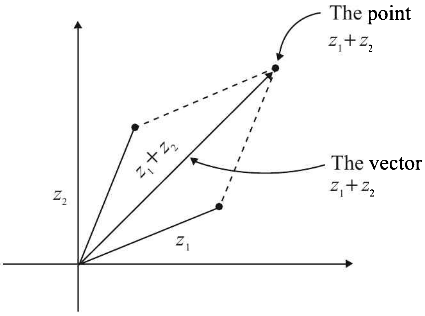
2. Arithmetic Operations

2.1 Addition and Subtraction

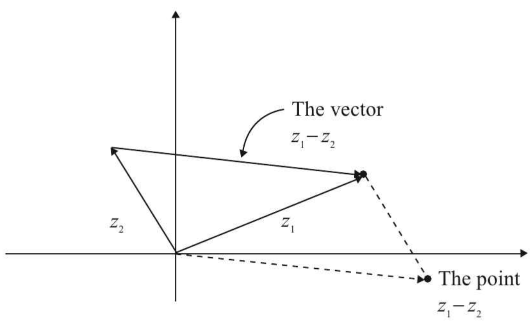
Let $z_1 = a_1 + ib_i$ and $z_2 = a_2 + ib_2$ be two complex numbers. We can obviously define the addition and subtraction operations as follows:

$$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$$

However, it is almost always more useful to think of the addition and subtraction of complex numbers in terms of the addition and subtraction of vectors. Thinking of z_1 and z_2 as vectors drawn from the origin to the points z_1 and z_2 respectively, we can use the vector addition and subtraction laws (parallelogram or triangle laws) to evaluate the vectors $z_1 + z_2$ and $z_1 - z_2$.



ADDITION

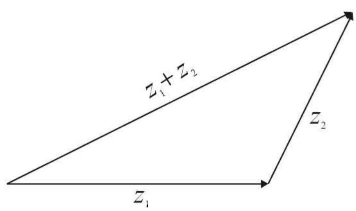


SUBTRACTION

We make the (really) important observation from the second figure above that $|z_1 - z_2|$ represents the distance between the points z_1 and z_2. This is an extremely powerful interpretation. For example, using this, look at how we interpret the following expressions:

$\lvert z - 1\rvert = 2$:	The distance of point z from point 1 is 2 units.
$\lvert z - i\rvert = \lvert z + i\rvert$:	Since $\lvert z + i\rvert$ can be written as $\lvert z - (-i)\rvert$, this means that the distance of point z from point i is the same as its distance from the point $-i$.

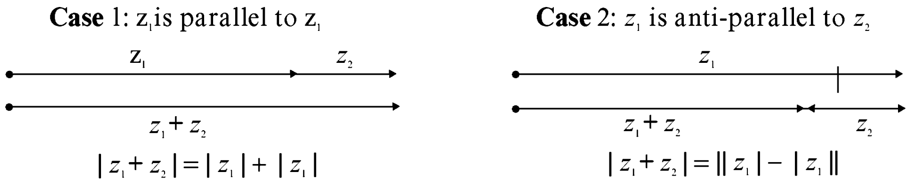
Interpreting addition and subtraction of complex numbers in terms of vectors easily leads to the triangle inequality:



Since the sum of two sides of any triangle (in this case: $|z_1|$ and $|z_2|$) must be greater than the third side (in this case: $|z_1 + z_2|$), while their difference must be less than the third, we have

$$\underbrace{\big||z_1| - |z_2|\big|}_{\text{Difference of the two sides}} \leq \underbrace{|z_1 + z_2|}_{\text{Third side}} \leq \underbrace{|z_1| + |z_2|}_{\text{Sum of the two sides}}$$

The equality signs occur when z_1 and z_2 are parallel or anti-parallel (in which case they will not, strictly speaking, form a triangle):



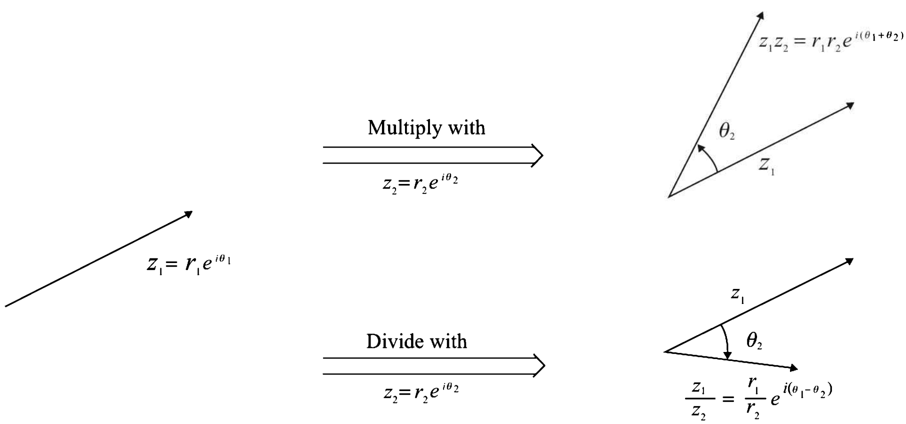
Of course, the triangle inequality can also be proven algebraically, but its geometric interpretation is much more intuitive to understand.

2.2 Multiplication and Division

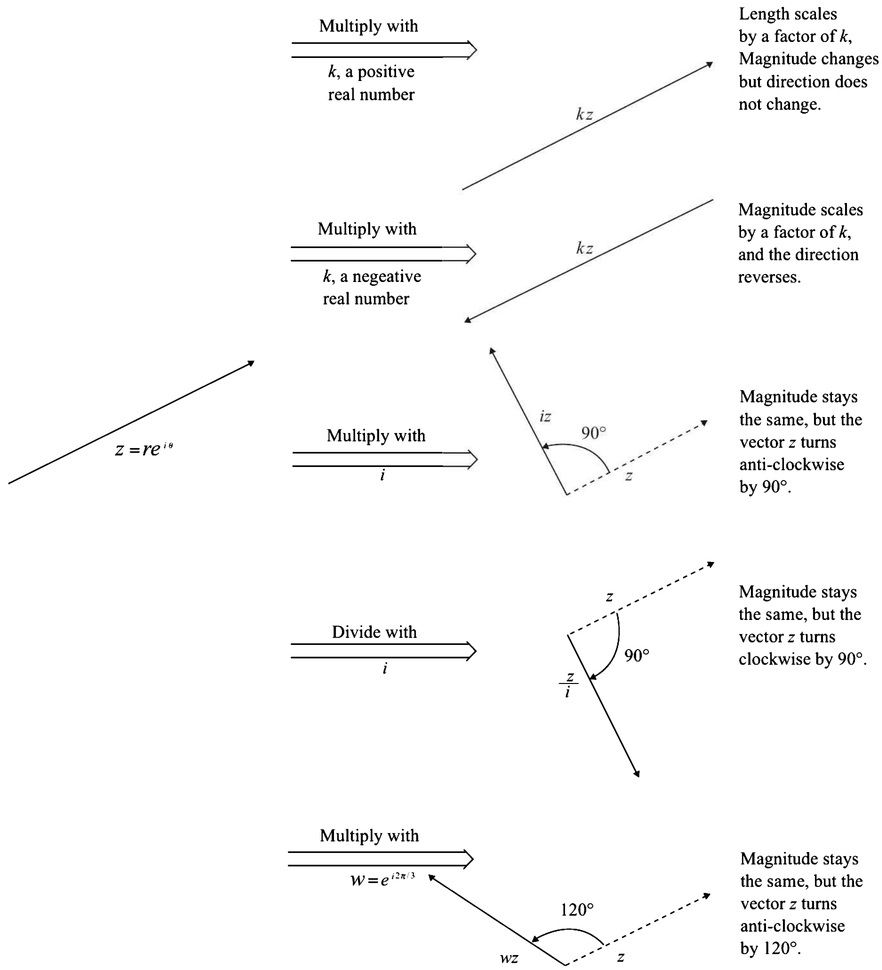
Whenever dealing with multiplication and division of complex numbers, things will be easier if you use the Polar or Euler representations for the numbers. Letting $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, we have:

	Result	Observations
Multiplication $z_1 z_2$	$r_1 r_2 e^{i(\theta_1+\theta_2)}$	Moduli multiply Arguments add
Division z_1/z_2	$\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$	Moduli divide Arguments subtract

Given these results, it will be helpful on many occasions to think of multiplication and division operations in the following manner: Thinking in terms of vectors, if you multiply (or divide) a complex number (vector) $z_1 = r_1 e^{i\theta_1}$ by another complex number $z_2 = r_2 e^{i\theta_2}$, you are essentially multiplying (or dividing) the modulus of the first vector by the second, and *rotating* the first vector anti-clockwise (or clockwise in the case of division) by angle θ:



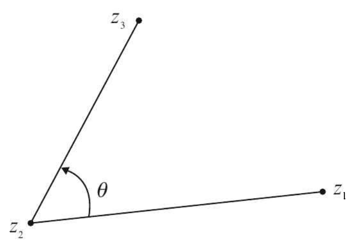
Obviously, if θ_2 is negative, the directions of rotation will be opposite to the ones shown. It is particularly useful to keep in mind the following special cases:



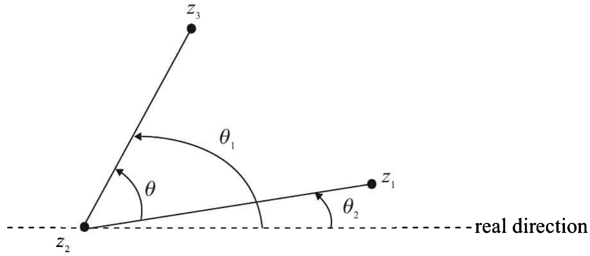
To summarize once again, always remember that multiplication and division of complex numbers leads to rotation and the scaling of lengths of the corresponding vectors in the complex plane.

3. The concept of Rotation

Rotation is nothing but a consequence of the way complex numbers multiply, and is used to relate complex numbers and angles that they make. Since it is so widely used, we will discuss it in some detail. Consider a configuration of complex numbers as shown below:



We know the angle θ. Our purpose is to write down an expression that relates all the four quantities z_1, z_2, z_3 and θ. Consider the vector $z_3 - z_2$. Let its argument be θ_1. Similarly, let the argument of the vector $z_1 - z_2$ be θ_2. Now, a little thought will show you that θ is simply $\theta_1 - \theta_2$.



Now, we write $z_3 - z_2$ and $z_1 - z_2$ in Euler's form:

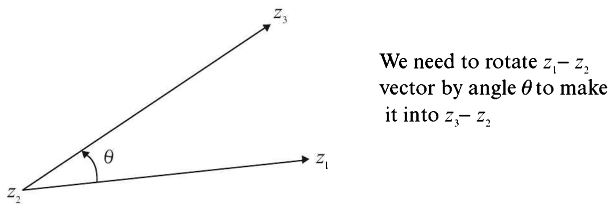
$$z_3 - z_2 = |z_3 - z_2|\, e^{i\theta_1} \tag{1}$$

$$z_1 - z_2 = |z_1 - z_2|\, e^{i\theta_2} \tag{2}$$

Since we know $\theta_1 - \theta_2$, we divide (1) by (2) to get:

$$\frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} e^{i\theta} \tag{3}$$

This is the relation we were looking for. It relates all the four terms z_1, z_2, z_3 and θ. You might have wondered why this method is called rotation. Well, you can think of this method in this way:



You are given the vector $z_1 - z_2$. You need to modify it into the vector $z_3 - z_2$. How can you do it? Obviously, there will be a change in modulus. Apart from that, you need to *rotate* the vector $z_1 - z_2$ anti-clockwise by angle θ too. This is where the term rotation comes from. Viewing the process in this way, we obtain the relation (3) as follows:

- Write down the unit vector in the direction of the original vector, the one that you need to rotate. In our case, it will be:

$$\frac{z_1 - z_2}{|z_1 - z_2|}.$$

- To rotate this unit vector by an angle θ (anti-clockwise; for clockwise, it will be $-\theta$), we multiply it by $e^{i\theta}$. For the current case, this turns the unit vector into a new unit vector along the direction of the vector $z_3 - z_2$:

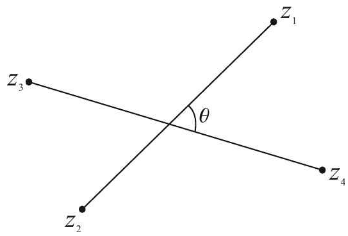
$$\frac{z_1 - z_2}{|z_1 - z_2|} e^{i\theta}.$$

- Finally, to turn this unit vector into the final vector that we need to obtain after rotation, we multiply the unit vector by the appropriate magnitude. Thus we obtain the required final vector after rotation

$$z_3 - z_2 = |z_3 - z_2| \cdot \frac{z_1 - z_2}{|z_1 - z_2|} e^{i\theta}.$$

This is the same relation that we obtained in (3).

Using this same approach, suppose that we now wish to relate four complex numbers in the following configuration:



Using rotation, we can obtain the vector $z_1 - z_2$ from the vector $z_4 - z_3$ as follows:

Unit vector along $z_4 - z_3$ direction (Initial unit vector): $\dfrac{z_4 - z_3}{|z_4 - z_3|}$

Unit vector along $z_1 - z_2$ direction (Final unit vector): $\dfrac{z_1 - z_2}{|z_1 - z_2|}$

Angle between the two: θ

$$\Rightarrow \quad \frac{z_1 - z_2}{|z_1 - z_2|} = \frac{z_4 - z_3}{|z_4 - z_3|} e^{i\theta}.$$

This is the relation we wished to obtain.

4. Powers of a Complex Number; Roots of Unity

In complex numbers, you will frequently encounter problems on the roots of unity (or more generally, of any complex number). It is essential to understand what the nth roots of a complex number signify (geometrically as well as algebraically), and for that purpose, you need to understand how rational powers of complex numbers are calculated. As this is another major concept in complex numbers (and a lot of students don't understand it clearly), we will again discuss this in some detail.

Given z, suppose you have to evaluate z^n where n is a rational number. Let us write z in polar (Euler) form:

$$z = re^{i\theta}$$

Suppose n is an integer. Then z^n is straightforward to evaluate

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Now, consider the case when n is a non-integer rational number. Let us take $n = \frac{1}{k}$, where k is an integer. You might wonder, what is the problem in following the same approach for $n = \frac{1}{k}$ as we did when n was an integer:

$$z^n = z^{1/k} = (re^{i\theta})^{1/k} = r^{1/k} e^{\frac{i\theta}{k}} \tag{1}$$

The problem is that the value, which we have obtained in (1), is *just one of the possible values* for z^n; there are other values too. To be precise, in this case, for example, the *n*th power of *z*, which is actually the *k*th root of *z*, will have *k* different values, the value in (1) being just one of those *k* values. How?

Consider $z = e^{i\theta}$ and $n = \frac{1}{4}$. How many values does $z^{1/4} = (e^{i\theta})^{1/4}$ have? $e^{i\theta/4}$ will be one of those values. To actually show that we can obtain other values, we write $e^{i\theta}$ as:

$$e^{i\theta} = e^{i(2p\pi+\theta)}, \quad p \in \mathbb{Z}$$

This is justified since $e^{i(2p\pi)} = 1$. Now,

$$(e^{i\theta})^{1/4} = \{e^{i(2p\pi+\theta)}\}^{1/4} = e^{i(\frac{2p\pi+\theta}{4})}$$

Let us give different integer values to *p* and see what we obtain:

$$p = 0 \Rightarrow \; z_0^{1/4} = e^{i\theta/4}$$

$$p = 1 \Rightarrow \; z_1^{1/4} = e^{i(\frac{\pi}{2}+\frac{\theta}{4})}$$

$$p = 2 \Rightarrow \; z_2^{1/4} = e^{i(\pi+\frac{\theta}{4})}$$

$$p = 3 \Rightarrow z_3^{1/4} = e^{i(\frac{3\pi}{2}+\frac{\theta}{4})}$$

$$p = 4 \Rightarrow z_4^{1/4} = e^{i(2\pi+\theta/4)} = e^{i2\pi} \cdot e^{i\,\theta/4} = e^{i\,\theta/4} = z_0^{1/4} \text{ and so on.}$$

Now we give negative integer values to *p*:

$$p = -1 \;\; \Rightarrow \;\; z_{-1}^{1/4} = e^{i(-\frac{\pi}{2}+\frac{\theta}{4})} = e^{i(-2\pi+\frac{3\pi}{2}+\frac{\theta}{4})} = e^{-i2\pi}e^{i(\frac{3\pi}{2}+\frac{\theta}{4})} = z_3^{1/4}$$

Similarly, $p = -2 \;\; \Rightarrow \;\; z_{-2}^{1/4} = z_2^{1/4}$ and so on.

We see that there are precisely 4 unique values of $z^{1/4}$, given by any four consecutive integer values of *p*. For example, we could take *p* from {0, 1, 2, 3} or {7, 8, 9, 10} or whatever you wish. The important thing to remember is that the four values of *p* should be consecutive (the value of the root follows a cycle of 4, *p* and *p* + 4 will give the same value for the root). Also, this discussion shows that the fourth root of a complex number has four unique values. What about the *n*th root? *n* different values.

This entire discussion is neatly summarized by the *De-Moivre's Theorem*. Let $z = re^{i\theta}$:

(a) If *n* is an integer, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$$= r^n(\cos n\theta + i\sin n\theta)$$

(b) If *n* is a non-integer rational number, say of the form $\dfrac{p}{q}$,

$$z^n = (re^{i\theta})^{\frac{p}{q}} = r^{p/q}e^{\frac{ip\theta}{q}} = r^{p/q}\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)$$

is only one of the values of z^n. There will be actually multiple values of z^n.

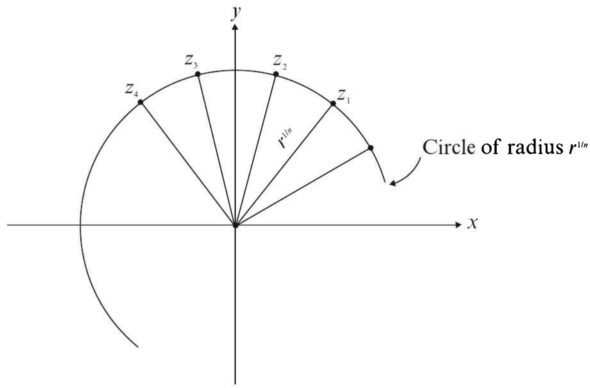
To evaluate the nth roots of an arbitrary complex number, we proceed as follows. Let, $z = re^{i\theta} = re^{i(2p\pi+\theta)}$:

$$\Rightarrow \quad z^{\frac{1}{n}} = \{re^{i(2p\pi+\theta)}\}^{\frac{1}{n}} = r^{1/n}e^{i\left(\frac{2p\pi+\theta}{n}\right)}.$$

This will have n unique values given by n successive integer values of p. We take p from the set $\{0, 1, 2, \ldots, (n-1)\}$. The n values that we obtain are listed out below:

$$r^{1/n}e^{\frac{i\theta}{n}},\ r^{1/n}e^{i\left(\frac{2\pi}{n}+\frac{\theta}{n}\right)},\ r^{1/n}e^{i\left(\frac{4\pi}{n}+\frac{\theta}{n}\right)},\ldots,r^{1/n}e^{i\left(\frac{2p\pi}{n}+\frac{\theta}{n}\right)},\ldots,r^{1/n}e^{i\left(\frac{2(n-1)\pi}{n}+\frac{\theta}{n}\right)}.$$

These n values are termed the nth roots of z. How will they lie on a plane? Notice that the angle between any two successive roots z_i and z_{i+1} is $\frac{2\pi}{n}$. Thus, the nth roots will lie on a circle of radius $r^{1/n}$ and will be *evenly spaced out*, the angle between any two successive roots being $\frac{2\pi}{n}$.



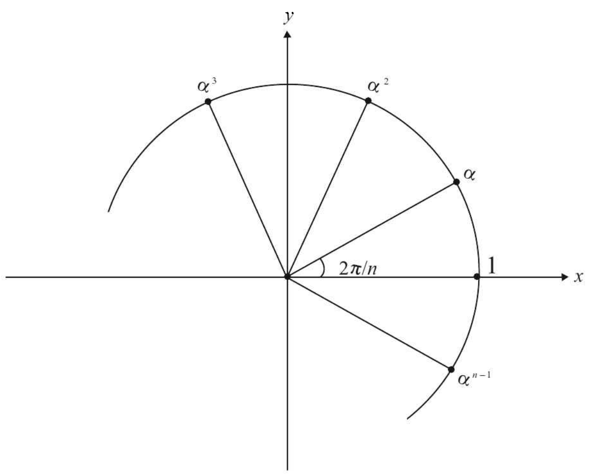
Now, we consider a special case, the nth roots of unity, or 1. In other words, we want the solutions to the equation $z^n = 1$. The Euler's form of 1 is e^{i0}. Therefore,

$$z = e^{i\frac{(2p\pi+0)}{n}} = e^{\frac{i2p\pi}{n}}$$

The n different values are

$$1, e^{\frac{i2\pi}{n}}, e^{\frac{i4\pi}{n}}, \ldots, e^{\frac{i2(n-1)\pi}{n}}$$

If we let $e^{i2\pi/n} = \alpha$, the n roots can be represented as $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$. These n roots will be evenly spaced out, lying on a circle of radius 1 centred at the origin.



The n roots of unity (unity is a root itself) lie on a unit circle centred at the origin; the angle between anytwo successive roots is $\frac{2\pi}{n}$.

From symmetry, notice that the sum of these *n* roots (vectors) will be 0.

$$1+\alpha+\alpha^2+\cdots+\alpha^{n-1}=0.$$

This can be proved analytically of course (try it!). We can also evaluate the product of these *n* roots:

$$P=1\cdot\alpha\cdot\alpha^2\ldots\alpha^{n-1}=\alpha^{1+2+3+\cdots+n-1}=\alpha^{\frac{n(n-1)}{2}}.$$

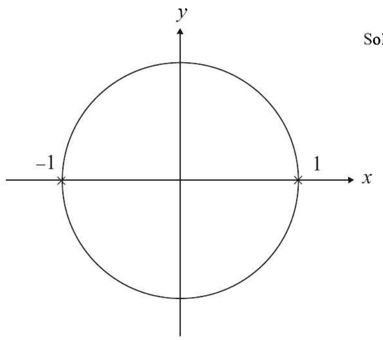
Since $\alpha=e^{i2\pi/n}$, $P=e^{\frac{i2\pi}{n}\cdot\left(\frac{n(n-1)}{2}\right)}=e^{i\pi(n-1)}$. Now, observe that if *n* is odd, *P* is 1, while if *n* is even, *P* is –1:

$$P=\begin{Bmatrix} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{Bmatrix}$$

We now summarize some particular cases of the *n*th roots of unity.

(i) $\boxed{n=2}$

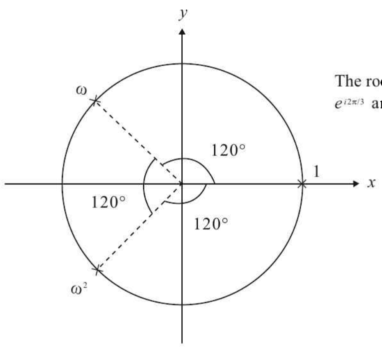
We want the square roots of unity, or, the solutions to $z^2=1$. One of the roots is 1. Where can we symmetrically place the other root? Obviously, at –1.



Solutions of $z^2=1$.

(ii) $\boxed{n=3}$

We want to solve $z^3=1$. One of the roots is 1. The other roots can be placed symmetrically if they are at angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, as shown below:



The roots apart from 1 are $e^{i2\pi/3}$ and $e^{i4\pi/3}$, labeled as ω and ω²

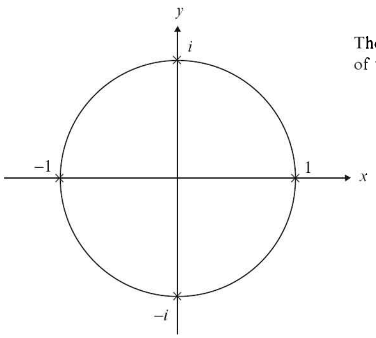
The two other roots are:

$$\omega=e^{i2\pi/3}=-\frac{1}{2}+\frac{i\sqrt{3}}{2},\qquad \omega^2=e^{i4\pi/3}=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

Notice that ω and ω^2 are conjugates of each other. Also, $1+\omega+\omega^2=0$ and $1\cdot\omega\cdot\omega^2=\omega^3=1$.

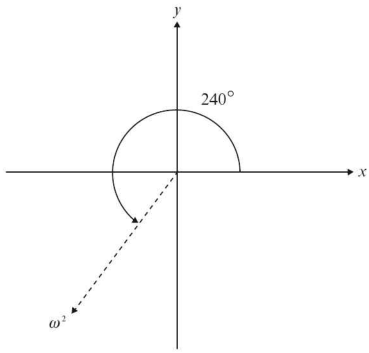
(iii) $\boxed{n = 4}$

We want to solve $z^4 = 1$. The four roots can be symmetrically placed as shown below:

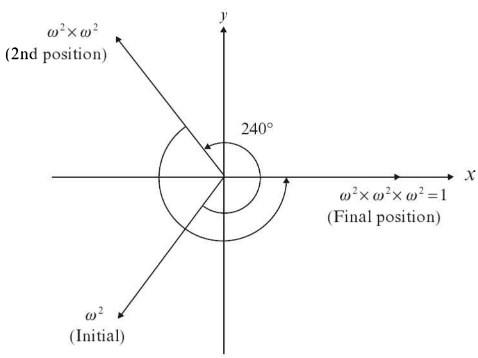


Note that $1 + i + (-1) + (-i) = 0$ and $1 \cdot (i) \cdot (-1) \cdot (-i) = -1$.

Higher order roots can be similarly evaluated. You must understand carefully the geometrical significance of the *n*th roots. Let us take one of the cube roots of unity for this purpose, say ω^2.

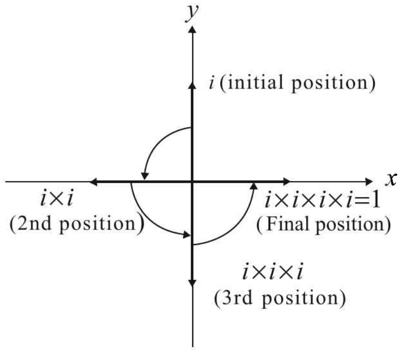


If you cube ω^2, you are essentially rotating this vector. The argument of ω^2 is 240°; when you cube ω^2, you will rotate ω^2 by 240° (for $\omega^2 \times \omega^2$) + another 240°(for $\omega^2 \times \omega^2 \times \omega^2$). Hence, ω^2, when cubed, will become the vector 1.



Thus, $(\omega^2)^3 = 1$.

Similarly, observe one of the fourth roots of unity, say *i*. The argument of *i* is 90°. When you raise *i* to power 4, you are essentially rotating the vector *i* by 90°(for $i \times i$) + 90°(for $i \times i \times i$) + 90° (for $i \times i \times i \times i$). Therefore, *i* when raised to power four will become the vector 1.



Thus, $i^4 = 1$.

This discussion should make you realise that when looking for the nth roots of unity, you are looking for vectors which when rotated by a certain fixed angle $(2\pi/n)$ a particular number $(n-1)$ of times, give the vector 1. Lets conclude with a final example:

Illustration 1:

Find the square roots and the cube roots of i.

Solution: First, write i in its Euler's form: $i = e^{\frac{i\pi}{2}}$

Square roots: $z^2 = e^{i\pi/2} = e^{i(2p\pi + \frac{\pi}{2})} \quad \Rightarrow \quad z = e^{i(p\pi + \frac{\pi}{4})}$

The two roots are given by two consecutive values of p, say $p = 0, 1$:

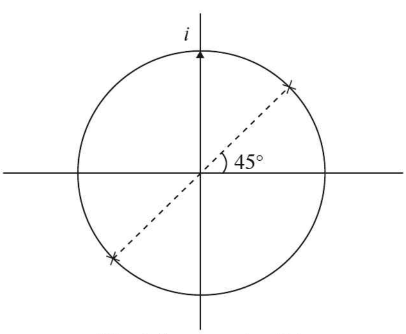
$$\Rightarrow \quad z_1 = e^{i\,\pi/4},\ z_2 = e^{i\,5\pi/4}$$

Cube roots: $z^3 = e^{i\left(2p\pi + \frac{\pi}{2}\right)} \quad \Rightarrow \quad z = e^{i\left(\frac{2p\pi}{3} + \frac{\pi}{6}\right)}$

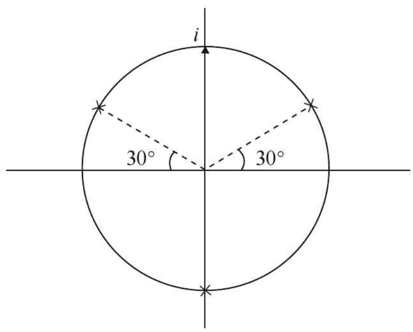
There are three roots; take $p = 0, 1, 2$:

$$\Rightarrow \quad z_1 = e^{i\pi/6},\ z_2 = e^{i5\pi/6},\ z_3 = e^{i3\pi/2}$$

Let us plot the square roots and cube roots of i on the plane:



(Two) Squre roots of i

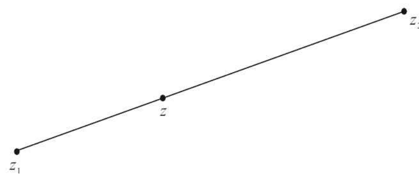


(Three) Cube roots of i

5. Geometry with Complex Numbers

Using complex numbers, we can write down equations of straight lines and circles, among other things, in the complex plane. To be able to do that properly, you must always keep the following facts in mind:

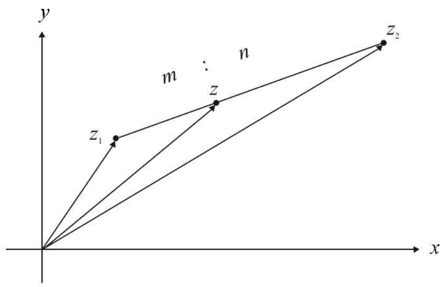
(a) $|z_1 - z_2|$ should be interpreted as the distance between the points z_1 and z_2, or equivalently, the length of the vector $z_1 - z_2$.
(b) You must know the difference between the point z and the vector z. Suppose we talk about the point $1 + i$. This has a fixed location in the plane, as you know, in the first quadrant. There's also the vector $1 + i$, which has an x-component of 1 and a y-component of 1, but this vector is not fixed. It is a free vector. We can move the vector $1+i$ anywhere around the plane as we wish, as long as we don't change its magnitude and direction.
(c) If z is purely real, then $z = \bar{z}$. If z is purely imaginary, then $z + \bar{z} = 0$.
(d) Suppose z_1 and z_2 are fixed in the plane.



We want to find a point z on the line segment joining z_1 and z_2, such that this point divides the line segment in the ratio $m : n$. z will be given by:

$$z = \frac{mz_2 + nz_1}{m+n}.$$

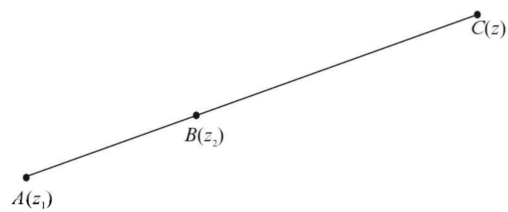
How? Lets consider z_1, z_2 and z with respect to a reference axis:



Now, from the figure, it should be clear that

$$z - z_1 = \frac{m}{m+n}(z_2 - z_1) \quad \Rightarrow \quad z = \frac{mz_2 + nz_1}{m+n}.$$

Such a point z is said to divide the line segment joining z_1 and z_2 *internally* in the ratio $m : n$. We could also have an external division as follows.



Here, z divides the line segment joining z_1 and z_2 externally in the ratio $m : n$, *i.e.*,

$$\frac{AC}{CB} = \frac{m}{n}.$$

Verify that z in this case is given by

$$z = \frac{mz_2 - nz_1}{m - n}.$$

To understand how complex numbers can be used to represent geometrical entities, it is best to consider some examples:

Illustration 2:

(a) Write the equation of a circle of radius 1 centred at $1 + i$.

(b) What relation should z satisfy so that it is closer to 1 than to i?

Solution: (a) Let z be a variable complex number representing the required locus (circle). We want that the distance of z from $1 + i$ must always equal 1.

$$\Rightarrow \quad |z - (1+i)| = 1$$

(b) The required condition can be translated into a mathematical form as follows:

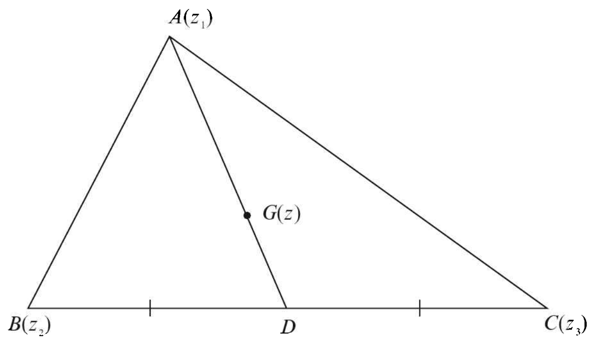
(Distance of z from 1) < (Distance of z from i)

$$\Rightarrow \quad |z - 1| < |z - i|$$

Illustration 3:

If z_1, z_2 and z_3 represent the vertices of an arbitrary triangle, find its centroid.

Solution: Let ABC be the triangle, and let G be its centroid. Let D be the mid point of BC.



Since D is the mid-point of BC (D divides BC in the ratio 1:1), D is given by $\frac{z_2+z_3}{2}$. From plane geometry, we also know that the centroid divides any median in the ratio 2:1. Thus,

$$AG : GD = 2 : 1$$

This implies that G is given by

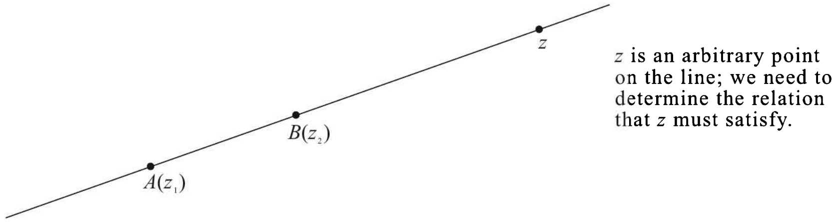
$$z = \frac{2\left(\frac{z_2+z_3}{2}\right) + 1(z_1)}{2+1} = \frac{z_1 + z_2 + z_3}{3}$$

Thus, the centroid is given by $= \dfrac{z_1 + z_2 + z_3}{3}$

Illustration 4:

Find the equation of the straight line joining the points $A(z_1)$ and $B(z_2)$.

Solution: The situation is sketched in the figure below:



Observe from the figure that the vectors $z - z_1$ and $z - z_2$ are either in the same direction (whenever z lies out side the line segment AB) or they are anti-parallel (whenever z lies between A and B). Thus, the vector $z - z_1$ can be obtained by multiplying the vector $z - z_2$ with a scalar (a real number) in all cases:

$$z - z_1 = s(z - z_2); \quad s \in \mathbb{R}$$

$$\Rightarrow \quad z - sz = z_1 - sz_2$$

$$\Rightarrow \quad z = \frac{1}{1-s} z_1 - \frac{s}{1-s} z_2 = tz_1 + (1-t)z_2, \quad \text{where } t = \frac{1}{1-s} \in \mathbb{R}.$$

Thus, we vary t over all real numbers in the relation above and we'll obtain all the corresponding points on the required line.This is the equation of the straight line in parametric form. We can also write the required equation without involving any parameter. Since $z - z_2$ is a scalar (real) multiple of $z_1 - z_2$,

$$\frac{z - z_2}{z_1 - z_2} \text{ is purely real} \quad \Rightarrow \quad \frac{z - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0. \qquad (1)$$

This is the required equation in non-parametric form. How do we measure the slope of this line from the expression in (1)? Observe that the required slope is $\tan\theta = \tan(\arg(z_1 - z_2))$. If we evaluate $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$, which is actually, $\frac{|z_1 - z_2|e^{i\theta}}{|z_1 - z_2|e^{-i\theta}} = e^{2i\theta}$, we do get a measure of θ. Thus, we can use $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$ as some form of slope. Lets call it the *complex slope* of the line given by (1). Now, (1) can be simplified further. Notice that $z_1\bar{z}_2 - \bar{z}_1 z_2$ can be written as $z_1\bar{z}_2 - \overline{z_1\bar{z}_2}$ which is equal to $2i\,\mathrm{Im}(z_1\bar{z}_2)$. Therefore, (1) becomes:

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + 2i\,\mathrm{Im}(z_1\bar{z}_2) = 0.$$

Multiplying both sides by i, we get

$$z(i(\bar{z}_1 - \bar{z}_2)) - \bar{z}(i(z_1 - z_2)) - 2\,\mathrm{Im}(z_1\bar{z}_2) = 0$$

$$\Rightarrow \quad az + \bar{a}\bar{z} + b = 0 \quad \left\{\begin{array}{l} \text{where } a = i(\bar{z}_1 - \bar{z}_2) \\ \text{so that } \bar{a} = -i(z_1 - z_2) \\ \text{and } b \text{ is real} \end{array}\right\}$$

The complex slope of this line as defined by us earlier, is $-\frac{\bar{a}}{a}$ $\left(\frac{-\text{coefficient of } \bar{z}}{\text{coefficient of } z}\right)$. This is the equation of a straight line in its most general (complex form). From this equation, try to write down the condition for the collinearity of three points in determinant form.

Illustration 5:

(a) If s_1 and s_2 be the complex slopes of two lines, find the condition on them so that the lines are
(i) parallel (ii) perpendicular

(b) Find the equation of a line perpendicular to the line $az + \bar{a}\bar{z} + b = 0$, passing through z_1.

Solution: (a) Let the actual slopes be $\tan\theta_1$ and $\tan\theta_2$. The complex slopes are $s_1 = e^{2i\theta_1}$ and $s_2 = e^{2i\theta_2}$:

(i) For parallel lines, $\theta_1 = \theta_2$. Therefore, $s_1 = s_2$.

(ii) For perpendicular lines, $\theta_1 - \theta_2 = \frac{\pi}{2}$:

$$\Rightarrow \quad 2(\theta_1 - \theta_2) = \pi \Rightarrow \quad e^{i2(\theta_1-\theta_2)} = e^{i\pi} = -1$$

$$\Rightarrow \quad s_1 \cdot s_2^{-1} = -1 \Rightarrow \quad s_1 + s_2 = 0.$$

(b) The complex slope of the original line is $s_1 = \frac{z-z_1}{\bar{z}-\bar{z}_1}$. The complex slope of the perpendicular line will be $s_2 = \frac{\bar{a}}{a}$. Since, the lines are perpendicular, $s_1 + s_2 = 0$:

$$\Rightarrow \quad \frac{z-z_1}{\bar{z}-\bar{z}_1} - \frac{\bar{a}}{a} = 0 \quad \Rightarrow \quad a(z-z_1) - \bar{a}(\bar{z}-\bar{z}_1) = 0$$

$$\Rightarrow \quad az - \bar{a}\bar{z} - (az_1 - \bar{a}\bar{z}_1) = 0 \quad \Rightarrow \quad az - \bar{a}\bar{z} - 2i\,\mathrm{Im}(az_1) = 0.$$

Illustration 6:

Find the general equation of a circle in complex form.

Solution: Let us consider an arbitrary circle with centre z_0 and radius r, whose equation will be $|z - z_0| = r$.

$$\Rightarrow \quad |z - z_0|^2 = r^2$$

$$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 - r^2 = 0$$

$$\Rightarrow \quad |z|^2 + az + \bar{a}\bar{z} + k = 0 \quad \text{where} \quad a = -\bar{z}_0 \text{ and } k = |z_0|^2 - r^2.$$

This is the general equation that we required. Given this form, we can easily deduce that the centre will be $-\bar{a}$ and the radius will be $\sqrt{|z_0|^2 - k}$.

Important Ideas and Tips

1. *Basic Meaning:* The term *complex* in complex numbers means that any such number is a combination, or complex, of two numbers, *a* and *bi*, where *a* represents a displacement along the *x*-direction, and *bi* reprsents a displacement along the *y*-direction.

2. *Geometrical Association:* The most important idea of this chapter is to make a geometrical association of any complex number $z = a + bi$ with the point (a, b) or the vector $a\hat{i} + b\hat{j}$ in the plane. Even purely real numbers should be seen from this perspective.

3. *The meaning of 'Imaginary':* It's incorrect to think that imaginary numbers are imaginary in the sense that they are mathematically invalid. The only meaning that should be attached to the word imaginary is that imaginary numbers are numbers which fall outside the set of real numbers. They are perfectly valid mathematical entities. The term imaginary is an unfortunate historical artifact.

4. *Real and Imaginary Parts.* Both the real and imaginary parts of a complex number are purely real. If $z = a + bi$, then $\text{Re}(z) = a$ and $\text{Im}(z) = b$. The imaginary part is not equal to bi, but simply b.

5. *Interpretation of Modulus:* The modulus of a complex number z, represented by $|z|$, should always be seen as the distance of point z from the origin, or the length of vector z. Most students, when given a complex number $z = a + bi$, and asked the question: *What is the modulus of* z?, will reply that the modulus is $\sqrt{a^2 + b^2}$. Technically, this is correct, but this is just a formula, a consequence the real meaning of the modulus is the distance of point z from the origin, or the length of vector z, and this should never be forgotten. Formulae are secondary; meanings are more important.

6. *Interpretation of Argument:* The argument of a complex number z, represented by arg(z) should always be seen as the angle that the line joining the origin to the point z (or vector z) makes with the positive direction of the x-axis. Once again, the meaning of arg(z) is more important to keep in mind than any formula for arg(z).

7. *Argument and Principal Argument:* A distinction must be made between argument and principal argument. The argument is any value of the angle, measured from the positive x-axis, which correctly describes the direction of vector z. For example, for a purely imaginary complex number (a point lying on the positive y-axis), the argument could be given as $\frac{\pi}{2}, 2\pi + \frac{\pi}{2}, -36\pi + \frac{\pi}{2}$, or even $10000000\pi + \frac{\pi}{2}$, etc. since all these angles essentially give the same direction. Thus, the argument of a complex number can numerically have infinite values, all corresponding to one direction. The principal argument is a special value of the argument which lies in the range $(-\pi, \pi]$. The only reason principal arguments are important is that they enable us to assign unique argument values to complex numbers.

8. *Mistakes in Calculating Arguments:* If $z = a + bi$, then arg(z) is not always equal to $\tan^{-1}(\frac{b}{a})$. For example, if $z = -1 - i$, then the formula $\tan^{-1}(\frac{b}{a})$ gives the argument as $\frac{\pi}{4}$, while the actual argument is $-\frac{3\pi}{4}$ (or $\frac{5\pi}{4}$). Thus, whenever you have to find the argument of a complex number, first determine the quadrant in which it lies, and then proceed to find the angle it makes with the positive x-axis.

9. *Mistakes in Conjugation.* A common misconception pertains to the conjugation operation. Let z_1 and z_2 be two complex numbers. What will be the conjugate of $z = z_1 + iz_2$? If you replied: $z_1 - iz_2$, your answer is incorrect! Note that z_1 and z_2 are themselves complex. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. The correct procedure to find the required conjugate is as follows:

$$
\begin{aligned}
\overline{z_1 + iz_2} &= \overline{(a_1 + ib_1) + i(a_2 + ib_2)} \\
&= \overline{a_1 + i(a_2 + b_1) + i^2 b_2} \\
&= \overline{(a_1 - b_2) + i(a_2 + b_1)} \\
&= (a_1 - b_2) - i(a_2 + b_1)
\end{aligned}
$$

10. *Arithmetic Operations:* Addition and subtraction of complex numbers should almost always be thought of from a vector perspective. This makes it very easy to interpret many complex expressions. Multiplication and division of complex numbers should be done using their Euler (exponential) forms, and should be seen as leading to rotation and scaling of vectors, as described in the preceeding discussion.

11. *Keystones of the Chapter:* It won't be an exaggeration to say that if you remember and understand the following three facts, most of the material of this chapter will start seeming obvious to you:

    - Complex numbers correspond to points or vectors in the plane.
    - Addition of complex numbers is similar to addition of vectors.
    - Multiplication of complex numbers corresponds to scaling and rotation of vectors: if you multiply a complex number z with another (say $re^{i\theta}$), z gets rotated by angle θ and scaled by $|z|$.

    Try to see these basic facts at work whenever you solve a complex numbers problem.

Complex Numbers

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

For complex numbers z and w, if $|z|^2 w - |w|^2 z = z - w$, then which of the following are possible?

(A) $z = w$ (B) $z = \bar{w}$ (C) $zw = 1$ (D) $z\bar{w} = 1$

Solution: The correct options are (A) and (D). The relation $|z|^2 w - |w|^2 z = z - w$ when rearranged gives:

$$\Rightarrow \quad \frac{z}{w} = \frac{1+|z|^2}{1+|w|^2} \tag{1}$$

$$\Rightarrow \quad \frac{z}{w} \text{ is purely real.}$$

Let $z = kw$, where $k \in \mathbb{R}$. Substituting for z in (1), we get

$$\frac{kw}{w} = \frac{1+k^2|w|^2}{1+|w|^2}$$

$$\Rightarrow \quad k + k|w|^2 = 1 + k^2|w|^2 \quad \Rightarrow \quad k - 1 = k|w|^2(k-1)$$

$$\Rightarrow \quad (k|w|^2 - 1)(k-1) = 0$$

This means that either $k = 1$ or $k = \frac{1}{|w|^2}$, so that

$$\Rightarrow \quad \frac{z}{w} = 1 \text{ or } \frac{z}{w} = \frac{1}{|w|^2} = \frac{1}{w\bar{w}} \quad \Rightarrow \quad z = w \text{ or } z\bar{w} = 1.$$

■

Example 2

If $|z-\frac{4}{z}|=2,$ then the difference between $\max(|z|)$ and $\min(|z|)$ is:

(A) 1 (B) 2 (C) 4 (D) 8

Solution: Applying the triangle inequality on $|z-\frac{4}{z}|$, we get:

$$\left||z|-\frac{4}{|z|}\right| \le \left|z-\frac{4}{z}\right| \le |z|+\frac{4}{|z|}$$

$$\Rightarrow \quad \left||z|-\frac{4}{|z|}\right| \le 2 \le |z|+\frac{4}{|z|}$$

The right side of this inequality is always satisfied (verify). We therefore use the left side of this inequality:

$$\left||z|-\frac{4}{|z|}\right| \le 2$$

$$\Rightarrow \quad -2 \le |z|-\frac{4}{|z|} \le 2$$

$$\Rightarrow \quad |z|^2+2|z|-4 \ge 0 \quad \text{and} \quad |z|^2-2|z|-4 \le 0$$

$$\Rightarrow \quad |z| \ge -1+\sqrt{5} \quad \text{and} \quad |z| \le 1+\sqrt{5}$$

$$\Rightarrow \quad \sqrt{5}-1 \le |z| \le \sqrt{5}+1$$

These are the required maximum and minimum values. The required difference is 2. Therefore, the correct option is (B). ■

Example 3

If z_1 and z_2 are fixed and z satisfies

$$|z-z_1|^2+|z-z_2|^2=k,$$

what are the possible values of k so that this equation represents a circle?

(A) $k>\frac{1}{4}\left||z_1|^2-|z_2|^2\right|$ (B) $k>\frac{1}{2}\left||z_1|^2-|z_2|^2\right|$ (C) $k>\frac{1}{2}|z_1-z_2|^2$

(D) $k>|z_1-z_2|^2$ (E) None of these

Solution: Let us try to reduce (simplify) the given equation:

$$|z-z_1|^2+|z-z_2|^2=k$$

$$\Rightarrow \quad 2|z|^2-2\,\mathrm{Re}(z\bar{z}_1+z\bar{z}_2)+|z_1|^2+|z_2|^2=k$$

$$\Rightarrow \quad |z|^2-\mathrm{Re}(z(\bar{z}_1+\bar{z}_2))=\frac{1}{2}(k-|z_1|^2-|z_2|^2)$$

Careful thinking will show that the LHS can be transformed into a perfect square with the following manipulation (adding a specific term on both sides of the equality):

$$\Rightarrow \quad |z|^2 - \mathrm{Re}(z(\bar{z}_1 + \bar{z}_2)) + \left|\frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}(k - |z_1|^2 - |z_2|^2) + \left|\frac{z_1 + z_2}{2}\right|^2$$

$$\Rightarrow \quad \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\left(k - \frac{|z_1 - z_2|^2}{2}\right) \quad \{\text{verify this step}\}$$

$$= \frac{1}{4}(2k - |z_1 - z_2|^2)$$

Thus, the centre is $z_0 = \frac{z_1 + z_2}{2}$ and the radius is $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$. For the radius to be a real non-zero number, the condition that k must satisfy is

$$k > \frac{1}{2}|z_1 - z_2|^2$$

These are the possible values of k. The correct option is (C). ■

Example 4

The sum of the imaginary parts of all the complex numbers, which simultaneously satisfy

$$\left|\frac{z - 12}{z - 8i}\right| = \frac{5}{3} \text{ and } \left|\frac{z - 4}{z - 8}\right| = 1, \text{is}$$

(A) 25 (B) 28 (C) 32 (D) 35

Solution: It is easy to observe that the first equation is that of a circle while the second is that of a straight line which is the perpendicular bisector of 4 and 8, *i.e.*, from the second equation, we know that z is of the form $z = 6 + xi$. Using this in the first equation, we get:

$$\left|\frac{-6 + xi}{6 + (x - 8)i}\right| = \frac{5}{3}$$

$$\Rightarrow \quad 9(6^2 + x^2) = 25(6^2 + (x - 8)^2) = 25(100 + x^2 - 16x)$$

$$\Rightarrow \quad 16x^2 - 400x + 2176 = 0 \quad \Rightarrow \quad x^2 - 25x + 136 = 0$$

$$\Rightarrow \quad (x - 8)(x - 17) = 0 \quad \Rightarrow \quad x = 8 \text{ or } 17$$

Thus, $z = 6 + 8i$ or $6 + 17i$. The required sum is 8 + 17 = 25. The correct option is (A). ■

Example 5

If $|\frac{1 - iz}{z - i}| = 1$, then which of the following is true?

(A) z is purely real.
(B) z is purely imaginary.
(C) z lies in the first or third quadrant.
(D) z lies in the second or fourth quadrant.

Solution: From the given relation, $|1-iz|^2=|z-i|^2$:

$$\Rightarrow \quad 1+|z|^2-2\,\text{Re}(iz)=|z|^2+1-2\,\text{Re}(i\bar{z}) \quad \Rightarrow \quad \text{Re}(iz)=\text{Re}(i\bar{z})$$

If $z=x+iy$, this means that

$$\text{Re}(i(x+iy))=\text{Re}(i(x-iy))$$

$$\Rightarrow \quad -y=y \quad \Rightarrow \quad y=0 \quad \Rightarrow \quad z \text{ is purely real.}$$

The correct option is (A). ■

Example 6

If $\arg(z^{1/3})=\frac{1}{2}\arg(z^2+\bar{z}z^{1/3})$, then the value of $|z|$ (where z is a non real complex number) is

(A) $\dfrac{1}{\sqrt{2}}$ (B) 1 (C) $\sqrt{2}$ (D) 2

Solution: Since the given relation contains only arguments, we can use the properties that arguments satisfy, to simplify this relation:

$$2\arg(z^{1/3})=\arg(z^2+\bar{z}z^{1/3})$$

$$\Rightarrow \quad \arg(z^{2/3})=\arg(z^2+\bar{z}z^{1/3}) \quad \Rightarrow \quad \arg(z^2+\bar{z}z^{1/3})-\arg(z^{2/3})=0$$

$$\Rightarrow \quad \arg\left(\frac{z^2+\bar{z}z^{1/3}}{z^{2/3}}\right)=0 \quad \Rightarrow \quad \arg\left(z^{4/3}+\frac{\bar{z}}{z^{1/3}}\right)=0$$

$$\Rightarrow \quad z^{4/3}+\frac{\bar{z}}{z^{1/3}}=\bar{z}^{4/3}+\frac{z}{\bar{z}^{1/3}} \quad \left(\begin{array}{l}\text{because if } \arg(z)=0,\\ z \text{ is purely real} \Rightarrow \quad z=\bar{z}\end{array}\right)$$

$$\Rightarrow \quad \bar{z}^{1/3}(z^{5/3}+\bar{z})=z^{1/3}(\bar{z}^{5/3}+z) \quad \Rightarrow \quad \bar{z}^{1/3}z^{1/3}z^{4/3}+\bar{z}^{4/3}=z^{1/3}\bar{z}^{1/3}\bar{z}^{4/3}+z^{4/3}$$

$$\Rightarrow \quad |z|^{2/3}\,z^{4/3}+\bar{z}^{4/3}=|z|^{2/3}\,\bar{z}^{4/3}+z^{4/3} \quad \Rightarrow \quad z^{4/3}(1-|z|^{2/3})-\bar{z}^{4/3}(1-|z|^{2/3})=0$$

$$\Rightarrow \quad (z^{4/3}-\bar{z}^{4/3})(1-|z|^{2/3})=0$$

Since z is a non-real complex number, $z\neq\bar{z}$, and so

$$z^{4/3}\neq\bar{z}^{4/3}.$$

Therefore,

$$|z|^{2/3}=1 \quad \Rightarrow \quad |z|=1$$

The correct option is (B). ■

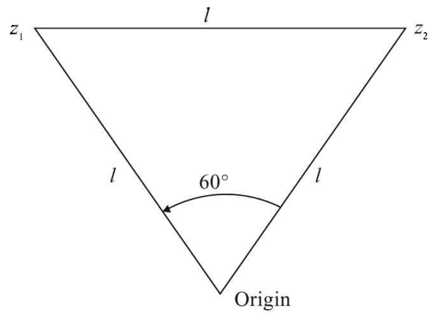
Example 7

If the complex numbers z_1, z_2 and the origin form an equilateral triangle, then which of the following is true?

(A) $z_1^2+z_2^2+z_1z_2=0$ (C) $z_1^2+z_2^2+2(z_1+z_2)=0$ (E) None of these

(B) $z_1^2+z_2^2-z_1z_2=0$ (D) $z_1^2+z_2^2-2(z_1+z_2)=0$

Solution: We can assume the following configuration for z_1 and z_2:



Observe that the vector z_1 can be obtained from the vector z_2 through rotation. Thus, we have

$$\frac{z_1}{l} = \frac{z_2}{l}e^{i\pi/3} \quad \Rightarrow \quad \frac{z_1}{z_2} = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{z_1^2}{z_2^2} = \frac{1}{4} - \frac{3}{4} + \frac{1}{2}i\sqrt{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = -1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + \frac{z_1}{z_2}$$

$$\Rightarrow \quad z_1^2 = -z_2^2 + z_1 z_2$$

$$\Rightarrow \quad z_1^2 + z_2^2 - z_1 z_2 = 0$$

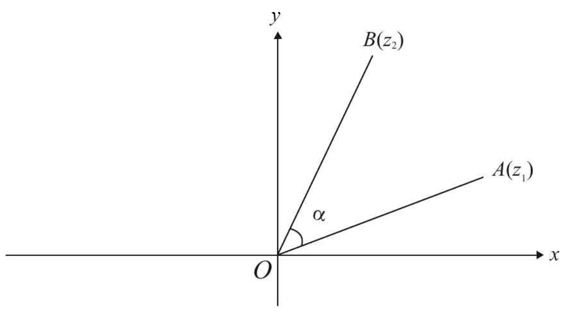
The correct option is (B). ■

Example 8

Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, the value of $\frac{p^2}{q}\sec^2\frac{\alpha}{2}$ will be:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution: The situation in the problem can be represented by the following diagram:



Using rotation $(OA \to OB)$, we get:

$$\frac{z_2}{z_1} = \frac{OB}{OA}e^{i\alpha} = e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{1}$$

Adding 1 to both sides of (1) (it will soon become clear why), we get

$$\frac{z_2}{z_1}+1 = 1+\cos\alpha + i\sin\alpha$$

$$= 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$= 2\cos\frac{\alpha}{2}e^{i\alpha/2}. \tag{2}$$

Since z_1 and z_2 are the roots of $z^2+pz+q=0$,

$$z_1+z_2 = -p \tag{3}$$

$$z_1 z_2 = q \tag{4}$$

Substituting (3) in (2) and then squaring, we get

$$\frac{p^2}{z_1^2} = 4\cos^2\frac{\alpha}{2}e^{i\alpha} = 4\cos^2\frac{\alpha}{2}\,\frac{z_2}{z_1} \qquad \text{(From (1))}$$

$$\Rightarrow \quad p^2 = 4\cos^2\frac{\alpha}{2}z_1 z_2 = 4q\cos^2\alpha/2 \qquad \text{(From (4))}$$

Thus, $\frac{p^2}{q}\sec^2\frac{\alpha}{2}$ is equal to 4. The correct option is (D). ■

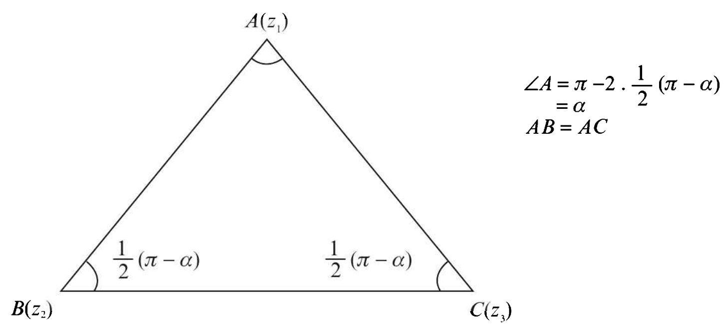
Example 9

If z_1, z_2 and z_3 represent the complex numbers A, B, C respectively and $\angle ABC = \angle ACB = \frac{1}{2}(\pi-\alpha)$, then the value of $\frac{(z_2-z_3)^2}{(z_3-z_1)(z_1-z_2)}$ is:

(A) $2\sin^2\alpha/2$ (C) $2\cos^2\alpha/2$ (E) None of these

(B) $4\sin^2\alpha/2$ (D) $4\cos^2\alpha/2$

Solution: Since $\angle ABC = \angle ACB$, the triangle ABC is isosceles:



Applying rotation about $z_1\,(\overrightarrow{AB} \to \overrightarrow{AC})$:

$$\frac{z_3 - z_1}{z_2 - z_1} = e^{i\alpha} = \cos\alpha + i\sin\alpha \qquad (1)$$

Since the final result we would like to obtain contains $\alpha/2$, we subtract 1 from both sides of (1):

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} - 1 = -1 + \cos\alpha + i\sin\alpha$$

$$\Rightarrow \quad \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$= -2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)$$

$$= 2i\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right) = 2i\sin\frac{\alpha}{2}e^{i\alpha/2}$$

Squaring both sides,

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}e^{i\alpha} = -4\sin^2\frac{\alpha}{2}\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

Cross-multiplying by $(z_2 - z_1)^2$ tells us that the correct option is (B). ■

Example 10

If 1, α_1, α_2..., α_{n-1} are the nth roots of unity, what are the possible values of $(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{n-1})$?

(A) 0 (B) 1 (C) n (D) 2^n (E) None of these

Solution: Since $x^n - 1 = 0$ has n roots (that have been specified in the question), we can write (by the factor theorem):

$$x^n - 1 = (x-1)(x-\alpha_1)\cdots(x-\alpha_{n-1})$$

$$\Rightarrow \quad \frac{x^n - 1}{x - 1} = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}) \qquad (1)$$

The form of the expression whose value we need to obtain hints that we should substitute $x = -1$ in (1):

$$\frac{(-1)^n - 1}{(-1) - 1} = (-1-\alpha_1)(-1-\alpha_2)\cdots(-1-\alpha_{n-1})$$

$$= (-1)^{n-1}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{n-1})$$

If n is even, the LHS is 0, so that the value of the expression becomes 0. If n is odd, the expression becomes

$$\frac{-2}{-2} = (-1)^{n-1}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{n-1})$$

$$\Rightarrow \quad (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{n-1}) = 1$$

Thus, the expression takes the value 0 or 1 depending on whether n is even or odd respectively. The correct options are (A) and (B). ■

Example 11

If $p \in \mathbb{Z}$, what is the sum of the pth powers of the nth roots of unity?

(A) 0 in all cases
(B) 0 if p is not a multiple of n
(C) n in all cases
(D) n if p is a multiple of n
(E) None of these

Solution: Let the nth roots of unity be

$$1, \alpha, \alpha^2, ..., \alpha^{n-1} \qquad \text{where} \quad \alpha = e^{i2\pi/n}$$

The sum of the pth powers of these roots is:

$$\begin{aligned} S_p &= (1)^p + (\alpha)^p + (\alpha^2)^p + \cdots + (\alpha^{n-1})^p \\ &= 1 + \alpha^p + (\alpha^p)^2 + \cdots + (\alpha^p)^{n-1} \end{aligned} \tag{1}$$

This is a GP with common ratio α^p:

$$\Rightarrow \quad S_p = \frac{(\alpha^p)^n - 1}{\alpha^p - 1} \tag{2}$$

If p is not a multiple of n, $\alpha^p \neq 1$ so that the expression for S_p is defined in (2). The numerator of S_p is $(\alpha^p)^n - 1 = (\alpha^n)^p - 1 = 1^p - 1 = 0$. Thus, $S_p = 0$.
Suppose now that p is a multiple of n. In that case, $\alpha^p = 1$, so that S_p is directly obtainable from (1).

$$\begin{aligned} S_p &= 1 + 1 + 1 + \cdots + 1 \ (n \text{ times}) \\ &= n \end{aligned}$$

Thus, $$S_p = \begin{cases} 0, & \text{if } p \text{ is not a multiple of } n \\ n, & \text{if } p \text{ is a multiple of } n \end{cases}$$

The correct options are (B) and (D). ■

Example 12

If α is the nth root of unity given by $\alpha = e^{i2\pi/n}$ and z_1 and z_2 are any two complex numbers, the value of $\dfrac{\sum_{p=0}^{n-1} \left|z_1 + \alpha^p z_2\right|^2}{\left|z_1\right|^2 + \left|z_2\right|^2}$ is

(A) $n-1$ (B) n (C) $n+1$ (D) None of these

Solution: $|z_1+\alpha^p z_2|^2=(z_1+\alpha^p z_2)(\bar{z}_1+\bar{\alpha}^p\bar{z}_2)$

$$=|z_1|^2+|z_2|^2+\alpha^p\bar{z}_1 z_2+\bar{\alpha}^p z_1\bar{z}_2$$

Now, $\sum_{p=0}^{n-1}\alpha^p=0$ (sum of the nth roots is 0)

$$\Rightarrow \sum_{p=0}^{n-1}\bar{\alpha}^p=0$$

Thus,

$$\sum_{p=1}^{n-1}|z_1+\alpha^p z_2|^2=n(|z_1|^2+|z_2|^2)+\bar{z}_1 z_2\sum_{p=0}^{n-1}\alpha^p+z_1\bar{z}_2\sum_{p=0}^{n-1}\bar{\alpha}^p$$

$$=n(|z_1|^2+|z_2|^2)$$

We see that the required ratio is n, so the correct option is (B). ■

Example 13

The sum of the real and imaginary parts of $\sum_{p=1}^{32}(3p+2)(\sum_{q=1}^{10}(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}))^p$ is:

(A) 0 (B) 16 (C) 32 (D) 48

Solution: Although the expression is enormous, the alert reader will quickly realise that this expression can be expressed in terms of the eleventh roots of unity:

$$\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)=-i\left(\cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11}\right)$$

$$=-i\alpha^q,\quad \text{where}\quad \alpha=\cos\frac{2\pi}{11}+i\sin\frac{2\pi}{11}=e^{i2\pi/11}$$

Thus, $$\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)=\sum_{q=1}^{10}(-i\alpha^q)=-i\sum_{q=1}^{10}\alpha^q$$

$$=-i\times -1 \quad (\because\ 1+\alpha+\ldots\alpha^{10}=0)$$

$$=i$$

The given expression reduces to

$$S=\sum_{p=1}^{32}(3p+2)i^p=3\sum_{p=1}^{32}pi^p+2\sum_{p=1}^{32}i^p$$

We observe that the term $\sum_{p=1}^{32}i^p$ is 0. Therefore,

$$S=3[(i-2-3i+4)+(5i-6-7i+8)+\cdots-31i+32]$$

$$=3[(-2i+2)+(-2i+2)+\cdots(-2i+2]=3\times 8\times 2(1-i)=48(1-i)$$

The sum of the real and imaginary parts is $48+(-48)=0$. The correct option is (A). ■

Example 14

Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-sided regular polygon such that:

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$$

The value of n is:

(A) 5 (B) 7 (C) 9 (D) 11

Solution: We let the vertices be represented by $z_1, z_2, ..., z_n$ (in an anti-clockwise order), so that z_1 coincides with the origin. Thus, we will have

$$z_2 = z_1 e^{i2\pi/n}, \quad z_3 = z_1 e^{i4\pi/n}, \quad z_4 = z_1 e^{i6\pi/n} \text{ and so on.}$$

$$\Rightarrow \quad A_1A_2 = |z_1 - z_1 e^{i2\pi/n}| = |z_1||1 - e^{i2\pi/n}| = |z_1| \left| \left(1 - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n} \right) \right|$$

$$= |z_1| \left| \left(2\sin^2\frac{\pi}{n} - 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n} \right) \right| = 2|z_1|\sin\frac{\pi}{n}.$$

Similarly, $$A_1A_3 = |z_1 - z_1 e^{i4\pi/n}| = 2|z_1|\sin\frac{2\pi}{n}$$

$$A_1A_4 = |z_1 - z_1 e^{i6\pi/n}| = 2|z_1|\sin\frac{3\pi}{n}.$$

Substituting for A_1A_2, A_1A_3 and A_1A_4 in the relation given in the question, we get

$$\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$$

$$\Rightarrow \quad \sin\frac{2\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{\pi}{n}\sin\frac{2\pi}{n} + \sin\frac{\pi}{n}\sin\frac{3\pi}{n}.$$

Writing $\sin\frac{2\pi}{n}$ as $2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ and cancelling out $\sin\frac{\pi}{n}$ from both sides, we get:

$$2\cos\frac{\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n}$$

$$\Rightarrow \quad \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n}$$

$$\Rightarrow \quad \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \quad \text{(why?)}$$

$$\Rightarrow \quad n = 7.$$

The correct option is (B). ■

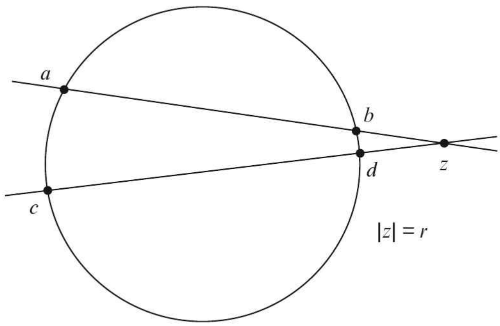
Example 15

Two different non-parallel lines meet the circle $|z| = r$ in the points a, b and c, d respectively. If these two lines meet in z, then z will be given by:

(A) $z = \dfrac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$ (B) $z = \dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$ (C) $z = \dfrac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}+c^{-1}d^{-1}}$

(D) $z = \dfrac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{ab+cd}$ (E) None of these

Solution: The situation given in the question is sketched in the figure below:



Since a, b, c, d lie on the circle $|z| = r$, we have

$$|a| = |b| = |c| = |d| = r$$

$$\Rightarrow \quad a\bar{a} = b\bar{b} = c\bar{c} = d\bar{d} = r^2 \tag{1}$$

Now, a, b and z are collinear $\Rightarrow \quad z - b = \lambda(b-a)$

$$\Rightarrow \quad \frac{z-b}{b-a} \text{ is purely real} \quad \Rightarrow \quad \frac{z-b}{b-a} = \frac{\bar{z}-\bar{b}}{\bar{b}-\bar{a}}$$

$$\Rightarrow \quad (\bar{b}-\bar{a})z - (b-a)\bar{z} + \bar{a}b - a\bar{b} = 0$$

$$\Rightarrow \quad \bar{z} = \frac{(\bar{b}-\bar{a})z + \bar{a}b - a\bar{b}}{b-a} \tag{2}$$

Similarly, since c, d and z are collinear,

$$\bar{z} = \frac{(\bar{d}-\bar{c})z + \bar{c}d - c\bar{d}}{d-c} \tag{3}$$

From (2) and (3)

$$\frac{(\bar{b}-\bar{a})z+\bar{a}b-a\bar{b}}{b-a}=\frac{(\bar{d}-\bar{c})z+\bar{c}d-c\bar{d}}{d-c} \tag{4}$$

Using (1) in (4), we obtain

$$\frac{(\frac{r^2}{b}-\frac{r^2}{a})z+\frac{r^2b}{a}-\frac{r^2a}{b}}{b-a}=\frac{(\frac{r^2}{d}-\frac{r^2}{c})z+\frac{r^2d}{c}-\frac{r^2c}{d}}{d-c}$$

$$\Rightarrow \quad \frac{-z}{ab}+\frac{b+a}{ab}=\frac{-z}{cd}+\frac{d+c}{cd}$$

$$\Rightarrow \quad z\left(\frac{1}{ab}-\frac{1}{cd}\right)=\frac{b+a}{ab}-\frac{d+c}{cd}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}-\frac{1}{d}=a^{-1}+b^{-1}-c^{-1}-d^{-1}$$

$$\Rightarrow \quad z=\frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

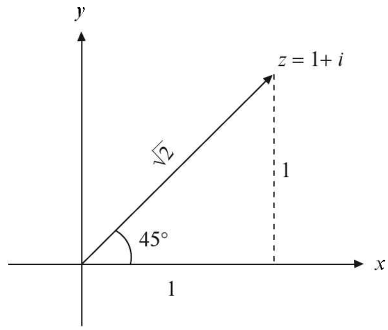
Example 16

Plot the following points on a plane and evaluate their polar forms:

(a) $z = 1 + i$	(c) $z = -3 - 3i$	(e) $z = 2i$	(g) $z = -4$
(b) $z = -1 + 2i$	(d) $z = 3 - 4i$	(f) $z = 3$	(h) $z = -5i$

Solution: This is a simple yet extremely useful exercise, as it makes the association between complex numbers and the corresponding points more concrete.

(a)

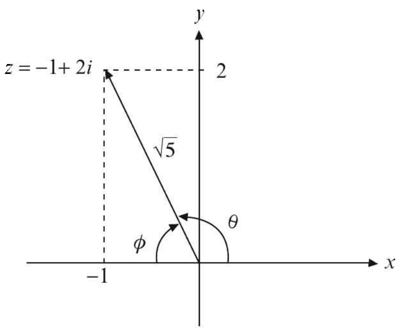


$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = 45° \text{ or } \frac{\pi}{4}$$

$$\therefore z = 1 + i \equiv \boxed{\left(\sqrt{2}, \frac{\pi}{4}\right)}$$

(b)



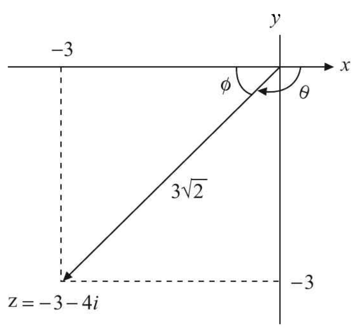
$$|z| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

$$\phi = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}2$$

$$\therefore \arg(z) = \theta = \pi - \tan^{-1}2$$

$$\therefore z = -1 + 2i \equiv \boxed{(\sqrt{5}, \pi - \tan^{-1}2)}$$

(c)



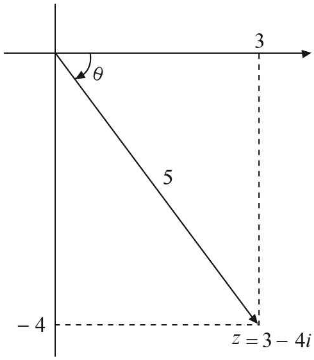
$$|z| = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$$

$$\phi = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$$

$$\therefore \arg(z) = -\theta = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

$$\therefore z = -3 - 3i \equiv \boxed{\left(3\sqrt{2}, \frac{-3\pi}{4}\right)}$$

(d)

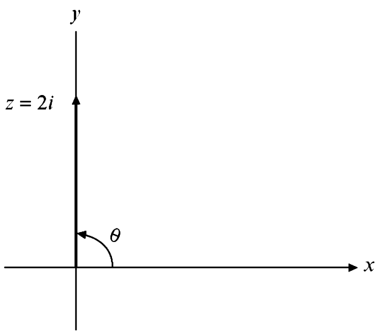


$|z| = \sqrt{(3)^2 + (-4)^2} = 5$

$\arg(z) = -\theta = -\tan^{-1}\dfrac{4}{3}$

$\therefore z = 3 - 4i \equiv \boxed{\left(5, -\tan^{-1}\dfrac{4}{3}\right)}$

(e)

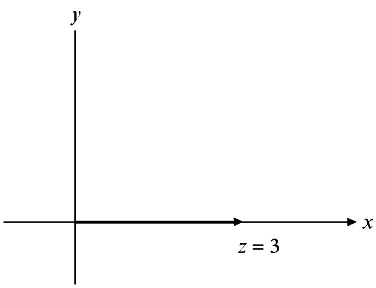


$|z| = 2$ $\left(\begin{array}{l}\text{distance of } z\\ \text{from the origin}\end{array}\right)$

$\arg(z) = \theta = \dfrac{\pi}{2}$

$\therefore z = 2i \equiv \boxed{\left(2, \dfrac{\pi}{2}\right)}$

(f)

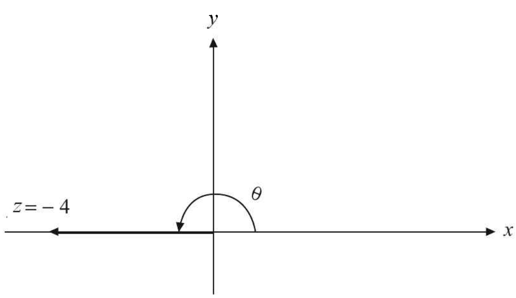


$|z| = 3$

$\arg(z) = 0$

$\therefore z = 3 \equiv \boxed{(3,\ 0)}$

(g)

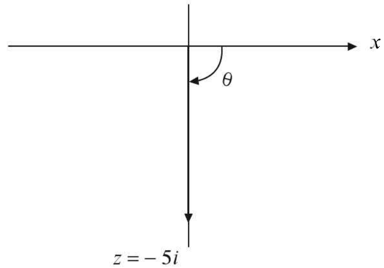


$|z| = 4$

$\arg(z) = \theta = \pi$

$\therefore z = -4 \equiv \boxed{(4, \pi)}$

(h)



$|z| = 5$

$\arg(z) = \theta = \dfrac{-\pi}{2}$

$\therefore z = -5i \equiv \boxed{\left(5, \dfrac{-\pi}{2}\right)}$

Parts (*f*) and (*g*) above were included particularly so that you develop a tendency of thinking of even purely real numbers as points on the plane, and realise the fact that the real set $\mathbb{R}$ is just a subset of $\mathbb{C}$. ■

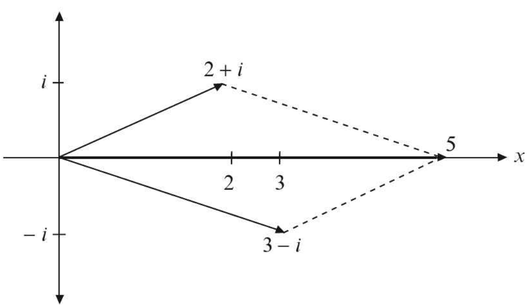
Example-17

Carry out the following operations graphically:

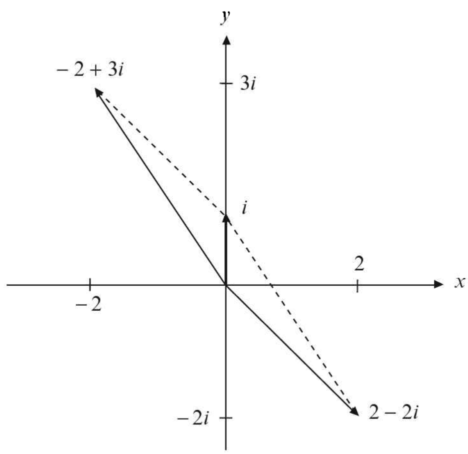
(a) $(2+i)+(3-i)$ (b) $(2-2i)+(-2+3i)$ (c) $(3-i)-(2-i)$ (d) $(3)-(5+2i)$

Solution: Observe carefully, how the parallelogram law is applied to each of the four parts:

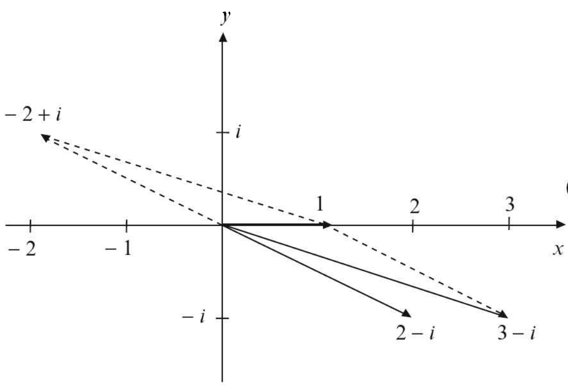
(a)



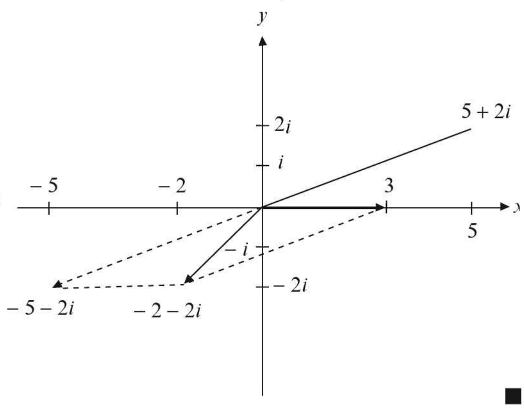
(b)



(c)



(d)



■

Example 18

Show the following operations graphically:

(a) $(1+i)\times(1-i)$

(b) $(2)\times(3+4i)$

(c) $\left(\dfrac{3}{13}-\dfrac{4}{13}i\right)\times\left(1+\dfrac{12}{5}i\right)$

(d) $\dfrac{2+i}{2-i}$

(e) $\dfrac{3+4i}{2}$

(f) $\dfrac{-4+3i}{3+4i}$

Solution: In each part, we represent the two complex numbers on whom we are applying the multiplication or division operation by z_1 and z_2, and the resulting complex number by z:

(a) $|z_1|=|1+i|=\sqrt{2}$

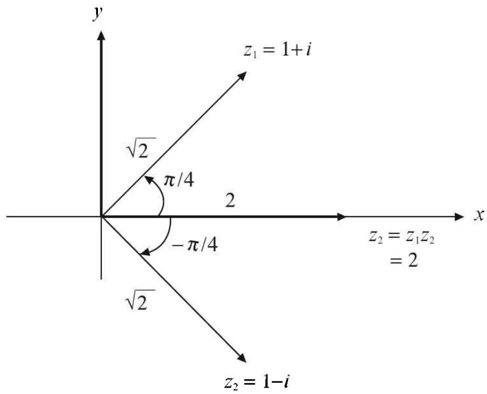
$\arg(z_1)=\dfrac{\pi}{4}$

$|z_2|=|1-i|=\sqrt{2}$

$\arg(z_2)=-\dfrac{\pi}{4}$

$\Rightarrow\quad |z|=|z_1|\,|z_2|=2$

$\arg(z)=\arg(z_1)+\arg(z_2)=0$



(b) $|z_1|=|2|=2$

$\arg(z_1)=0$

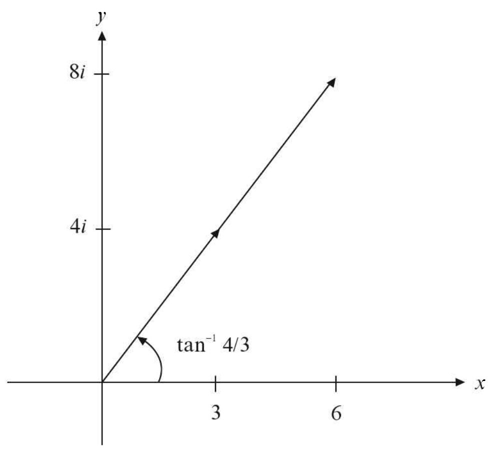
$|z_2|=|3+4i|=5$

$\arg(z_2)=\tan^{-1}\dfrac{4}{3}$

$\Rightarrow\quad |z|=|z_1||z_2|=10$

$\arg(z)=\arg(z_1)+\arg(z_i)$

$=\tan^{-1}\dfrac{4}{3}$



Observe how a complex number (a 'vector') when multiplied by a purely real number (a 'scalar') retains its direction; only its length gets modified.

(c) $|z_1| = \left|\frac{3}{13} - \frac{4i}{13}\right| = \frac{5}{13}$

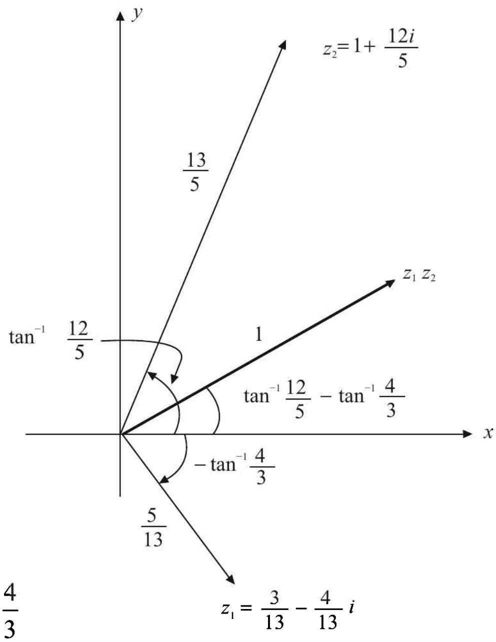
$\arg(z_1) = \arg\left(\frac{3}{13} - \frac{4i}{13}\right) = -\tan^{-1}\left(\frac{4}{3}\right)$

$|z_2| = \left|1 + \frac{12}{5}i\right| = \frac{13}{5}$

$\arg(z_2) = \arg\left(1 + \frac{12i}{5}\right) = \tan^{-1}\left(\frac{12}{5}\right)$

$\Rightarrow |z| = |z_1||z_2| = 1$

$\arg(z) = \arg(z_1) + \arg(z_2) = \tan^{-1}\frac{12}{5} - \tan^{-1}\frac{4}{3}$



(d) $|z_1| = |2 + i| = \sqrt{5}$

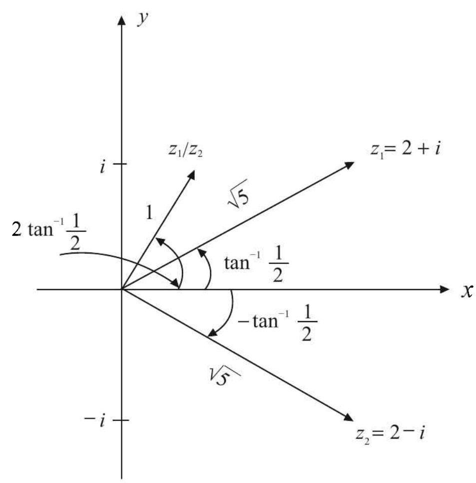
$\arg(z_1) = \tan^{-1}\frac{1}{2}$

$|z_2| = |2 - i| = \sqrt{5}$

$\arg(z_2) = -\tan^{-1}\frac{1}{2}$

$\Rightarrow |z| = \frac{|z_1|}{|z_2|} = 1$

$\arg(z) = \arg(z_1) - \arg(z_2) = 2\tan^{-1}\frac{1}{2}$



(e) $|z_1| = |3+4i| = 5$

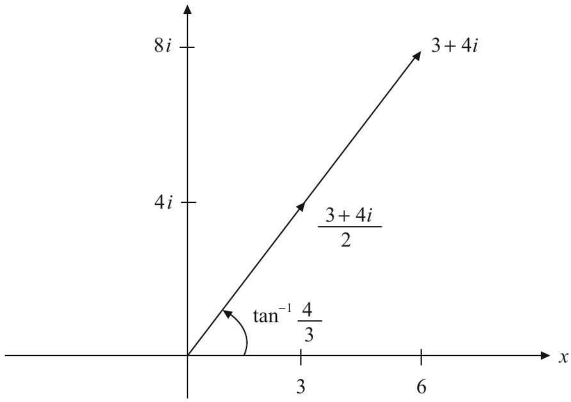
$$\arg(z_1) = \tan^{-1}\frac{4}{3}$$

$$|z_2| = 2$$

$$\arg(z_2) = 0$$

$$\Rightarrow |z| = \frac{|z_1|}{|z_2|} = \frac{5}{2}$$

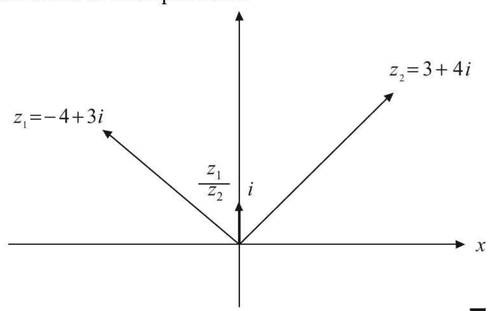
$$\arg(z) = \arg(z_1) - \arg(z_2) = \tan^{-1}\frac{4}{3}$$



As in multiplication by a real number, notice that division by a real number also causes just a change in the magnitude of the vector without a change in its direction. This fact is trivial, since division is but another form of multiplication.

(f) Notice that $(-4+3i) = i(3+4i)$ so that

$$\frac{-4+3i}{3+4i} = i$$



■

Example 19

If z_1 and z_2 are complex numbers such that $|z_1| < 1 < |z_2|$, prove that

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$$

Solution: We can equivalently show that $|1 - z_1\bar{z}_2| < |z_1 - z_2|$ or $|1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$. This is convenient because we know how to expand $|z|^2\ (= z\bar{z})$:

$$\Rightarrow \quad |1 - z_1\bar{z}_2|^2 - |z_1 - z_2|^2$$

$$= \{1 + |z_1|^2|z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)\} - \{|z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)\}$$

$$= 1 + |z_1|^2|z_2|^2 - |z_1|^2 - |z_2|^2$$

$$= (1 - |z_1|^2)(1 - |z_2|^2) < 0 \quad (\text{because } |z_1| < 1 < |z_2|)$$

$$\Rightarrow \quad |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$$

Hence, we get the desired result. ■

Example 20

If $|z|=1$, show that $\dfrac{i(1-z)}{1+z}=\tan\left(\dfrac{\arg(z)}{2}\right)$.

Solution: We will solve this using both an analytical and a geometrical approach.

Analytical Approach:

Let $z=re^{i\theta}=e^{i\theta}$ (since $r=1$)

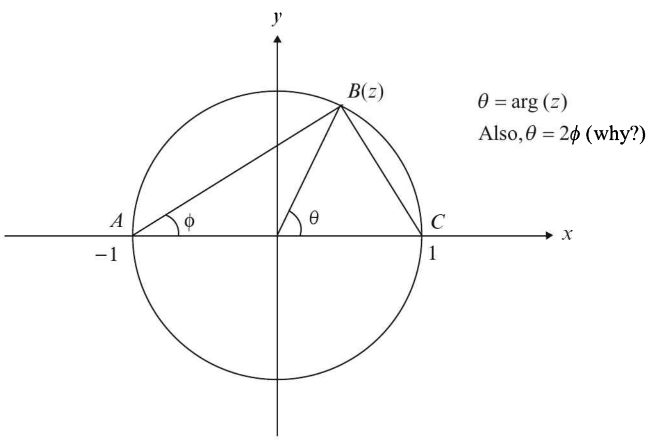
$$\Rightarrow \quad \frac{i(1-z)}{1+z}=\frac{i(1-e^{i\theta})}{1+e^{i\theta}}=\frac{i(1-\cos\theta-i\sin\theta)}{1+\cos\theta+i\sin\theta}$$

$$=\frac{\sin\theta+i(1-\cos\theta)}{1+\cos\theta+i\sin\theta}=\frac{2\sin\theta}{2+2\cos\theta} \quad \text{(how?)}$$

$$=\tan\frac{\theta}{2}=\tan\left(\frac{\arg(z)}{2}\right)$$

Geometrical Approach:

Let z lie anywhere on a unit circle centred at the origin:



Applying rotation $(AB \to BC)$,

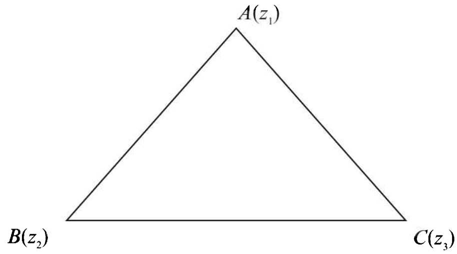
$$\frac{z-1}{z+1}=\frac{BC}{AB}e^{i\pi/2}=\frac{BC}{AB}\,i=i\tan\phi \quad \Rightarrow \quad \tan\phi=\frac{1}{i}\left(\frac{z-1}{z+1}\right)=\frac{i(1-z)}{1+z}$$

Since $\phi=\frac{\theta}{2}$, we get $\tan\frac{\theta}{2}=\frac{i(1-z)}{1+z}$ which is the desired result. ■

Example 21

Show that if $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$, z_1, z_2 and z_3 represent the vertices of an equilateral triangle.

Solution: Let z_1, z_2 and z_3 represent the vertices A, B and C of the triangle ABC. We need to show that ABC is equilateral.



We multiply the given relation by $z_3 - z_1$ on both sides to obtain:

$$\underbrace{\frac{z_3-z_1}{z_1-z_2}+\frac{z_3-z_1}{z_2-z_3}}_{\text{combine}}+1=0 \quad \Rightarrow \quad \frac{z_3-z_1}{z_1-z_2}+\frac{z_2-z_1}{z_2-z_3}=0 \quad \Rightarrow \quad \frac{z_3-z_1}{z_2-z_1}=\frac{z_1-z_2}{z_3-z_2}$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z_3-z_1}{z_2-z_1}\right)=\text{Arg}\left(\frac{z_1-z_2}{z_3-z_2}\right) \quad \Rightarrow \quad \angle A=\angle B$$

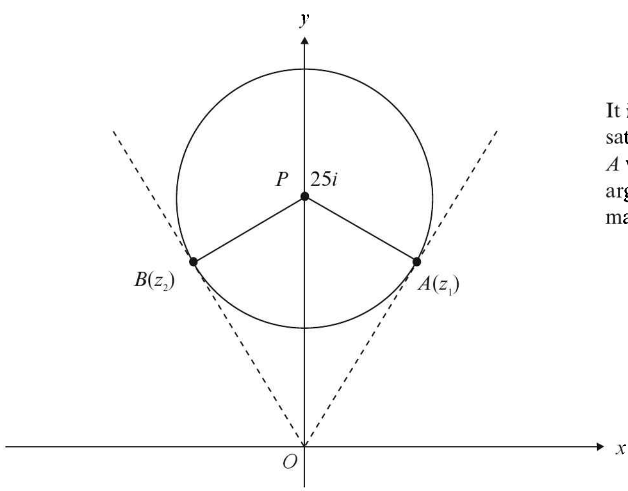
Similarly, we can prove that $\angle B=\angle C$ and therefore, $\angle A=\angle B=\angle C$. Thus, ABC is equilateral.

■

Example 22

If $|z-25i|\le 15$, find $\max(\arg(z))$ and $\min(\arg(z))$.

Solution: From the given relation, it is clear that z must lie inside (or on) a circle of radius 15 centred at $25i$. To obtain max(arg(z)) and min(arg(z)), what we can do is draw two tangents to the circle from the origin:



It is clear that of all the points satisfying the given relation, A will have the minimum argument while B will have the maximum argument.

Now, $OP = 25, \qquad AP = 15$

$$\Rightarrow \qquad \angle POA = \sin^{-1}\left(\frac{15}{25}\right) = \sin^{-1}\left(\frac{3}{5}\right) = \angle POB$$

Therefore, $$\arg(z_1) = \frac{\pi}{2} - \sin^{-1}\frac{3}{5}$$

and $$\arg(z_2) = \frac{\pi}{2} + \sin^{-1}\frac{3}{5}$$

These are the minimum and maximum values respectively. ■

Example 23

Find the largest and the smallest value of $|z|$ if z satisfies $|z + \frac{1}{z}| = a$.

Solution: We apply the triangle inequality on $|z + \frac{1}{z}|$:

$$a = \left|z + \frac{1}{z}\right| \geq \left||z| - \frac{1}{|z|}\right| \quad \Rightarrow \quad \overbrace{-a \leq |z|}^{\text{Ineq A}} - \underbrace{\frac{1}{|z|} \leq a}_{\text{Ineq B}}$$

Ineq A: $$|z|^2 + a|z| - 1 \geq 0$$

$\Rightarrow$ The roots of the quadratic expression are $\dfrac{-a \pm \sqrt{a^2+1}}{4}$.

$\Rightarrow$ Since $|z| > 0$, we have the solution as

$$|z| \geq \frac{-a + \sqrt{a^2+1}}{4}$$

Ineq B: $$|z|^2 - |z| - 1 \leq 0$$

$\Rightarrow$ The roots of this quadratic expression are $\dfrac{a \pm \sqrt{a^2+1}}{4}$.

$$\Rightarrow \qquad |z| \leq \frac{a + \sqrt{a^2+1}}{4}$$

Thus,

$$\frac{-a + \sqrt{a^2+1}}{4} \leq |z| \leq \frac{a + \sqrt{a^2+1}}{4}$$

These are the required largest and smallest values, between which $|z|$ can lie. ■

Example 24

If n is a positive integer, prove that $|\text{Im}(z^n)| \le n\,|\text{Im}(z)||z|^{n-1}$.

Solution:

$$\text{Im}(z^n) = \frac{z^n - \overline{z^n}}{2i} = \frac{z^n - \bar{z}^n}{2i}$$

Similarly,

$$\text{Im}(z) = \frac{z - \bar{z}}{2i}$$

$$\Rightarrow \quad \left|\frac{\text{Im}(z^n)}{\text{Im}(z)}\right| = \left|\frac{z^n - \bar{z}^n}{z - \bar{z}}\right| = |\bar{z}|^{n-1}\left|\frac{\left(\frac{z}{\bar{z}}\right)^n - 1}{\left(\frac{z}{\bar{z}}\right) - 1}\right|$$

$$= |\bar{z}|^{n-1}|\alpha^{n-1} + \alpha^{n-2} + \ldots 1| \qquad \left(\text{we let } \alpha = \frac{z}{\bar{z}}\right)$$

$$\le |\bar{z}^{\,n-1}(|\alpha|^{n-1} + |\alpha^{n-2}| + \cdots + 1)$$

$$= |z|^{n-1}\cdot n \qquad (|\alpha| = 1)$$

$$\Rightarrow \quad |\text{Im}(z^n)| \le n\,|\text{Im}(z)||z|^{n-1}$$

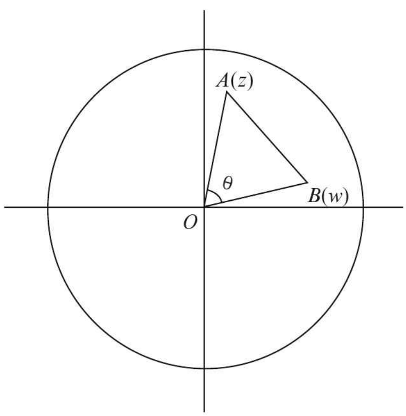
■

Example 25

If $|z| \le 1$ and $|w| \le 1$, show that

(i) $|z - w|^2 \le (|z| - |w|)^2 + (\arg(z) - \arg(w))^2$ (ii) $|z + w|^2 \ge (|z| + |w|)^2 - (\arg(z) - \arg(w))^2$

Solution: Assume some arbitrary values for z and w, and plot them on the plane (both z and w will lie inside the unit circle as shown below):



Note that:

$OA = |\,z\,|$
$OB = |\,w\,|$
$AB = |\,z - w\,|$
$\angle AOB = \theta = \arg(z) - \arg(w)$

Applying the cosine rule on triangle OAB above, we obtain:

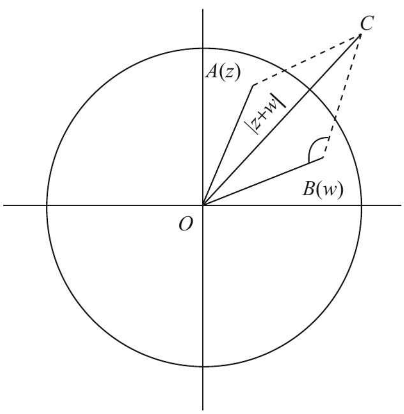
$$AB^2 = OA^2 + OB^2 - 2OA\cdot OB\cdot\cos\theta$$

$$|z-w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta = (|z|-|w|)^2 + 2|z||w| - 2|z||w|\cos\theta$$

$$= (|z|-|w|)^2 + 2|z||w|(1-\cos\theta) = (|z|-|w|)^2 + 4|z||w|\sin^2\frac{\theta}{2}$$

$$\le \left(|z|-|w|\right)^2 + \theta^2 \quad \left(\begin{array}{l}\text{because } |z|\le 1,\ \ |w|\le 1 \text{ and}\\ \sin\frac{\theta}{2}\le\frac{\theta}{2} \text{ so that} \sin^2\frac{\theta}{2}\le\frac{\theta^2}{4}\end{array}\right)$$

This proves the first part. To prove the second part, we apply the cosine rule again as shown below:



Note that:

$OA = BC = |z|$
$OB = |w|$
$OC = |z+w|$
$\angle AOB = \theta$
$\angle OBC = \pi - \theta$

Applying the cosine rule on triangle OBC, we obtain

$$OC^2 = OB^2 + BC^2 - 2OB\cdot BC\cdot\cos(\pi-\theta)$$

$$\Rightarrow \qquad |z+w|^2 = |z|^2 + |w|^2 + 2|z||w|\cos\theta = (|z|+|w|)^2 - 2|z||w|(1-\cos\theta)$$

$$= \left(|z|+|w|\right)^2 - 4|z||w|\sin^2\frac{\theta}{2} \quad \left(\begin{array}{l}\text{because, as in the previous}\\ \text{part}, 4|z||w|\sin^2\frac{\theta}{2}\le\theta^2\end{array}\right)$$

$$\ge (|z|+|w|)^2 - \theta^2$$

■

Example 26

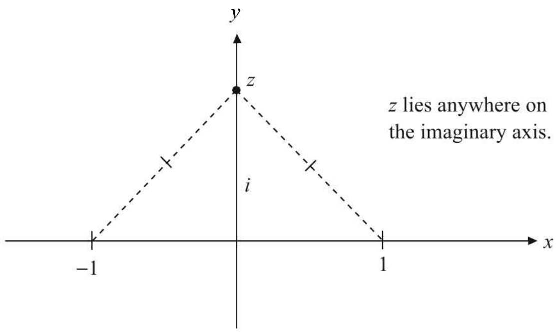
Plot the region/locus represented by z if z satisfies:

(a) $|z-1| = |z+1|$ **(b)** $|z-1+i| = |z+1-i|$ **(c)** $2 < \mathrm{Re}(z) < 3;\ 2 < \mathrm{Im}(z) < 3$ **(d)** $|z-i| + |z+i| = 3$

Solution: **(a)** As far as possible, we must look for geometrical interpretations of equations involving complex numbers. This equation, in particular, says that the distance of z from 1 (you must learn to view every number as a point on the complex plane; for example, 1 is a point which lies on the real axis) must be equal to the distance of z from -1, because

$$|z-1|=|z-(-1)|$$

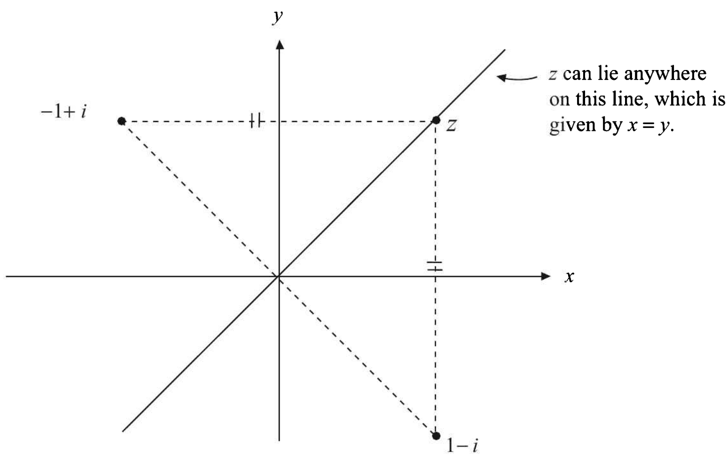
From plane geometry, z must lie on the perpendicular bisector of 1 and -1, or equivalently z must lie anywhere on the imaginary axis.



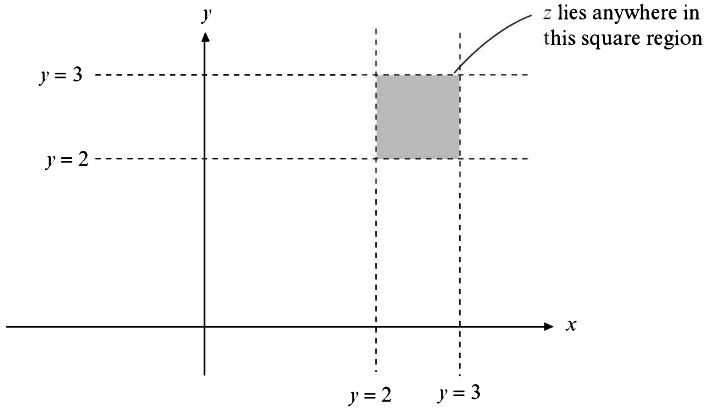
(b) As in part (a), this equation, which can be written as

$$|z-(1-i)|=|z-(-1+i)|$$

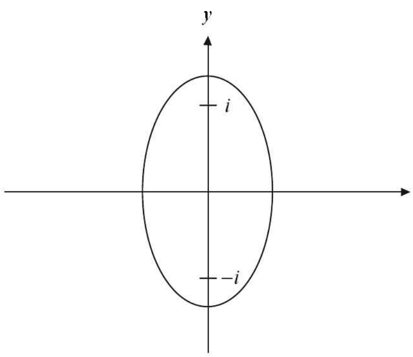
means that z is equidistant from $1-i$ and $-1+i$, *i.e.*, z lies on the perpendicular bisector of these two points:



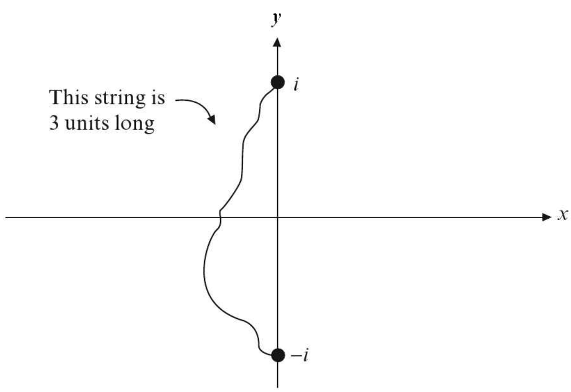
(c) Since $2 < \mathrm{Re}(z) < 3$, z lies anywhere in the region between the vertical lines $x = 2$ and $x = 3$. Also, since $2 < \mathrm{Im}(z) < 3$, z must also lie in the region between the horizontal lines $y = 2$ and $y = 3$.



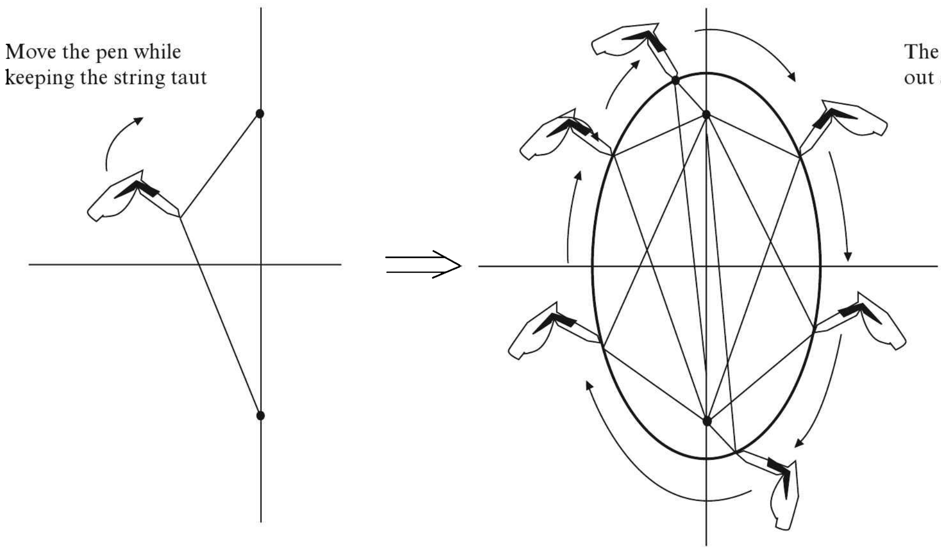
(d) To solve this part, a little knowledge of co-ordinate geometry would be helpful. The given equation says that the sum of the distances of z from i and from $-i$ must equal 3. z would therefore trace out an elliptical path in the plane with i and $-i$ as its two foci, as shown in the figure below:



For those not conversant with co-ordinate geometry, here's a rough explanation. Suppose that you fix two pegs at the points i and $-i$ and tie a 3-unit long string between the two pegs.



Now, with a pen, pull this string "away" from the pegs so that it becomes taut, and then, keeping the string taut, trace out a complete revolution on the plane with the tip of the pen (the taut string will automatically guide the pen):



■

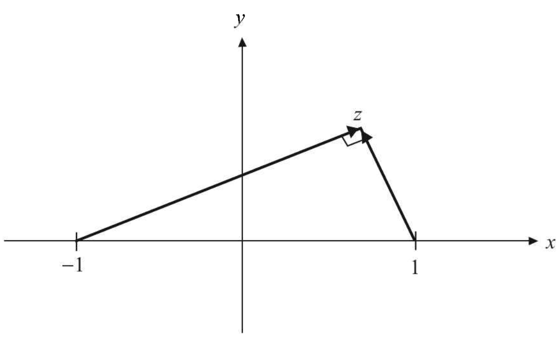
Example 27

Plot the locus of z if z satisfies $\left|\arg\left(\frac{z-1}{z+1}\right)\right| = \frac{\pi}{2}$.

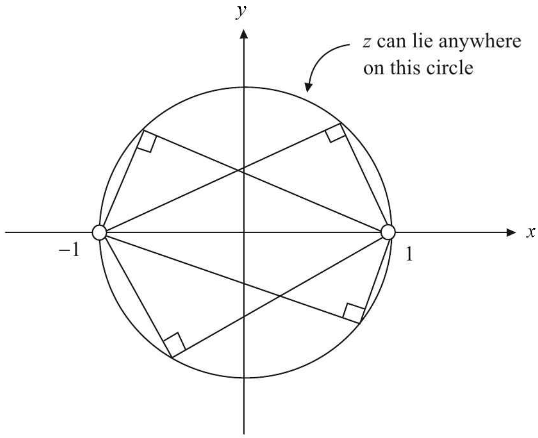
Solution: We have,

$$\arg\left(\frac{z-1}{z+1}\right) = \pm\frac{\pi}{2} \quad \Rightarrow \quad \arg(z-1) - \arg(z+1) = \pm\frac{\pi}{2} \tag{1}$$

$z-1$ is the vector drawn from the point 1 to the point z. Similarly, $z+1$ is the vector drawn from -1 to z. The angle between these two vectors, as (1) tells us, is $\pm\frac{\pi}{2}$.



Since the angle in a semicircle is a right angle, z can lie anywhere on a circle with 1 and –1 as the end-points of a diameter. 1 and –1 themselves cannot lie on this circle because either z – 1 or z + 1 becomes a zero vector if z = 1 or –1, and the argument of a zero vector cannot be uniquely defined.



■

Example 28

Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$.

Solution: We let $z = x + iy$. Using the given relation, we get,

$$x - iy = i(x+iy)^2 = i(x^2 - y^2 + 2ixy) = -2xy + i(x^2 - y^2).$$

Comparing the real and imaginary parts, we get,

$$-2xy = x \qquad \text{and} \qquad x^2 - y^2 = -y.$$

From the first equation, we get,

$$x(1+2y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = -1/2$$

Now we use these two values in the second equation:

$$x = 0 \quad \Rightarrow \quad 5y = 0 \text{ or } 1$$
$$y = -1/2 \quad \Rightarrow \quad x = \pm\sqrt{3}/2.$$

Thus, we get the following solutions for x and y:

$$(0,\ 0),\ (0,\ 1),\ \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right), \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$$

Since we want a non-zero solution, we neglect the solution (0, 0). The valid solutions are: $i,\ \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i.$ ■

Example 29

If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$ and $\cos\gamma + 2\cos\beta + 3\cos\gamma = 0$, find:

(a) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma$

(b) $\sin(\beta+\gamma) + 2\sin(\alpha+\gamma) + 3\sin(\alpha+\beta)$

Solution: Construct three complex number z_1, z_2 and z_3, such that

$$z_1 = \cos\alpha + i\sin\alpha; \quad z_2 = \cos\beta + i\sin\beta; \quad z_3 = \cos\gamma + i\sin\gamma$$

Note that $|z_1| = |z_2| = |z_3| = 1$. Also, from the given relations,

$$z_1 + 2z_2 + 3z_3 = 0 \qquad (1)$$

and $$\frac{1}{z_1} + \frac{2}{z_2} + \frac{3}{z_3} = 0 \qquad (2) \qquad \left(\text{because } \bar{z}_i = \frac{1}{z_i};\ i = 1, 2, 3\right)$$

(a) From (1), since $z_1 + 2z_2 + 3z_3 = 0$, we have

$$z_1^3 + 8z_2^3 + 27z_3^3 = 3 \cdot z_1 \cdot 2z_2 \cdot 3z_3$$

$$\Rightarrow \quad e^{i3\alpha} + 8e^{i3\beta} + 27e^{i3\gamma} = 18e^{i(\alpha+\beta+\gamma)}$$

$$\Rightarrow \quad \cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$$

(By comparing the real parts)

(b) From (2),

$$z_2 z_3 + 2z_1 z_3 + 3z_1 z_2 = 0$$

$$\Rightarrow \quad e^{i(\beta+\gamma)} + 2e^{i(\alpha+\gamma)} + 3e^{i(\alpha+\beta)} = 0.$$

Comparing the imaginary parts on both sides,

$$\sin(\beta+\gamma) + 2\sin(\alpha+\gamma) + 3\sin(\alpha+\beta) = 0.$$

■

Example 30

Let ω, ω^2 be the complex cube roots of unity. Let z_1, z_2, z_3 be complex numbers such that

$$z_1 + z_2 + z_3 = A$$

$$z_1 + \omega z_2 + \omega^2 z_3 = B$$

$$z_1 + \omega^2 z_2 + \omega z_3 = C.$$

Evaluate z_1, z_2 and z_3 in terms of A, B, and C.

Solution: We know that $1 + \omega + \omega^2 = 0$. We have to use this to somehow express each of z_1, z_2 and z_3 independently in terms of A, B and C. Label the three equations as (I), (II) and (III). (I) + (II) + (III) gives

$$3z_1 = A + B + C \quad \Rightarrow \quad z_1 = \frac{A+B+C}{3}$$

(I) + ω^2(II) + ω(III) gives

$$3z_2 = A + \omega^2 B + \omega C \quad \Rightarrow \quad z_2 = \frac{A + B\omega^2 + C\omega}{3}.$$

Finally, (I) + ω(II) + ω^2(III) gives

$$3z_3 = A + B\omega + C\omega^2 \quad \Rightarrow \quad z_3 = \frac{A + B\omega + C\omega^2}{3}.$$

■

Example 31

Let a complex number α, $\alpha \neq 1$ be a root of $z^{p+q} - z^p - z^q + 1 = 0$, where p and q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0$, or $1 + \alpha + \alpha^2 + \cdots + \alpha^{q-1} = 0$, but not both together.

Solution: The given equation can be written as

$$(z^p - 1)(z^q - 1) = 0$$

Therefore, α is a root of either $z^p - 1 = 0$ or $z^q - 1 = 0$. In other words, α is either a pth or a qth root of unity. Now,

$$z^p - 1 = (z - 1)(z^{p-1} + z^{p-2} + \cdots + z + 1).$$

Substituting $z = \alpha$ gives

$$0 = (\alpha - 1)(\alpha^{p-1} + \alpha^{p-2} + \cdots + \alpha + 1).$$

Since $\alpha \neq 1$, we get

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0. \tag{1}$$

Similarly, if $z^q - 1 = 0$, we get

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{q-1} = 0. \tag{2}$$

We now need to show that (1) and (2) cannot hold simultaneously. In other words, α cannot be a pth and a qth root of unity at the same time (given the condition that p and q are distinct primes). This is easy to prove. If α is a pth root of unity, we have:

$$\alpha = e^{i\frac{2m\pi}{p}}. \tag{3}$$

If α is a qth root of unity,

$$\alpha = e^{i\frac{2n\pi}{q}} \tag{4}$$

Observe the right hand sides of (3) and (4) carefully. The terms in the exponents, $\frac{m}{p}$ (where $m < p$) and $\frac{n}{q}$ (where $n < q$) can never be equal. Why? Lets assume they are equal:

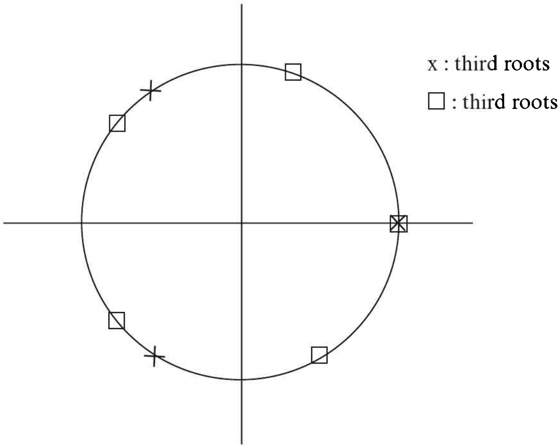
$$\frac{m}{p} = \frac{n}{q} \quad \Rightarrow \quad pn = mq$$

Since p is a prime, different from q, it cannot have m and q as factors. Since $n < q$, n cannot have q as a factor. Thus pn cannot have m or q as factors, which is a contradiction.

$\Rightarrow$ (3) and (4) cannot be equal

$\Rightarrow$ (1) and (2) cannot be simultaneously satisfied.

Here's an example, for $p = 3$ and $q = 5$:



Since 3 and 5 are distinct primes, × and □ do not overlap (except at 1) $\Rightarrow$ A complex number $\alpha \neq 1$ *cannot* be a 3rd and 5th root of unity at the same time. This reasoning can be generalized to p and q.

■

Example 32

Find all the roots of the equation $z^{12} - 56z^6 - 512 = 0$ whose imaginary parts are non-negative.

Solution: We let $z^6 = x$ so that the given equation reduces to a quadratic.

$$x^2 - 56x - 512 = 0 \text{ or } (x-64)(x+8) = 0$$

$$\Rightarrow \quad x = 64, -8 \quad \Rightarrow \quad z^6 = 64, -8$$

(a) $\boxed{z^6 = 64}$

$$z^6 = 64 = 2^6 e^{i(2p\pi+0)} \quad \Rightarrow \quad z = 2e^{i\left(\frac{2p\pi+0}{6}\right)}$$

Verify that for $p = 0, 1, 2, 3$, we get non-negative imaginary parts for z:

$$z = 2, \ 2e^{i\pi/3}, \ 2e^{i2\pi/3}, \ 2e^{i\pi}$$

(b) $\boxed{z^6 = -8}$

$$z^6 = -8 = 2^3 e^{i\pi} = 2^3 e^{i(2p\pi+\pi)}$$

$$\Rightarrow \quad z = 2^{1/2} e^{i\left(\frac{2p\pi+\pi}{6}\right)}.$$

Verify that for $p = 0, 1, 2$, we get non-negative imaginary parts for z:

$$z = \sqrt{2}e^{i\pi/6}, \sqrt{2}e^{i\pi/2}, \sqrt{2}e^{i5\pi/6}.$$

Thus we get seven values of z that satisfy the given condition. ■

Example 33

If both ω and ω^2 (non-real cube roots of unity) satisfy the equation

$$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x}=\frac{2}{x},$$

show that $x = 1$ also satisfies this equation.

Solution: The given equation can be re-written (upon re-arranging into the standard form) as

$$2x^4+(a+b+c+d)x^3-(abc+acd+abd+bcd)x-2abcd=0$$

ω and ω^2 are roots of this equation. Let α and β be the other two roots.

$$\Rightarrow \quad \omega+\omega^2+\alpha+\beta=-\frac{(a+b+c+d)}{2}.$$

Since $\omega+\omega^2=-1$, this reduces to

$$\alpha+\beta=1-\frac{(a+b+c+d)}{2}. \tag{1}$$

Also, the coefficient of x^2 is 0:

$$\Rightarrow \quad \alpha\beta+\alpha\omega+\alpha\omega^2+\beta\omega+\beta\omega^2+\omega^3=0$$

$$\Rightarrow \quad \alpha\beta-\alpha-\beta+1=0$$

$$\Rightarrow \quad (\alpha-1)(\beta-1)=0$$

$$\Rightarrow \quad \alpha=1 \text{ or } \beta=1.$$

If $\alpha=1$, β can be determined from (1), and vice-versa. Thus, x=1 is a root of the given equation. ■

Example 34

Plot the fifth roots of $16(-\sqrt{3}+i)$ on the plane.

Solution: We first write $z=16(-\sqrt{3}+i)$ in its Euler form.

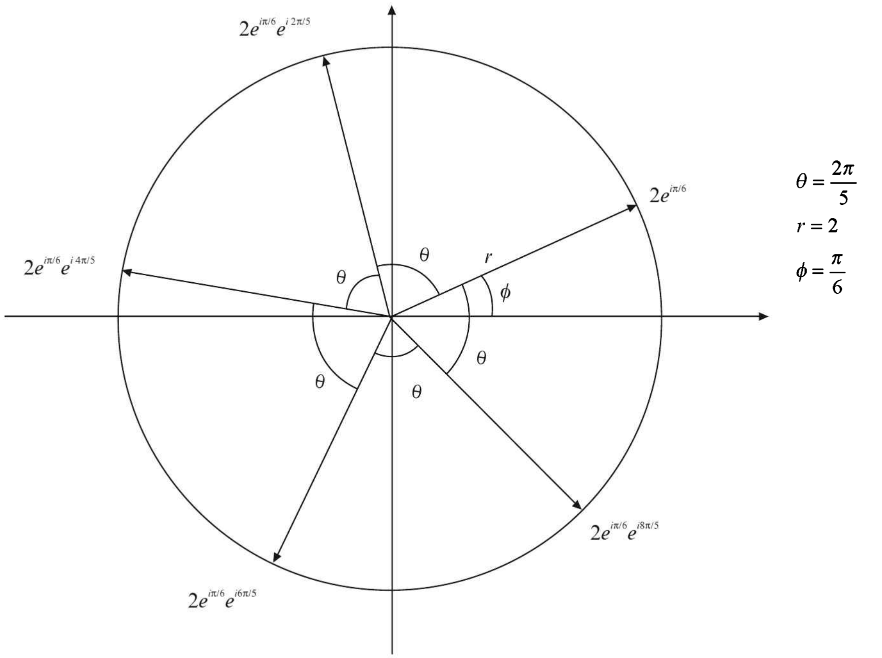
$$z=16(-\sqrt{3}+i)=32e^{i5\pi/6}=32e^{i\left(2p\pi+\frac{5\pi}{6}\right)},p\in\mathbb{Z}$$

$$\Rightarrow \quad z^{1/5}=2e^{i\left(\frac{2p\pi}{5}+\frac{\pi}{6}\right)},p\in\mathbb{Z}$$

To obtain the roots, we let p take on five consecutive integer values, say p = 0, 1, 2, 3, 4. The roots obtained are:

$$2e^{i\pi/6},\,2e^{i\pi/6}e^{i2\pi/5},\,2e^{i\pi/6}e^{i4\pi/5},\,2e^{i\pi/6}e^{i6\pi/5},\,2e^{i\pi/6}e^{i8\pi/5}.$$

These five roots will lie evenly spaced out at angles of 72° between any two consecutive roots. The first root is at an angle of 30°.

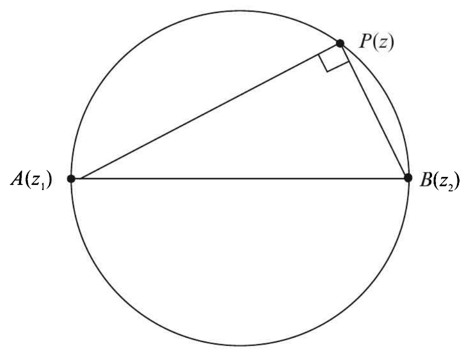


■

Example 35

Let $A(z_1)$ and $B(z_2)$ be arbitrary points in the complex plane. Find the equation of the circle having AB as a diameter.

Solution: Let $P(z)$ be an arbitrary point lying on the required circle as shown in the figure below:



We can now take two approaches:

(i) We know that the angle in a semi-circle is a right angle. Therefore,

$$AP^2 + PB^2 = AB^2$$
$$\Rightarrow \quad |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

This is a possible equation of the required circle. All points lying on the circle will satisfy this equation.

(ii) Applying rotation $(\overrightarrow{PA} \longrightarrow \overrightarrow{PB})$, we get

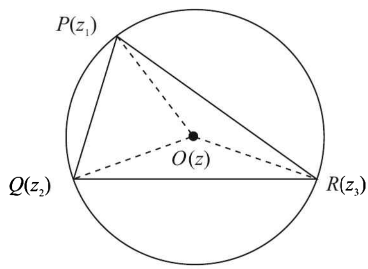
$$\frac{z_2 - z}{z_1 - z} = \frac{|z_2 - z|}{|z_1 - z|} e^{i\pi/2} = ik, \qquad k \in \mathbb{R}$$
$$\Rightarrow \quad \frac{z_2 - z}{z_1 - z} \text{ is purely imaginary}$$
$$\Rightarrow \quad \frac{z - z_2}{z - z_1} + \frac{\overline{z} - \overline{z}_2}{\overline{z} - \overline{z}_1} = 0$$
$$\Rightarrow \quad (z - z_1)(\overline{z} - \overline{z}_2) + (\overline{z} - \overline{z}_1)(z - z_2) = 0$$

This is another possible equation of the circle. ■

Example 36

Find the circumcentre of the triangle whose vertices are given by the complex numbers z_1, z_2 and z_3.

Solution:



We have to find z, the circumcentre O of triangle PQR. By virtue of being the circumcentre, z is equidistant from z_1, z_2 and z_3. Therefore,

$$|z - z_1| = |z - z_2| = |z - z_3|$$
$$\Rightarrow \quad (z - z_1)(\overline{z} - \overline{z}_1) = \underbrace{(z - z_2)(\overline{z} - \overline{z}_2)}_{\text{Equality A}} = \underbrace{(z - z_3)(z - \overline{z}_3)}_{\text{Equality B}}$$

From the first two terms in the equality above (Equality A), we get:

$$z\overline{z} - z\overline{z}_1 - z_1\overline{z} + z_1\overline{z}_1 = z\overline{z} - z\overline{z}_2 - z_2\overline{z} + z_2\overline{z}_2$$
$$\Rightarrow \quad \overline{z}(z_2 - z_1) = z(\overline{z}_1 - \overline{z}_2) + |z_2|^2 - |z_1|^2 \qquad (1)$$

Similarly, from equality B, we get:

$$\bar{z}(z_3 - z_2) = z(\bar{z}_2 - \bar{z}_3) + |z_3|^2 - |z_2|^2. \tag{2}$$

Dividing (1) by (2), we get:

$$\frac{z_2 - z_1}{z_3 - z_2} = \frac{z(\bar{z}_1 - \bar{z}_2) + |z_2|^2 - |z_1|^2}{z(\bar{z}_2 - \bar{z}_3) + |z_3|^2 - |z_2|^2}.$$

Solving for z, we get:

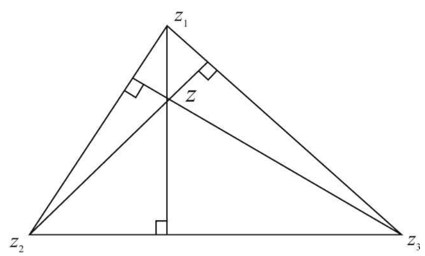
$$z = \frac{|z_1|^2 (z_2 - z_3) + |z_2|^2 (z_3 - z_1) + |z_3|^2 (z_1 - z_2)}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)}.$$

■

Example 37

Find the orthocentre of the triangle whose vertices are z_1, z_2 and z_3.

Solution: Let z be the required orthocentre:



From the figure, it is clear that:

$$\arg\left(\frac{z_1 - z}{z_3 - z_2}\right) = \frac{\pi}{2} \qquad \Rightarrow \qquad \frac{z_1 - z}{z_3 - z_2} \text{ is purely imaginary}$$

$$\Rightarrow \quad \frac{z_1 - z}{z_3 - z_2} + \frac{\bar{z}_1 - \bar{z}}{\bar{z}_3 - \bar{z}_2} = 0 \qquad \Rightarrow \qquad \bar{z} - \bar{z}_1 = \frac{(\bar{z}_3 - \bar{z}_2)(z_1 - z)}{z_3 - z_2} \tag{1}$$

Similarly, $$\arg\left(\frac{z_2 - z}{z_1 - z_3}\right) = \frac{\pi}{2} \qquad \Rightarrow \qquad \bar{z} - \bar{z}_2 = \frac{(\bar{z}_1 - \bar{z}_3)(z_2 - z)}{z_1 - z_3} \tag{2}$$

Subtracting (2) from (1), we obtain

$$\bar{z}_2 - \bar{z}_1 = \frac{(\bar{z}_3 - \bar{z}_2)(z_1 - z)}{z_3 - z_2} - \frac{(\bar{z}_1 - \bar{z}_3)(z_2 - z)}{z_1 - z_3}. \tag{3}$$

A sequence of manipulations to separate z in terms of the other constants in (3) will give

$$z = \frac{|z_1|^2 (z_2 - z_3) + |z_2|^2 (z_3 - z_1) + |z_3|^2 (z_1 - z_2) + z_1(\bar{z}_2 - \bar{z}_3) + z_2(\bar{z}_3 - \bar{z}_1) + z_3(\bar{z}_1 - \bar{z}_2)}{(\bar{z}_1 z_2 - z_1\bar{z}_2) + (\bar{z}_2 z_3 - z_2\bar{z}_3) + (\bar{z}_3 z_1 - z_3\bar{z}_1)}$$

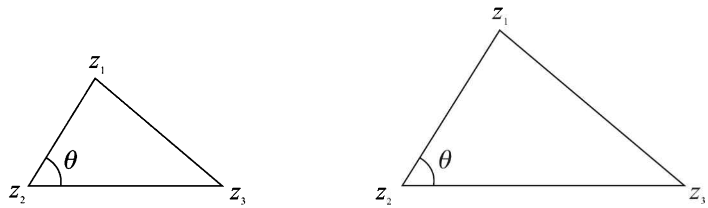
This is the complex representation of the orthocenter of the triangle. The last step is an exercise which the reader must complete. ■

Example 38

Show that the triangles whose vertices are z_1, z_2, z_3 and Z_1, Z_2, Z_3 are directly similar if:

$$\begin{vmatrix} z_1 & Z_1 & 1 \\ z_2 & Z_2 & 1 \\ z_3 & Z_3 & 1 \end{vmatrix} = 0$$

Solution:



Since the triangles are directly similar, the vector $z_1 - z_2$ will be a scalar multiple of $Z_1 - Z_2$; the vector $z_2 - z_3$ will be the (same) scalar multiple of $Z_2 - Z_3$ and so on:

$$z_1 - z_2 = \lambda(Z_1 - Z_2), \qquad z_2 - z_3 = \lambda(Z_2 - Z_3)$$

$$\Rightarrow \quad \frac{z_1 - z_2}{z_2 - z_3} = \frac{Z_1 - Z_2}{Z_2 - Z_3}$$

$$\Rightarrow \quad z_1(Z_2 - Z_3) + z_2(Z_3 - Z_1) + z_3(Z_1 - Z_2) = 0$$

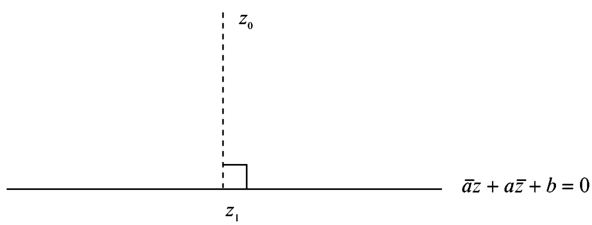
$$\Rightarrow \quad \begin{vmatrix} z_1 & Z_1 & 1 \\ z_2 & Z_2 & 1 \\ z_3 & Z_3 & 1 \end{vmatrix} = 0.$$

■

Example 39

Show that the perpendicular distance of a point z_0 from the line $\bar{a}z + a\bar{z} + b = 0 \ \ (b \in \mathbb{R})$ is $\frac{|a\bar{z}_0 + \bar{a}z_0 + b|}{2|a|}$.

Solution:



Let z_1 be the foot of the perpendicular dropped from z_0 onto the given line. We need to evaluate $|z_0 - z_1|$. Now, since z_1 lies on the given line, we have

$$\bar{a}z_1 + a\bar{z}_1 + b = 0 \tag{1}$$

Also, the complex slopes of the given line and the perpendicular must add to 0:

$$\frac{-a}{\bar{a}}+\frac{z_0-z_1}{\bar{z}_0-\bar{z}_1}=0$$

$$\Rightarrow \quad -a(\bar{z}_0-\bar{z}_1)+\bar{a}(z_0-z_1)=0$$

$$\Rightarrow \quad -a\bar{z}_0+\bar{a}z_0+a\bar{z}_1-\bar{a}z_1=0 \tag{2}$$

From (1) + (2),

$$-a\bar{z}_0+\bar{a}z_0+2a\bar{z}_1+b=0$$

$$\Rightarrow \quad \bar{z}_1=\frac{a\bar{z}_0-\bar{a}z_0-b}{2a}$$

$$\Rightarrow \quad \bar{z}_1-\bar{z}_0=\frac{-a\bar{z}_0-\bar{a}z_0-b}{2a}$$

$$\Rightarrow \quad |z_1-z_0|=\frac{|a\bar{z}_0+\bar{a}z_0+b|}{2\,|a|}$$ ■

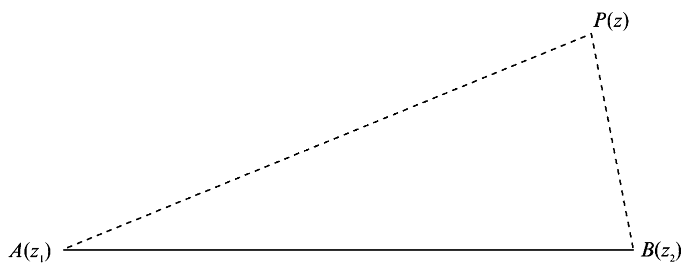
Example 40

Prove that $|\frac{z-z_1}{z-z_2}|=k$ represents a circle if $k\neq 1$ and a line if $k=1$.

Solution: For $k=1$, the proof is trivial. The given equation reduces to

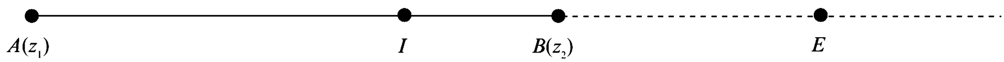
$$|z-z_1|=|z-z_2|$$

which implies that z lies on the perpendicular bisector of the line segment joining z_1 and z_2. Let us now consider the case when $k\neq 1$.



We need to find the locus of a point P which moves such that $\frac{AP}{PB} = k$. Note that one such point also lies on the line segment AB itself which divides AB internally in the ratio $k : 1$. Also, another such point lies on the extended line of the line segment AB, which will externally divide AB in the ratio $k : 1$. If $k > 1$, this external point will lie on the 'right' side of B (in the figure below), and if $k < 1$, this point will lie to the 'left' of A.

Assuming $k > 1$, we let I be the point of internal division and E the point of external division of the segment AB in the ratio $k : 1$.



Thus, $\frac{AI}{IB} = \frac{AE}{EB} = k$. I is given by $\frac{kz_2+z_1}{k+1}$ while E is given by $\frac{kz_2-z_1}{k-1}$. Now, we need the locus of the points z which satisfy

$$|z - z_1| = k\,|z - z_2|$$

$$\Rightarrow \quad |z|^2 + |z_1|^2 - z\bar{z}_1 - \bar{z}z_1 = k^2\,|z|^2 + k^2\,|z_2|^2 - k^2 z\bar{z}_2 - k^2\bar{z}z_2$$

$$\Rightarrow \quad (k^2-1)\,|z|^2 - z(k^2\bar{z}_2 - \bar{z}_1) - \bar{z}(k^2 z_2 - z_1) + k^2\,|z_2|^2 - |z_1|^2 = 0$$

$$\Rightarrow \quad |z|^2 - z\frac{(k^2\bar{z}_2 - \bar{z}_1)}{k^2-1} - \bar{z}\frac{(k^2 z_2 - z_1)}{k^2-1} + \frac{k^2\,|z_2|^2 - |z_1|^2}{k^2-1} = 0$$

$$\Rightarrow \quad |z|^2 - \frac{z(k^2\bar{z}_2 - \bar{z}_1)}{k^2-1} - \frac{\bar{z}(k^2 z_2 - z_1)}{k^2-1} + \frac{|k^2 z_2 - z_1|^2}{(k^2-1)^2} = \frac{|k^2 z_2 - z_1|^2}{(k^2-1)^2} - \frac{k^2\,|z_2|^2 - |z_1|^2}{k^2-1}$$

Introduction of a new term

$$\Rightarrow \quad \left|z - \frac{k^2 z_2 - z_1}{k^2-1}\right|^2 = \frac{k^2(|z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2)}{(k^2-1)^2} = \frac{k^2\,|z_1 - z_2|^2}{(k^2-1)^2}$$

$$\Rightarrow \quad \left|z - \frac{k^2 z_2 - z_1}{k^2-1}\right| = \frac{k\,|z_1 - z_2|}{k^2-1}.$$

This is the equation of a circle with centre at $\frac{k^2 z_2 - z_1}{k^2-1}$ and radius $\frac{k|z_1-z_2|}{k^2-1}$. Note that the mid point of I and E is:

$$\frac{1}{2}\left(\frac{kz_2 + z_1}{k+1} + \frac{kz_2 - z_1}{k-1}\right) = \frac{k^2 z_2 - z_1}{k^2-1}.$$

Thus, the centre of this circle is actually the mid-point of IE and the radius is $\frac{k}{k^2-1}$ times the original line-segment AB. ■

Example 41

If $\theta = \frac{8\pi}{11}$, find the value of $\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta$.

Solution: Consider the eleventh roots of unity, *i.e.*, the solutions to the equation

$$z^{11} - 1 = 0.$$

The roots are $e^{i2k\pi/11}$, $k = 0, 1, ..., 10$. The roots can also be expressed in terms of $e^{i8k\pi/11}$, *i.e.*, if we let $\alpha = e^{i8k\pi/11}$ and take 11 consecutive values of k, we will still be able to list down all the roots Why? Take $\alpha = e^{i2k\pi/11}$ and take 11 consecutive values of k; and then take $\alpha = e^{i8k\pi/11}$ and again take 11 consecutive values of k. You will get the same set of 11 (eleventh roots of unity) values. Make sure you understand this point. Thus, we let $\alpha = e^{i8k\pi/11}$ and let k take the values –5, –4, ..., 4, 5 (eleven consecutive values):

$$\Rightarrow \quad \sum_{k=-5}^{5} e^{i8k\pi/11} = 0 \quad \Rightarrow \quad \sum_{\substack{k=-5 \\ k\neq 0}}^{5} e^{i8k\pi/11} = -1$$

$$\Rightarrow \quad 2(\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta) = -1 \; (\because e^{i\phi} + e^{-i\phi} = 2\cos\phi)$$

Thus, the required sum is $-\frac{1}{2}$. ■

Complex Numbers

PART-C: Advanced Problems

P1. Which of the following relations are true for an arbitrary complex number z?

(A) $\overline{(e^{z})} = e^{\bar{z}}$ (B) $\overline{(\ln z)} = \ln \bar{z}$ (C) $\overline{(\cos z)} = \cos \bar{z}$ (D) $\overline{(z_1^{z_2})} = \bar{z}_1^{\bar{z}_2}$

P2. Let z_1, z_2, z_3 be three complex numbers such that $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$. The value of $|9z_1z_2 + 4z_1z_3 + z_2z_3|$ is:

(A) 3 (B) 4 (C) 5 (D) 6 (E) None of these

P3. The possible values of n for which the curve $\arg\left(\frac{z+1}{z+2}\right) = \pm\frac{\pi}{2}$ and the curve $\operatorname{Im}(z) = n\operatorname{Re}(z)$ intersect is

(A) $\left[-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$ (B) $\left[-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right]$ (C) $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$ (D) $\left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right]$

(E) None of these

P4. Let $z = x + iy$, where x, y are integers. For $n > 2$, what is the number of solutions to the equation $z^2 + \bar{z}^2 = 10^n$?

(A) $2(n^2 - 1)$ (C) $2(n-2)(n+1)$ (E) None of these

(B) $2(n^2 - 4)$ (D) $2(n-1)(n+2)$

P5. Let α_k, $k = 1$ to n, be the nth roots of unity. The value of $\sum_{k=1}^{n} \frac{\alpha_k}{2-\alpha_k}$ is:

(A) $\frac{n}{2^n}$ (B) $\frac{n-1}{2^n - 1}$ (C) $\frac{n}{2^n - 1}$ (D) $\frac{n+1}{2^n - 1}$ (E) None of these

P6. Let $f(x) = \Pi_{j=0}^{n-1}(2\cos 2^j x - 1)$, where $n \geq 1$. Let $k \in \mathbb{Z}$. The value of $f\left(\frac{2\pi k}{2^n \pm 1}\right)$ is:

(A) 0 (B) 1 (C) n (D) 2^n (E) None of these

P7. z_1, z_2 and z_3 are three points on a circle centered at the origin. A point, z, is chosen on the circle such that the line joining z and z_1 is perpendicular to the line joining z_2 and z_3. Which of the following is true?

(A) $zz_1 + z_2z_3 = 0$ (C) $z^2 - z_1^2 + z_2z_3 = 0$ (E) None of these

(B) $z^2 + z_1^2 + z_2z_3 = 0$ (D) $zz_1 - z_2z_3 = 0$

P8. $P(z_1)$ lies on a circle with OP as diameter, where O is the origin. Points $Q(z_2)$ and $R(z_3)$ are taken on the circle such that $\angle POQ = \angle QOR = \theta$. The ratio, $\frac{z_1 z_3}{z_2^2}$, is given by:

(A) $\dfrac{\cos 2\theta}{\cos^2\theta}$ (B) $\dfrac{\sin 2\theta}{\sin^2\theta}$ (C) $\dfrac{\cos^2\theta}{\cos 2\theta}$ (D) $\dfrac{\sin^2\theta}{\sin 2\theta}$ (E) None of these

P9. The number of values of z satisfying $|z-2i|=2$ and $|z-1-3i|+|z+1-3i|=2\sqrt{2}$ is:

(A) 1 (B) 2 (C) 3 (D) 4

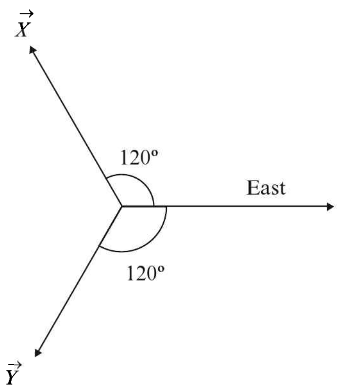
P10. What are the real values of the parameter 'a' for which at least one complex number z satisfies both the equality $|z+\sqrt{2}| = a^2-3a+2$ and the inequality $|z+i\sqrt{2}| < a^2$?

(A) $[-1,\infty)$ (B) $[0,\infty)$ (C) $[1,\infty)$ (D) $[2,\infty)$ (E) None of these

P11. Assume that $A_i\,(i=1,2,\ldots,n)$ are the vertices of a regular polygon inscribed in a circle of radius unity. The value of $\frac{|A_1A_2|^2+|A_1A_3|^2+\ldots+|A_1A_n|^2}{n}$ is:

(A) $\dfrac{1}{2}$ (B) 1 (C) 2 (D) 4

P12. From the origin, a person moves 1 unit east, then $\frac{1}{2}$ unit parallel to $\vec{X}$, then $\frac{1}{2^2}$ unit parallel to $\vec{Y}$, then $\frac{1}{2^3}$ unit east again, and so on.



The distance of the person from the origin after time t, where t tends to infinity, will be

(A) $\dfrac{2}{\sqrt{5}}$ (B) $\dfrac{3}{\sqrt{5}}$ (C) $\dfrac{2}{\sqrt{7}}$ (D) $\dfrac{3}{\sqrt{7}}$ (E) None of these

SUBJECTIVE TYPE EXAMPLES

P13. Can there be a complex number z, such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$?

P14. If $11z^{10} + 10iz^9 + 10iz - 11 = 0$, find the possible values of $|z|$.

P15. Three points represented by complex numbers a, b, c lie on a circle with centre as the origin and radius r. The tangent at C cuts the chord joining the points a, b at z. Express z in terms of a, b, c.

P16. If the points representing complex numbers $z_1, z_2, ..., z_n$ in the Argand plane lie on one side of a line drawn through the origin, then prove that the same is true for the points representing complex numbers $\frac{1}{z_1}, \frac{1}{z_2}, \cdots, \frac{1}{z_n}$. Also, which of the following is true?

(a) $z_1 + z_2 + \cdots + z_n \neq 0$ (b) $\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n} \neq 0$

P17. Let z_1, z_2, z_3 be three complex numbers such that $|z_1| = a$, $|z_2| = b$, $|z_3| = c$ and $z_1 \neq z_2 \neq z_3$. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then:

(a) Prove that z_1, z_2, z_3 lie on a circle with the centre at the origin.
(b) Find the relation between $\arg(\frac{z_3}{z_2})$ and $\arg(\frac{z_3 - z_1}{z_2 - z_1})$.

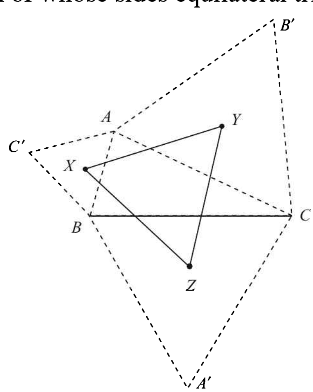
P18. Find all possible values of $a \in \mathbb{R}$ if there exists at least one z, which satisfies $|z| = 3$, $|z - (a(1+i) - i)| \leq 3$ and $|z + 2a - (a+1)i| > 3$ simultaneously.

P19. (a) A particle is moving initially (in the Argand plane) on a circle centered at the origin with unit radius. At $t = 0$, its position is $P(1 + 0i)$. At $t = 0$, the center of the particle's movement is abruptly shifted to $\frac{1}{2} + i0$. From now onwards, whenever the particle rotates anti-clockwise through angle θ, the center of its movement is shifted abruptly to the mid-point of the segment joining the particle to the previous center of its movement. Where will the particle converge at $t = \infty$?
(b) Where will the particle converge if the direction of rotation is alternated, *i.e.*, the particle rotates first through angle θ, then through $-\theta$, and so on, everything else being the same as in Part (a)?

P20. The radius of a circle which circumscribes a regular n-sided polygon with vertices $A_1, A_2, ..., A_n$ is R.

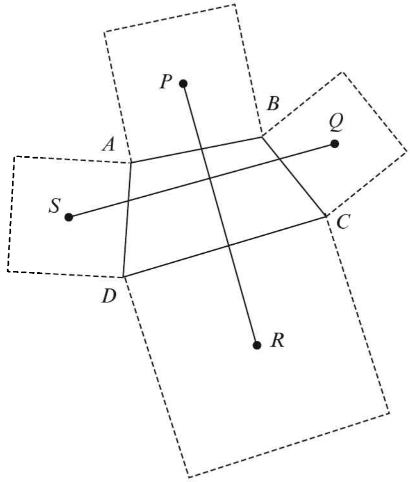
(a) Find the sum of the squares of all the sides and all the diagonals.
(b) Find the sum of all the sides and all the diagonals.
(c) Find the product of all the sides and all the diagonals.

P21. Consider a triangle ABC, on each of whose sides equilateral triangles are drawn.



Prove that the centroids X, Y, Z of the three equilateral triangles themselves form an equilateral triangle.

P22. On each side of a quadrilateral ABCD, squares are drawn. The centers of the opposite squares are joined.



Show that PR and QS are equal in length and perpendicular to one another.

P23. Let M be a point on the circle circumscribing a regular n-sided polygon with vertices $A_1, A_2, \ldots, A_n$. Prove the following:

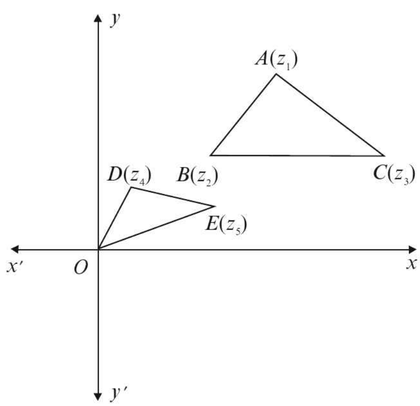
(a) If n is even,

$$|MA_1|^2 + |MA_3|^2 + \cdots = |MA_2|^2 + |MA_4|^2 + \cdots$$

(b) If n is odd,

$$|MA_1| + |MA_3| + \cdots = |MA_2| + |MA_4| + \cdots$$

P24. Let A, B, C, D and E be points on the complex plane representing the complex numbers z_1, z_2, z_3, z_4 and z_5 respectively. If $(z_3 - z_2)z_4 = (z_1 - z_2)z_5$, can we say that the triangles ABC and DOE are similar?



P25. In the Argand plane, z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with $AC = BC$ and equal angle θ. If z_4 is the incentre of the triangle, determine the values of the following in terms of θ.

(a) $\dfrac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (b) $\dfrac{(z_2 - z_1)^2}{(z_1 + z_2 - 2z_3)(z_1 + 2z_4)}$

P26. If $z_1, z_2, z_3, z_1', z_2'$ and z_3' are complex numbers such that $(z_1 - z_2)(z_1' - z_2') = (z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1')$, then can we say that the triangles whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' are equilateral?

P27. If α is a non-real complex nth root of unity, find the values of:

(a) $1 + 2\alpha + 3\alpha^2 + \cdots + n\alpha^{n-1}$

(b) $1 + 3\alpha + 5\alpha^2 + \cdots + (2n-1)\alpha^{n-1}$

P28. Given $z = \cos\left(\frac{2\pi}{2n+1}\right) + i\sin\left(\frac{2\pi}{2n+1}\right)$, where n is a positive integer, find the equation whose roots are:

$\alpha = z + z^3 + z^5 + \cdots + z^{2n-1}$ and $\beta = z^2 + z^4 + \cdots + z^{2n}$

P29. Let $p_k\ (k = 1, 2, ..., n)$ be the nth roots of unity. Let $z = a + ib$ and $A_k = \text{Re}(z)\,\text{Re}(p_k) + i\{\text{Im}(z)\,\text{Im}(p_k)\}$

(a) Show that A_k lies on an ellipse.

Let S be the focus of the ellipse on the positive major axis.

(b) Find $\sum_{k=1}^{n} A_k S$ (c) Find $\sum_{k=1}^{n} (A_k S)^2$

P30. A cubic equation $f(x) = 0$ has one real root α and two complex roots $\beta \pm i\gamma$. Points A, B, C represent roots $\alpha, \beta + i\gamma$ and $\beta - i\gamma$ respectively on the Argand diagram. Show that the roots of the equation $f'(x) = 0$ (the left hand side is the derivative of f) are complex if A falls inside one of the two equilateral triangles described on base BC.

P31. If a, b are complex constants and t a real parameter, what does the equation $z = a\cos t + b\sin t$ represent in the following cases?

(a) a is purely real, b is purely imaginary
(b) $|a - b| = |a + b|$

P32. Points $D(z_1)$, $E(z_2)$ and $F(z_3)$ lie on a circle centered at the origin O. The tangents to the circle at D, E and F intersect at A, B and C. A, B, C are opposite to D, E, F respectively.

(a) Find the complex representation of A.
(b) Determine the equation of the line AO in a simplified, symmetrical form.

P33. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of the point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$.

P34. Sum the following series:

(a) $\cos\alpha + \dfrac{1}{2}\cos(\alpha + \beta) + \dfrac{1}{2^2}\cos(\alpha + 2\beta) + \cdots\infty$

(b) $\cos(n\alpha + \beta) + \cos((n-1)\alpha + 2\beta) + \cos((n-2)\alpha + 3\beta) + \cdots + \cos(\alpha + n\beta)$

(c) $\dfrac{1}{2}\cos(\alpha + \beta) + \dfrac{2}{2^2}\cos(\alpha + 2\beta) + \dfrac{3}{2^3}\cos(\alpha + 3\beta) + \cdots\infty$

P35. Consider the equation $z^2 + z(a+\alpha) + b\alpha = 0, (\alpha > 0 \text{ and } \alpha \in \mathbb{R})$.

(a) Sketch the locus of roots of this equation in the Argand plane.
(b) Find the maximum value of the angle made by the line joining the roots to the origin.

P36. Find the total area of the region satisfying the following constraints:

$$\sin \log_a |z| > 0, \quad a > 1, \quad |z| < 1$$

P37. Let a_n, b_n be complex numbers and N a positive natural number such that:

$$a_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} b_k e^{\frac{i2kn\pi}{N}}, \quad n = 0 \text{ to } N-1$$

(a) Prove that

$$b_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k e^{\frac{-i2kn\pi}{N}}$$

(b) Hence, prove that

$$\sum_{n=0}^{N-1} |a_n|^2 = \sum_{n=0}^{N-1} |b_n|^2$$

Complex Numbers

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. All the four relations are true. We will prove all four one by one. Let $z = x + iy = re^{i\theta}$.

(A) $\overline{(e^z)} = \overline{(e^{x+iy})} = \overline{e^x \cdot e^{iy}} = e^x \cdot e^{-iy} = e^{x-iy} = e^{\bar{z}}$

You might find the operation of raising a real number to an arbitrary complex number kind of strange. But with time, you will realise that what we are doing is mathematically consistent and therefore makes sense. For example, we can even determine sines and cosines of complex numbers (as in part c below)!

(B) $\overline{(\ln z)} = \overline{\ln(re^{i\theta})} = \overline{(\ln r + \ln e^{i\theta})} = \overline{\ln r + i\theta} = \ln r - i\theta = \ln r + \ln e^{-i\theta} = \ln re^{-i\theta} = \ln \bar{z}$

(C) We can write $\cos z$ as $\frac{e^{iz} + e^{-iz}}{2}$

$$\Rightarrow \quad \overline{\cos z} = \overline{\left(\frac{e^{iz} + e^{-iz}}{2}\right)} = \frac{e^{-i\bar{z}} + e^{i\bar{z}}}{2} \qquad \text{(From part (a))}$$

$$= \cos \bar{z}.$$

(D) $z_1^{z_2}$ can be written as $e^{z_2 \ln z_1}$. Therefore,

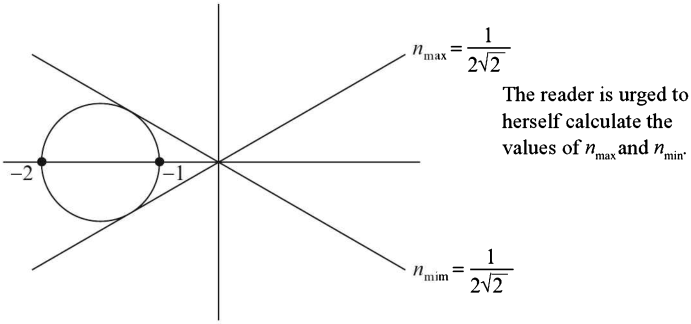
$$\Rightarrow \quad \overline{\left(z_1^{z_2}\right)} = \overline{(e^{z_2 \ln z_1})} = e^{\overline{z_2 \ln z_1}} = e^{\bar{z}_2 \overline{(\ln z_1)}} = e^{\bar{z}_2 \ln \bar{z}_1} = \bar{z}_1^{\bar{z}_2}.$$

S2. We have $z_1 = \frac{1}{\bar{z}_1}$, $z_2 = \frac{4}{\bar{z}_2}$ and $z_3 = \frac{9}{\bar{z}_3}$, so that:

$$1 = |z_1 + z_2 + z_3| = \left| \frac{1}{\bar{z}_1} + \frac{4}{\bar{z}_2} + \frac{9}{\bar{z}_3} \right| = \frac{|9z_1 z_2 + 4z_1 z_3 + z_2 z_3|}{|z_1||z_2||z_3|} \qquad \text{(how?)}$$

Thus, the required value is $|z_1||z_2||z_3|$, that is, 6. The correct option is (D).

S3. The first curve is a circle with –1, –2 as end-points of a diameter. The second one is the straight line $y = nx$.



The possible values of n thus form the set $[-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}]$. The correct option is (A).

S4. Using $z \to x + iy$ in the given relation reduces it to:

$$2(x^2 - y^2) = 10^n = 2^n \cdot 5^n \quad \Rightarrow \quad (x+y)(x-y) = 2^{n-1} \cdot 5^n$$

This implies that both $x + y$ and $x - y$ must be even, and therefore we should partition $2^{n-1} \cdot 5^n$ into two parts such that both parts get at least one factor of '2'. This can be done in $(n-2)(n+1)$ ways (why?). But since, for every (x, y) as a solution, $(-x, -y)$ is also a solution, the total number of solutions is $2(n-2)(n+1)$. The correct option is (C).

S5. We consider the partial fraction expansion of $\frac{n}{x^n - 1}$:

$$\frac{n}{x^n - 1} = \frac{a_1}{x - \alpha_1} + \frac{a_2}{x - \alpha_2} + \cdots + \frac{a_n}{x - \alpha_n} \tag{1}$$

where α_i are the nth roots of unity. Now, we evaluate a_j:

$$a_j = \lim_{x \to \alpha_j} \frac{n(x - \alpha_j)}{x^n - 1} = \lim_{x \to \alpha_j} \frac{n(x - \alpha_j)}{x^n - \alpha_j^n} = \frac{1}{\alpha_j^{n-1}} = \alpha_j \quad \text{(Note carefully how the limit is applied)}$$

Finally we substitute $x = 2$ in (1) to obtain the required sum:

$$\sum_{k=1}^{n} \frac{\alpha_j}{2 - \alpha_j} = \frac{n}{2^n - 1}$$

The correct option is (C).

S6. The trick is to write $2\cos 2^j x$ as $e^{i2^j x} + e^{-i2^j x}$. Letting e^{ix} as α, we have:

$$f(x) = \left(\alpha + \frac{1}{\alpha} - 1\right)\left(\alpha^2 + \frac{1}{\alpha^2} - 1\right) \cdots \left(\alpha^{2^{n-1}} + \frac{1}{\alpha^{2^{n-1}}} - 1\right)$$

$$= \frac{(\alpha^2 - \alpha + 1)(\alpha^4 - \alpha^2 + 1) \cdots (\alpha^{2^n} - \alpha^{2^{n-1}} + 1)}{\alpha^{1+2+2^2+\cdots+2^{n-1}}}$$

$$= \frac{1}{\alpha^{2^n - 1}} \left\{ \frac{\alpha^3 + 1}{\alpha + 1} \times \frac{\alpha^6 + 1}{\alpha^2 + 1} \times \cdots \times \frac{(\alpha^{2^{n-1}})^3 + 1}{\alpha^{2^{n-1}} + 1} \right\}$$

Multiplying this by $\frac{\alpha^3-1}{\alpha-1}$, we have

$$f(x)\times\frac{\alpha^3-1}{\alpha-1}=\frac{1}{\alpha^{2^n-1}}\times\frac{(\alpha^{2^n})^3-1}{\alpha^{2^n}-1} \qquad \text{(How?)}$$

$$\Rightarrow \quad f(x)=\frac{1}{\alpha^{2^n-1}}\cdot\frac{\alpha-1}{\alpha^3-1}\cdot\frac{(\alpha^{2^n})^3-1}{\alpha^{2^n}-1}$$

We have therefore succeeded in finding a closed form expression for $f(x)$. The rest is straightforward. When $x=\frac{2\pi k}{2^n-1}$, α^{2^n-1} becomes $e^{i(2\pi k)}$, that is 1. Thus,

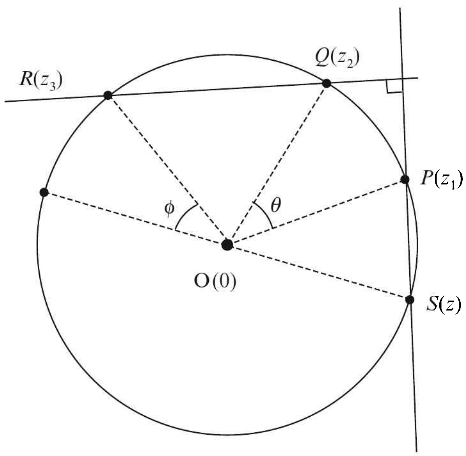
$$\alpha^{2^n}=\alpha \;\Rightarrow\; f\left(\frac{2\pi k}{2^n-1}\right)=1$$

On the other hand, when $x=\frac{2\pi k}{2^n+1}$, then α^{2^n+1} is 1, *i.e.*,

$$\alpha^{2^n}=\frac{1}{\alpha} \Rightarrow\; f\left(\frac{2\pi k}{2^n+1}\right)=1$$

In both cases, the answer is 1. The correct option is (B).

S7. Consider the following figure carefully:



We observe that $\theta=\phi$, because of the perpendicularity of the two secants. Now,

$$\angle SOR+\angle SOQ-\angle SOP=\pi \qquad \text{(by using the fact that } \theta=\phi).$$

Therefore,

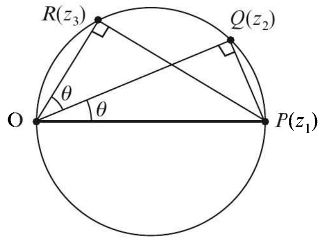
$$(\arg(z_3)-\arg(z))+(\arg(z_2)-\arg(z))-(\arg(z_1)-\arg(z))=\pi$$

$$\Rightarrow \quad \arg(z_3)+\arg(z_2)=\pi+\arg(z_1)+\arg(z)$$

$$\Rightarrow \quad \arg(z_2z_3)=\pi+\arg(zz_1)$$

$$\Rightarrow \quad z_2z_3=-zz_1 \quad\Rightarrow\quad zz_1+z_2z_3=0$$

The correct option is (A).

S8. 

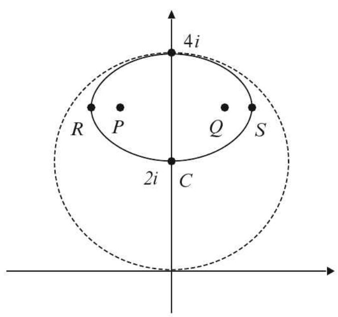
In terms of $\overrightarrow{OP}(z_1)$, z_2 and z_3 are easily obtainable by rotation about the point O. Thus,

$$z_2 = z_1 \cos\theta\, e^{i\theta};\ z_3 = z_1 \cos 2\theta e^{i2\theta}$$

From this, one can easily show that $\frac{z_1 z_3}{z_2^2}$ equals $\frac{\cos 2\theta}{\cos^2\theta}$.

This means that the correct option is (A).

S9. The first equation represents a circle centered at the point $C(2i)$, with a radius of 2 units. To understand what the second equation represents, we note that the distance between the points $P(1+3i)$ and $Q(-1+3i)$ is 2 units, and $2 < 2\sqrt{2}$; thus, the second equation represents an ellipse with its foci as the points P and Q. We note that the extremities of the minor axis of this ellipse are $2i$ and $4i$, since the sum of distances of both these points from P and Q is $2\sqrt{2}$. We also note that the extremities of the major axis of this ellipse are $R(-\sqrt{2}+3i)$ and $S(\sqrt{2}+3i)$ (why?), and the distances of these points from $C(2i)$ is $\sqrt{3}$, which is less than 2; this implies that these points R and S lie within the circle:



It is clear that there is only one point of intersection, namely $4i$, and the ellipse is internally tangent to the circle at this point. Thus, the correct option is (A).

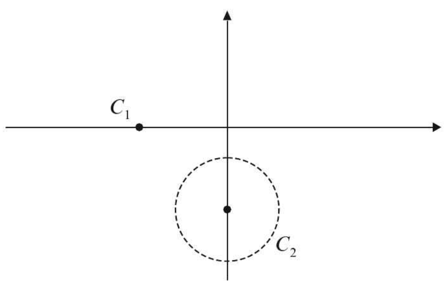
S10. The first expression represents a circle C_1 with center at $z_1 = -\sqrt{2}$ and radius $r_1 = a^2 - 3a + 2$. The second expression represents the interior of a circle C_2 with center at $z_2 = -i\sqrt{2}$ and radius $r_2 = a^2$. Thus, we must have:

$$a^2 - 3a + 2 \geq 0 \ \Rightarrow\ a \leq 1 \quad \text{or} \quad a \geq 2$$

For at least one complex number z to satisfy both expressions, the circumference of the first circle must pass through the interior of the second circle. We note that even the touching of the two circles will not suffice because the second expression is a strict inequality.

Case 1: $a \leq 1$

Initially, consider the scenario when $a = 1$. In this case, C_1 is a point circle while C_2 is a circle of radius 1.



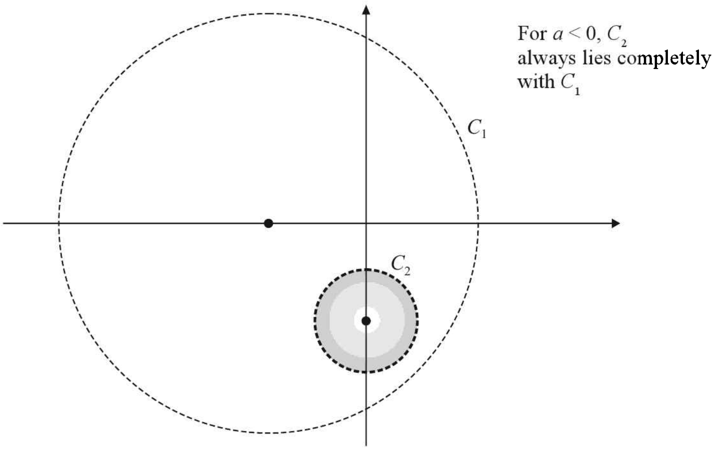
As a is decreased, the radius of C_1 increases while the radius of C_2 decreases (why?). When $a = 0$, the radius of C_1 is 2 while the radius of C_2 is 0. In this case, the second inequality has no solution, even though C_1 passes through the center of C_2.

When a is between 0 and 1, the sum of the two radii,

$$r_{\text{sum}} = r_{C_1} + r_{C_2} = a^2 - 3a + 2 + a^2 = 2a^2 - 3a + 2,$$

is less than distance between the two centers, so C_1 never passes through the interior of C_2.

We conclude that no solution exists for $a \in [0, 1]$. As a is decreased below 0, the radius of C_1 further increases, while the radius of C_1 also starts increasing. However, the increase happens in such a way that C_2 *always* lies completely within C_1:



This can be confirmed by observing that the difference between the two radii is greater than the distance between their centers for $a < 0$:

$$\begin{aligned} r_{\text{difference}} &= r_{C_1} - r_{C_2} = (a^2 - 3a + 2) - a^2 \\ &= -3a + 2 > 2 \quad \text{for } a < 0. \end{aligned}$$

Thus, even for $a < 0$, no solution exists.

Case 2: $a \geq 2$

Consider the scenario when $a = 2$. In this case, C_1 is a point circle while C_2 is a circle with radius 4. Thus, the center of C_1, *i.e.*, $z_1 = -\sqrt{2}$, is clearly a solution. As a is increased, the radii of C_1 as well as C_2 increase. A little thinking will show that in this case, C_1 will *always* pass through the interior of C_2. Let us consider why. The only way this cannot happen is if C_2 lies completely inside C_1, *i.e.*,

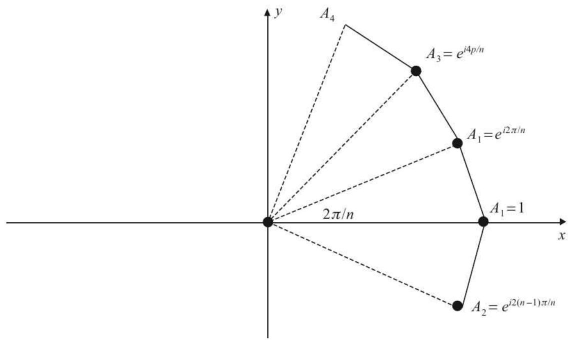
$$\begin{aligned} r_{\text{difference}} &= -3a + 2 > 2 \\ &\Rightarrow \quad a < 0. \end{aligned}$$

For $a > 2$, C_2 *cannot* lie completely inside C_1, *i.e.*, C_1 will *always* pass through the interior of C_1. The required solution is thus:

$$a \in [2, \infty).$$

We see that the correct option is (D).

S11. There is no loss of generality in assuming that one of the vertex, say A_1, lies at the point 1.



Thus, $A_1A_2 = |1 - e^{i2\pi/n}| = \left|1 - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right| = \left|2\sin^2\frac{\pi}{n} - 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right|$

$$= 2\sin\frac{\pi}{n}\left|\sin\frac{\pi}{n} - i\cos\frac{\pi}{n}\right| = 2\sin \pi/n.$$

Similarly, $\qquad A_1A_r = |1 - e^{i2(r-1)\pi/n}| = 2\sin\frac{(r-1)\pi}{n}$

Now, $\qquad |A_1A_2|^2 + |A_1A_3|^2 + \cdots + |A_1A_n|^2$

$$= 4\sin^2\frac{\pi}{n} + 4\sin^2\frac{2\pi}{n} + \cdots + 4\sin^2\frac{(n-1)\pi}{n}$$

$$= 2\left\{\left(1 - \cos\frac{2\pi}{n}\right) + \left(1 - \cos\frac{4\pi}{n}\right) + \cdots + \left(1 - \cos\frac{2(n-1)\pi}{n}\right)\right\}$$

$$= 2\left\{(n-1) - \left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots \cos\frac{2\cdot(n-1)\pi}{n}\right)\right\}$$

$$= 2\{(n-1) - (-1)\} \qquad \left(\begin{array}{l}\text{Why? Because } \sum_{r=0}^{n-1} e^{i2\pi r/n} = 0 \Rightarrow \sum_{r=1}^{n-1} e^{i2\pi r/n} = -1 \\ \qquad \Rightarrow \mathrm{Re}\left(\sum_{r=1}^{n-1} e^{i2\pi r/n}\right) = -1\end{array}\right)$$

$$= 2n$$

The required ratio is 2. The correct option is (C).

S12: Observe that if the east direction is taken to be the positive real axis, then $\vec{X}$ and $\vec{Y}$ will be parallel to ω and ω^2. Thus, the motion of the person can be expressed as the following geometric progression:

$$1 + \frac{\omega}{2} + \left(\frac{\omega}{2}\right)^2 + \left(\frac{\omega}{2}\right)^3 + \cdots\infty = \frac{1}{1 - \frac{\omega}{2}} = \frac{2}{2 - \omega}$$

The distance of the person from the origin after an infinite amount of time will be:

$$d_\infty = \left|\frac{2}{2-\omega}\right| = \frac{2}{\sqrt{7}}$$

where the modulus can be calculated by substituting the Cartesian form of ω and simplifying. The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

S13. We make use of the triangle inequality on the relation $\sum_{r=1}^{n} a_r z^r = 1$, and the fact that $|a_r| < 2$.

$$\begin{aligned} 1 = a_1 z + a_2 z^2 + \cdots + a_n z^n &\le |a_1||z| + |a_2||z|^2 + \cdots + |a_n||z|^n \\ &< 2(|z| + |z|^2 + \cdots + |z|^n) < 2(|z| + |z|^2 + \cdots \infty) \\ &= \frac{2|z|}{1-|z|} \\ &\Rightarrow \quad |z| > \frac{1}{2}. \end{aligned}$$

Thus, $|z|$ cannot be less than $\frac{1}{3}$.

S14. Rearrange the given equation to $z^9 = \dfrac{11-10iz}{11z+10i}$.

Assuming $z = a + ib$, we have

$$|z|^9 = \left|\frac{11-10iz}{11z+10i}\right| = \sqrt{\frac{11^2 + 220b + 10^2(a^2+b^2)}{11^2(a^2+b^2) + 220b + 10^2}}$$

Now, if $|z| > 1$, *i.e.*, if $a^2 + b^2 > 1$, then

$$11^2 + 220b + 10^2(a^2+b^2) - (11^2(a^2+b^2) + 220b + 10^2)$$

$$= (11^2 - 10^2)(1 - a^2 - b^2) < 0$$

This means that

$$11^2 + 220b + 10^2(a^2+b^2) < 11^2(a^2+b^2) + 220b + 10^2$$

$$\Rightarrow \sqrt{\frac{11^2 + 220b + 10^2(a^2+b^2)}{11^2(a^2+b^2) + 220b + 10^2}} < 1$$

$\Rightarrow |z|^9 < 1,$ which is a contradiction.

Similarly, we can show that if we assume $|z| < 1$, we reach the implication $|z|^9 > 1$, again a contradiction.

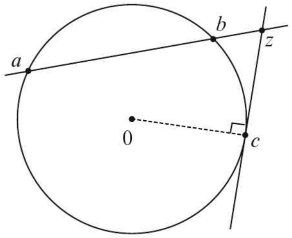
Therefore, the only possibility is $|z| = 1$.

S15. Firstly, we note that

$$|a|^2 = |b|^2 = |c|^2 = r^2.$$

Now, z must satisfy the following two relations:

$$\frac{z-b}{\bar{z}-\bar{b}} = \frac{b-a}{\bar{b}-\bar{a}}; \qquad \frac{z-c}{\bar{z}-\bar{c}} + \frac{c}{\bar{c}} = 0$$



From these two relations, we eliminate $\bar{z}$ to obtain z:

$$z = \frac{c(a\bar{b} - \bar{a}b + 2\bar{c}(b-a))}{c(\bar{b}-\bar{a}) + \bar{c}(b-a)}$$

Finally, we use $\bar{a} = \frac{r^2}{a}$, $\bar{b} = \frac{r^2}{b}$ and $\bar{c} = \frac{r^2}{c}$ to obtain $z = \frac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$.

S16. Recall that any line passing through the origin will have an equation of the form $az + \bar{a}\,\bar{z} = 0$. Assume that all z_i lie in the half-plane for which $az_i + \bar{a}\,\bar{z}_i > 0$. Dividing both sides by $|z_i|^2 = z_i\bar{z}_i$, we have:

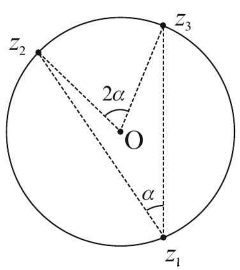
$$\frac{a}{\bar{z}_i} + \frac{\bar{a}}{z_i} > 0 \text{ for all } i.$$

Thus, all the points $\frac{1}{z_i}$ lie on the same side of the line $\bar{a}z + a\bar{z} = 0$.

Now, $az_i + \bar{a}\,\bar{z}_i > 0$, which implies that $a\Sigma z_i + \bar{a}\,\Sigma\bar{z}_i > 0$. This tells us that $\Sigma\, z_i$ cannot be 0 (why?). Similarly, we can show that $\Sigma\frac{1}{z_i}$ is also non-zero. Thus, both the statements are true.

S17: (a) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ implies that $(a+b+c)\,(a-b)\,(b-c)\,(c-a) = 0$, which in this situation can only mean that $a = b = c$ (this is because the situation is symmetric with respect to a, b and c, and all three of these are positive). Thus, z_1, z_2 and z_3 lie on a circle centered at the origin.

(b)



From this figure:

$$\arg\left(\frac{z_3}{z_2}\right) = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

S18. Since $|z| = 3$, z must lie on a circle of radius 3 centred at the origin. Now, the distance of z from $a(1+i) - i$ must not be greater than 3, *i.e.*, z must lie inside a circle of radius 3 centred at $a(1+i) - i$. Thus, the $|z| = 3$ circle and the latter circle must intersect (or at least touch) in order that both the relations $|z| = 3$ and $|z - (a(1+i) - 1)| \le 3$ are satisfied. This means that the distance between the centres of the two circles must be less than the sum of the radii, *i.e.*, 6.

$$\Rightarrow \quad |a(1+i) - 1| \le 6 \qquad \Rightarrow \quad |(a-1) + ai| \le 6$$

$$\Rightarrow \quad (a-1)^2 + a^2 \le 36 \qquad \Rightarrow \quad 2a^2 - 2a - 35 \le 0$$

$$\Rightarrow \quad \frac{1-\sqrt{71}}{2} \le a \le \frac{1+\sqrt{71}}{2} \tag{1}$$

By an analogous argument, for the relation $|z + 2a - (a+1)i| > 3$ to be satisfied simultaneously, z must lie outside a circle of radius 3 centred at $2a - (a+1)i$. Thus, the distance between the centres of the two circles $|z| = 3$ and this circle must be greater than the sum of the radii.

$$\Rightarrow \quad |2a-(a+1)i| > 6 \quad \Rightarrow \quad 5a^2+2a+1 > 36$$

$$\Rightarrow \quad a < \frac{-1-4\sqrt{11}}{5} \quad \text{or} \quad a > \frac{-1+4\sqrt{11}}{5} \qquad (2)$$

The intersection of (1) and (2) gives

$$\left[\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right]$$

S19. The trick in this question is to understand that because the particle's radius of motion is reducing with each step, the particle and its center of motion will eventually *converge*, because the radius will go to 0. Thus, to get the *final* position of the particle, we can determine the *final* position of its center of motion. The complex numbers representing the positions of the center of motion will form a GP, which can be summed easily:

(a) $S = \frac{1}{2} + \frac{1}{4}e^{i\theta} + \frac{1}{8}e^{i2\theta} + \frac{1}{16}e^{i3\theta} + \cdots\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{i\theta}} = \frac{1}{2-e^{i\theta}}$

(b) $S = \frac{1}{2} + \frac{1}{4}e^{i\theta} + \frac{1}{8} + \frac{1}{16}e^{i\theta} + \cdots\infty = \frac{2+e^{i\theta}}{3}$

These answers can be converted to their respective Cartesian forms to determine the coordinates of the final position of the particle in both cases. The answers are:

(a) $\frac{2-\cos\theta}{\sqrt{5-4\cos\theta}} + i\frac{\sin\theta}{\sqrt{5-4\cos\theta}}$

(b) $\frac{2+\cos\theta}{3} + i\frac{\sin\theta}{3}$

S20. Assume that the vertices are located at $z_{k+1} = Re^{i\frac{2\pi k}{n}}$, for $k = 0, 1, 2, \ldots, n-1$. Note that $z_1 = R$.
(a) Fix one vertex, say A_1. Let us first calculate the sum:

$$S_1 = |A_1A_2|^2 + |A_1A_3|^2 + \cdots + |A_1A_n|^2$$

$$= \sum_{j=2}^{n} |z_1 - z_j|^2 = \sum_{j=2}^{n} (|z_1|^2 + |z_j|^2 - 2|z_1||z_j|\cos(\theta_1 - \theta_j))$$

$$= (n-1)R^2 + (n-1)R^2 - 2R^2 \sum_{j=2}^{n} \cos(\theta_1 - \theta_j)$$

$$= (2n-2)R^2 + 2R^2 = 2nR^2 \quad \text{(Observe this step carefully)}$$

The total sum S can now be obtained by multiplying S_1 by $\frac{n}{2}$, that is, $S = n^2R^2$.

(b) Once again, we fix one vertex, say A_1, and calculate the sum

$$S_1 = |A_1\,A_2| + |A_1\,A_3| + \cdots + |A_1\,A_n|$$

$$= 2R\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}\right)$$

$$= 2R\cot\frac{\pi}{2n} \qquad \text{(How?)}$$

The total sum S is given by $S = \frac{n}{2}S_1 = nR\cot\frac{\pi}{2n}$.

(c) Once again, we calculate:

$$P_1 = |A_1\ A_2| \times |A_1\ A_3| \times \cdots \times |A_1\ A_n|$$

$$P_1 = |(z_1 - z_2)(z_1 - z_3)\cdots(z_1 - z_n)|$$

$$= \lim_{z \to R} \left| \frac{z^n - R^n}{z - R} \right| = nR^{n-1} \quad \text{(How did we convert the product into a limit?)}$$

The required product is:

$$P = P_1^{n/2} = n^{n/2}\, R^{\frac{n(n-1)}{2}}.$$

S21. This very interesting result is known as the Napolean's theorem. Although there are other ways to prove it, one of the best is through the use of complex numbers. We let the complex numbers a, b, c, a', b', c' represent the vertices A, B, C, A', B', C' respectively. Now, we make use of the fact that if z_1, z_2, z_3 represent the vertices of an equilateral triangle, then $z_1 + \omega z_2 + \omega^2 z_3 = 0$. Thus,

$$a + b\omega + c'\omega^2 = 0, \qquad b + c\omega + a'\omega^2 = 0, \qquad c + a\omega + b'\omega^2 = 0 \tag{1}$$

Now, if complex numbers x, y, z represent the centroids X, Y, Z respectively, then

$$x = \frac{a' + b + c}{3}, \quad y = \frac{b' + c + a}{3}, \quad z = \frac{c' + a + b}{3} \tag{2}$$

All we need to do after this is to use (1) and (2) to show that:

$$x + y\omega + z\omega^2 = 0.$$

This is straightforward and hence left as an exercise for the reader. The result implies that x, y, z form the vertices of an equilateral triangle.

S22. Let the complex numbers $2a, 2b, 2c, 2d$ represent the *sides* $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DA}$ of the quadrilateral, and let A be the origin. Note that $a + b + c + d$ must be 0. Now, we write P, Q, R, S in terms of a, b, c, d:

$$P = a + ia, \qquad Q = 2a + b + ib, R = 2a + 2b + c + ic, \qquad S = 2a + 2b + 2c + d + id$$

Understand carefully how this is done. For example, to reach Q from A, you first travel along $\overrightarrow{AB}$, and then cover half of $\overrightarrow{BC}$, and then travel another $\frac{BC}{2}$ units perpendicular to BC. We now have

$$\overrightarrow{PR} = a(1 - i) + 2b + c(1 + i), \quad \overrightarrow{QS} = b(1 - i) + 2c + d(1 + i)$$

From these two relations and the fact that $a + b + c + d = 0$, it can be easily shown that $\overrightarrow{PR} = i\overrightarrow{QS}$, which immediately implies that PR and QS are equal in length and perpendicular to each other.

S23. Assume that the vertices are located at $z_{k+1} = R\,e^{i\frac{2\pi k}{n}}$, for $k = 0, 1, 2, \ldots, n - 1$. Assume that the point M is $R\,e^{i\theta}$.

(a) Assume $n = 2m$:

$$\sum_{k \text{ is odd}} |MA_k|^2 = mR^2 + mR^2 - 2R^2 \sum_{k \text{ is odd}} \cos\left(\theta - \frac{2k\pi}{n}\right) \quad \left(\begin{array}{l}\text{Refer to Problem 20}\\ \text{to understand this step}\end{array}\right)$$

$$= 2R^2\left(m - \sum_{k \text{ is odd}} \cos\left(\theta - \frac{k\pi}{m}\right)\right)$$

$$= 2R^2\left(m - \frac{\sin\left(m \cdot \frac{2\pi}{m}\right)\cos\left(\theta - \frac{\pi}{m} + \frac{(m-1)2\pi}{2m}\right)}{\sin\frac{\pi}{m}}\right) = 2mR^2 = nR^2$$

Similarly, $\sum_{k \text{ is even}} |MA_k|^2 = nR^2$, so that the two are equal.

(b) Assume $n = 2m + 1$. Then,

$$S_1 = |MA_1| + |MA_3| + \cdots$$

$$= 2R\left(\sin\frac{\theta}{2} + \sin\left(\frac{\theta}{2} + \frac{2\pi}{2m+1}\right) + \cdots + \sin\left(\frac{\theta}{2} + \frac{2m\pi}{2m+1}\right)\right)$$

$$= \frac{2R\sin\frac{(m+1)\pi}{2m+1}\sin\left(\frac{\theta}{2} + \frac{m\pi}{2m+1}\right)}{\sin\frac{\pi}{2m+1}}.$$

Similarly, the other sum S_2 can be shown to posses the same value, so that $S_1 = S_2$.

In both the parts, we have directly summed the sinusoidal series involved. The reader is urged to verify these steps.

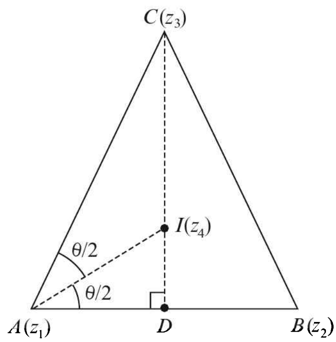
S24. $(z_3 - z_2)z_4 = (z_1 - z_2)z_5 \quad \Rightarrow \quad \dfrac{z_1 - z_2}{z_3 - z_2} = \dfrac{z_4}{z_5} = \dfrac{z_4 - 0}{z_5 - 0}.$ (1)

(1) clearly tells us that not only is $\frac{AB}{CB}$ equal to $\frac{DO}{EO}$, the angles $\angle ABC$ and $\angle DOE$ are also equal. Thus, the two triangles are similar by the SAS similarity criterion.

S25. **(a)** Note that:

$$AB = 2AD = 2AI\cos\frac{\theta}{2}$$

$$AC = AD\sec\theta = AI\cos\frac{\theta}{2}\sec\theta$$



Thus,

$$z_2 - z_1 = 2(z_4 - z_1)\cos\tfrac{\theta}{2}e^{-i\frac{\theta}{2}} \tag{1}$$

$$z_3 - z_1 = (z_4 - z_1)\cos\frac{\theta}{2}\sec\theta\, e^{i\frac{\theta}{2}} \tag{2}$$

Multiplying (1) and (2), we have:

$$\begin{aligned}(z_2-z_1)(z_3-z_1) &= 2\cos^2\frac{\theta}{2}\sec\theta(z_4-z_1)^2\\ &= \sec\theta(1+\cos\theta)(z_4-z_1)^2\end{aligned}$$

The required answer is therefore $\sec\theta(1+\cos\theta)$.

(b) Following a similar approach, we can show that the answer in this case will be $-\cot\theta\cot(\frac{\theta}{2})$.

S26. We write the equalities provided as follows:

$$\frac{z_1-z_2}{z_3-z_2}=\frac{z_3'-z_2'}{z_1'-z_2'} \quad\text{and}\quad \frac{z_2-z_3}{z_1-z_3}=\frac{z_1'-z_3'}{z_2'-z_3'}.$$

Subtracting 1 from both sides of the second relation, and subsequently a comparison of the resulting expression with the first yields:

$$\frac{z_2-z_1}{z_3-z_1}=\frac{z_3-z_2}{z_1-z_2}.$$

Thus, the angles formed at the vertices z_1 and z_2 are equal. In principle, you are done, because by symmetry arguments, the rest follows easily: the angles at z_2 and z_3 are equal, so that the triangle formed by z_1, z_2, z_3 is equilateral. By symmetry, so is the triangle z_1', z_2', z_3'.

S27. The given series are arithematic-geometric progressions. We can sum an AGP using the standard AGP summing technique, or using differentiation. For part (a), we discuss both these methods:

(a) (I) AGP summing technique:

$$S = 1+2\alpha+3\alpha^2+\cdots+n\alpha^{n-1}$$

$$\alpha S = \alpha+2\alpha^2+\cdots+(n-1)\alpha^{n-1}+n\alpha^n$$

$$\Rightarrow\quad (\alpha-1)S = n\alpha^n \quad\Rightarrow\quad S=\frac{n\alpha^n}{\alpha-1} \qquad \text{(Verify this step)}$$

But since $\alpha^n=1$, $S=\frac{n}{\alpha-1}$. Now, all that remains is to reduce S to the standard complex form, which is straightforward. Assuming $\alpha=e^{\frac{i2\pi r}{n}}$, we have

$$S=-\frac{n}{2}-i\frac{n}{2}\cot\frac{r\pi}{n}, \quad r=1,2,\ldots,n-1$$

(II) Differentiation:

$$1+x+x^2+\cdots+x^n=\frac{x^n-1}{x-1}$$

Differentiating this, and substituting $x=\alpha$, we once again have

$$1+2\alpha+3\alpha^2+\cdots+n\alpha^{n-1}=\frac{n}{\alpha-1}$$

(b) $$S = 1 + 3\alpha + 5\alpha^2 + \cdots + (2n-1)\alpha^{n-1}$$

$$\alpha S = \alpha + 3\alpha^2 + \cdots + (2n-3)\alpha^{n-1} + (2n-1)\alpha^n$$

$$\Rightarrow \quad (\alpha - 1)S = (2n-1)\alpha^n - 2(\alpha + \alpha^2 + \cdots + \alpha^{n-1}) - 1$$

$$= (2n-1) - 2(-1) - 1 = 2n$$

$$\Rightarrow \quad S = \frac{2n}{\alpha - 1} = -n - in\cot\frac{r\pi}{n}, \quad r = 1, 2, \ldots, n-1$$

S28. $z = \cos\left(\frac{2\pi}{2n+1}\right) + i\sin\left(\frac{2\pi}{2n+1}\right) = e^{i\left(\frac{2\pi}{2n+1}\right)} \quad \Rightarrow \quad z^{2n+1} = 1$

The sum of roots S is:

$$S = \alpha + \beta = z + z^2 + z^3 + \cdots + z^{2n}$$

$$= z\left(\frac{z^{2n} - 1}{z - 1}\right) = \frac{z^{2n+1} - z}{z - 1} = \frac{1 - z}{z - 1} = -1.$$

The product of roots P is:

$$P = \alpha\beta = (z + z^3 + \cdots + z^{2n-1})(z^2 + z^4 + \cdots + z^{2n})$$

$$= z^3(1 + z^2 + \cdots + z^{2n-2})(1 + z^2 + \cdots + z^{2n-2})$$

$$= z^3\left(\frac{z^{2n} - 1}{z^2 - 1}\right)^2 = z\left(\frac{z^{2n+1} - z}{z^2 - 1}\right)^2$$

$$= \frac{z}{(z+1)^2} = \frac{1}{z + \frac{1}{z} + 2}$$

$$= \frac{1}{e^{i\theta} + e^{-i\theta} + 2} \qquad \left(\theta = \frac{2\pi}{2n+1}\right)$$

$$= \frac{1}{2(1 + \cos\theta)} = \frac{1}{4}\sec^2\frac{\theta}{2} = \frac{1}{4}\sec^2\left(\frac{\pi}{2n+1}\right).$$

The required equation is

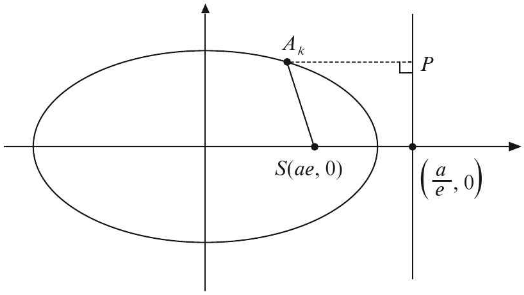
$$x^2 + x + \frac{1}{4}\sec^2\left(\frac{\pi}{2n+1}\right) = 0.$$

S29. **(a)** Since $p_k = e^{i\theta_k}$, where $\theta_k = \frac{2(k-1)\pi}{n}$, we have:

$$A_k = x_k + iy_k \,(\text{say}) = a\cos\theta_k + ib\sin\theta_k.$$

Thus, $\frac{x_k}{a} = \cos\theta_k$, $\frac{y_k}{b} = \sin\theta_k$, which implies that $\frac{x_k^2}{a^2} + \frac{y_k^2}{b^2} = 1$, *i.e.*, A_k lies on an ellipse.

(b) S will be the point $(ae, 0)$. However, to write the distance A_kS, it is easier to write it as e times the distance from the corresponding directrix:



$$A_kS = e(A_kP) = e\left(\frac{a}{e} - x_k\right) = a - ex_k$$

Thus, $\sum_{k=1}^{n} A_kS = \sum_{k=1}^{n} (a - ex_k) = na - e\sum_{k=1}^{n} x_k = na$ (how?)

(c) $\sum_{k=1}^{n} (A_kS)^2 = \sum_{k=1}^{n} \left(a - ex_k\right)^2 = na^2 - 2ae\sum_{k=1}^{n} x_k + e^2\sum_{k=1}^{n} x_k^2$

$$= na^2 + a^2e^2\sum_{k=1}^{n}\cos^2\theta_k \quad = na^2 + a^2e^2\sum_{k=1}^{n}\left(\frac{1+\cos 2\theta_k}{2}\right)$$

$$= na^2 + \frac{na^2e^2}{2} + \frac{1}{2}a^2e^2\sum_{k=1}^{n}\cos 2\theta_k.$$

By considering the equation $z^{n/2} - 1 = 0$ and its roots, we can show that $\sum_{k=1}^{n}\cos 2\theta_k = 0$, so that:

$$\sum_{k=1}^{n}(A_kS)^2 = na^2 + \frac{na^2}{2}\left(1 - \frac{b^2}{a^2}\right) = \frac{n}{2}(3a^2 - b^2).$$

S30. The function $f(x)$ is given by:

$$f(x) = M(x-\alpha)(x-(\beta+i\gamma))(x-(\beta-i\gamma)) \quad (M \in C)$$

$$= M(x-\alpha)(x^2 - 2\beta x + \beta^2 + \gamma^2)$$

The function $f'(x)$ is thus:

$$f'(x) = M\{3x^2 - (2\alpha + 4\beta)x + \beta^2 + \gamma^2 + 2\alpha\beta\}$$

$f'(x) = 0$ has complex roots if:

$$(2\alpha + 4\beta)^2 < 12(\beta^2 + \gamma^2 + 2\alpha\beta)$$

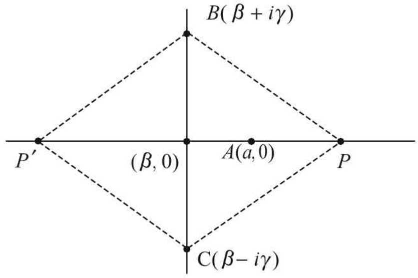
$$\Rightarrow \quad \alpha^2 + 4\beta^2 + 4\alpha\beta < 3\beta^2 + 3\gamma^2 + 6\alpha\beta$$

$$\Rightarrow \quad (\alpha - \beta)^2 < 3\gamma^2$$

$$\Rightarrow \quad -\sqrt{3}\gamma < \alpha - \beta < \sqrt{3}\gamma \quad \Rightarrow \quad \beta - \sqrt{3}\gamma < \alpha < \beta + \sqrt{3}\gamma \qquad (1)$$

Consider the plot below, where PBC and $P'BC$ are the two equilateral triangles drawn on the base BC: It is straightforward to show that:

$$P \equiv (\beta + \sqrt{3}\gamma, 0)$$
$$P' \equiv (\beta - \sqrt{3}\gamma, 0)$$



Thus, A will lie within the two equilateral triangles (note that A must lie on the real axis) if

$$\beta - \sqrt{3}\gamma < \alpha < \beta + \sqrt{3}\gamma$$

We have already shown that this holds, in (1).

S31. We proceed by considering the general case. Let $a = p + iq$, $b = r + is$, and $z = x + iy$. Thus,

$$x = p\cos t + r\sin t, \quad y = q\cos t + s\sin t$$

Solving for cos t and sin t, we have:

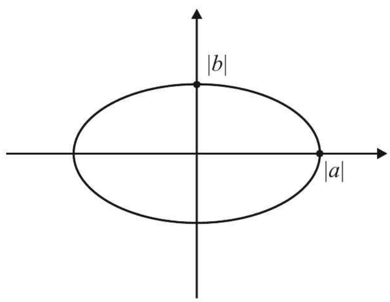
$$\cos t = \frac{sx - ry}{ps - qr}, \quad \sin t = \frac{py - qx}{ps - qr}.$$

Eliminating t, we have:

$$(q^2 + s^2)x^2 + (p^2 + r^2)y^2 - 2(pq + rs)xy = (ps - qr)^2. \tag{1}$$

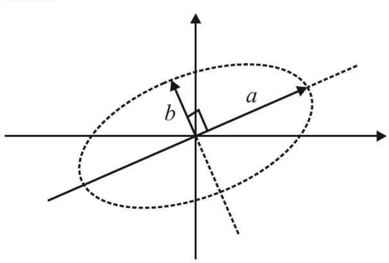
(a) If a is purely real and b is purely imaginary, we have $q = r = 0$, and thus (1) reduces to:

$$s^2x^2 + p^2y^2 = p^2s^2$$
$$\Rightarrow \quad \frac{x^2}{p^2} + \frac{y^2}{s^2} = 1$$
$$\Rightarrow \quad \frac{x^2}{|a|^2} + \frac{y^2}{|b|^2} = 1$$



This represents an ellipse with axes parallel to the co-ordinate axes.

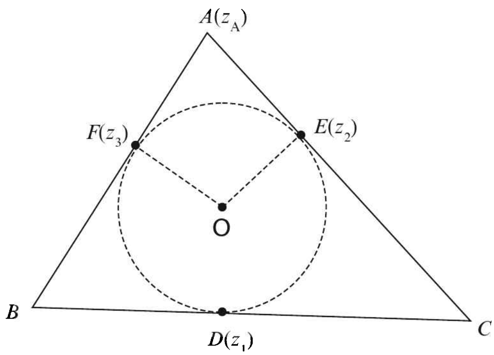
(b) The given relation implies that a and b are perpendicular (if we think of them as vectors). In this case again, a little thought will show that (1) represents an ellipse, albeit with its axes tilted with respect to the co-ordinate axes.



It is left to the reader to rigorously show from (1) that this will indeed be the case.

S32. Let A be represented by z_A. Note that $AE \perp OE$ and $AF \perp OF$.
Also, if the radius of the circle is r, then:

$$|z_1|^2 = |z_2|^2 = |z_3|^2 = r^2$$



(a) Since $AE \perp OE$, we have:

$$\frac{z_A - z_2}{\bar{z}_A - \bar{z}_2} + \frac{z_2}{\bar{z}_2} = 0 \quad \Rightarrow \quad z_A\bar{z}_2 + \bar{z}_A z_2 = 2|z_2|^2 = 2r^2 \tag{1}$$

Similarly, since $AF \perp OF$, we'll have

$$z_A\bar{z}_3 + \bar{z}_A z_3 = 2r^2 \tag{2}$$

From (1) and (2), we can eliminate $\bar{z}_A$:

$$z_A(\bar{z}_2 z_3 - z_2\bar{z}_3) = 2r^2(z_3 - z_2)$$

$$\Rightarrow \quad z_A\left(\frac{z_3}{z_2} - \frac{z_2}{z_3}\right) = 2(z_3 - z_2) \qquad \{\text{using } z_2\bar{z}_2 = z_3\bar{z}_3 = r^2\}$$

$$\Rightarrow \quad z_A = \frac{2z_2z_3}{z_2 + z_3}.$$

(b) If any point on AO is represented by z, then:

$$\frac{z}{\bar{z}} = \frac{z_A}{\bar{z}_A} \quad \text{(the line passes through the origin)}$$

$$\Rightarrow \quad z\left(\frac{2\bar{z}_2\bar{z}_3}{\bar{z}_2 + \bar{z}_3}\right) = \bar{z}\left(\frac{2z_2z_3}{z_2 + z_3}\right)$$

$$\Rightarrow \quad z\left(\frac{2\bar{z}_2\bar{z}_3}{\frac{r^2}{z_2} + \frac{r^2}{z_3}}\right) = \bar{z}\left(\frac{2z_2z_3}{z_2 + z_3}\right)$$

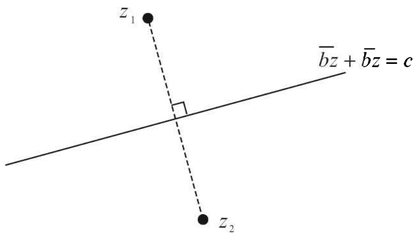
$$\Rightarrow \quad z\,\bar{z}_2\bar{z}_3 = \bar{z}\,r^2 = \bar{z}\sqrt{r^4} = \bar{z}\sqrt{z_2\bar{z}_2z_3\bar{z}_3}$$

$$\Rightarrow \quad z\sqrt{\bar{z}_2\bar{z}_3} = \bar{z}\sqrt{z_2z_3}$$

S33. Fact 1: The mid-point of the segment joining z_1 and z_2 lies on the given line

$$\bar{b}\left(\frac{z_1+z_2}{2}\right)+b\left(\frac{\bar{z}_1+\bar{z}_2}{2}\right)=c$$

$$\Rightarrow \quad \bar{b}(z_1+z_2)+b(\bar{z}_1+\bar{z}_2)=2c \tag{1}$$



Fact 2: The segment joining z_1 and z_2 is perpendicular to the given line:
This means that the sum of the two complex slopes must be 0:

$$\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}+\left(-\frac{b}{\bar{b}}\right)=0$$

$$\Rightarrow \quad \bar{b}(z_1-z_2)-b(\bar{z}_1-\bar{z}_2)=0 \tag{2}$$

Subtracting (2) from (1) gives us the desired result:

$$b\bar{z}_1+\bar{b}z_2=c$$

S34. (a) $S_1=\mathrm{Re}\left(e^{i\alpha}+\frac{1}{2}e^{i(\alpha+\beta)}+\frac{1}{2^2}e^{i(\alpha+2\beta)}+\cdots\infty\right)$

The series represents on infinite geometric progression with common ratio $\frac{e^{i\beta}}{2}$. Thus,

$$S_1=\mathrm{Re}\left(\frac{e^{i\alpha}}{1-\frac{e^{i\beta}}{2}}\right)=\mathrm{Re}\left(\frac{2e^{i\alpha}}{2-e^{i\beta}}\right)=\frac{4\cos\alpha-2\cos(\alpha-\beta)}{5-4\cos\beta}$$

where the final answer has been obtained after some simple algebraic manipulations.

(b) $S_2=\mathrm{Re}(e^{i(n\alpha+\beta)}+e^{i((n-1)\alpha+2\beta)}+e^{i((n-2)\alpha+3\beta)}+\cdots+e^{i(\alpha+n\beta)})$

This is once again a geometric progression with n terms , and common ratio $e^{i(\beta-\alpha)}$. Therefore,

$$S_2=\mathrm{Re}\left\{e^{i(n\alpha+\beta)}\left(\frac{e^{in(\beta-\alpha)}-1}{e^{i(\beta-\alpha)}-1}\right)\right\},$$

which upon manipulation yields

$$S_2=\frac{\cos(n\beta+\alpha)+\cos(n\alpha+\beta)-\cos(n+1)\alpha-\cos(n+1)\beta}{2-2\cos(\beta-\alpha)}.$$

(c) $S_3 = \text{Re}\left(\frac{1}{2}e^{i(\alpha+\beta)} + \frac{2}{2^2}e^{i(\alpha+2\beta)} + \frac{3}{2^3}e^{i(\alpha+3\beta)} + \cdots\infty\right) = \text{Re}(\tilde{S}_3)$

$\tilde{S}_3$ is an AGP with the AP being $\{1, 2, 3,...\}$ and the GP being $\{\frac{1}{2}e^{i(\alpha+\beta)}, \frac{1}{2^2}e^{i(\alpha+2\beta)},...\}$. We have:

$$\frac{e^{i\beta}}{2}\tilde{S}_3 = \frac{1}{2}e^{i(\alpha+2\beta)} + \frac{2}{2^2}e^{i(\alpha+3\beta)} + \frac{3}{2^3}e^{i(\alpha+3\beta)} + \cdots\infty.$$

Thus,

$$\frac{e^{i\beta}}{2}\tilde{S}_3 - \tilde{S}_3 = \frac{1}{2}e^{i\alpha}\left(e^{i\beta} + \frac{e^{i2\beta}}{2} + \frac{e^{i3\beta}}{2^2} + \cdots\infty\right) = \frac{1}{2}e^{i\alpha}\left(\frac{e^{i\beta}}{1-\frac{e^{i\beta}}{2}}\right)$$

$$\Rightarrow \quad \tilde{S}_3 = \frac{-2e^{i(\alpha+\beta)}}{(e^{i\beta}-2)^2}.$$

Now, S_3 can be evaluated by taking the real part of $\tilde{S}_3$. An alternate approach would be to differentiate S_1 in part –(a) as follows:

$$S_1 = \text{Re}\left(e^{i\alpha}\left(1 + \frac{e^{i\beta}}{2} + \frac{e^{i2\beta}}{2^2} + \cdots\infty\right)\right)$$

$$= \text{Re}(e^{i\alpha}(1 + z + z^2 + \cdots\infty)), \quad z = \frac{e^{i\beta}}{2}$$

$$\Rightarrow \quad \frac{dS_1}{dz} = \text{Re}\left(\frac{d}{dz}\left(\frac{e^{i\alpha}}{1-z}\right)\right) = \text{Re}\left(\frac{-e^{i\alpha}}{(1-z)^2}\right) = \text{Re}\left(\frac{S_3}{z}\right)$$

$$\Rightarrow \quad S_3 = \text{Re}\left(\frac{-ze^{i\alpha}}{(1-z)^2}\right) \qquad \text{(Make sure you understand these steps)}$$

Evaluating this real part is left to the reader as an exercise.

S35. (a) Observe that $\alpha = \dfrac{-z(z+a)}{z+b}$. Since α is a positive real, we have

$$\arg\left(\frac{z(z+a)}{z+b}\right) = \pi$$

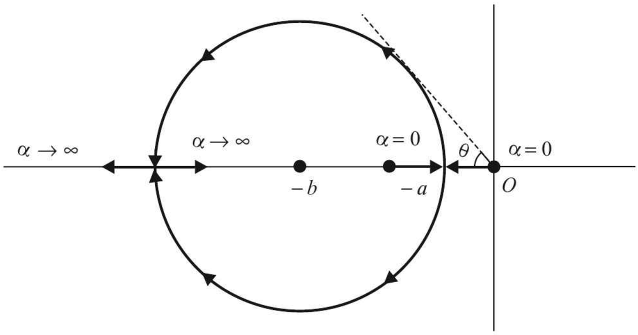
$$\Rightarrow \quad \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x+b} = \pi \quad \text{(Assuming } z = x + iy\text{)}.$$

Simplifying this yields

$$y(x^2 + y^2 + 2bx + ab) = 0.$$

This implies that either $y = 0$, which means the roots are purely real, or, $x^2 + y^2 + 2bx + ab = 0$, which is a circle of radius $\sqrt{b^2 - ab}$ with center $(-b, 0)$.

When $\alpha = 0$, $z = 0$ or $-a$. As α increases, the two roots move towards each other. When α goes above a certain value, the roots become complex and move on the circle, but after another threshold value, they again become real. As $\alpha \to \infty$, $z \to -b, \infty$.



(b) From the figure, we can easily deduce that the maximum angle θ is given by $\sin^{-1}(\frac{\sqrt{b^2-ab}}{b})$.

S36.

$$\sin \log_a |z| > 0 \;\Rightarrow\; \log_a |z| \in (2n\pi, (2n+1)\pi)$$

$$\Rightarrow \; |z| \in (a^{2n\pi}, a^{(2n+1)\pi})$$

Since $|z| < 1$, we have.

$$|z| \in (a^{-2n\pi}, a^{-(2n-1)\pi}),\, n \in \mathbb{Z}, n \geq 1 \qquad \text{(Why?)}$$

The region which z represents will consist of concentric rings, one corresponding to each value of n. The area A_n of the nth ring is:

$$A_n = \pi(a^{-(4n-2)\pi} - a^{-4n\pi}) \qquad \text{(Verify this)}$$

The required total area A is:

$$A = \sum_{n=1}^{\infty} A_n = \pi\left(\sum_{n=1}^{\infty} a^{-(4n-2)\pi} - \sum_{n=1}^{\infty} a^{-4n\pi}\right) = \pi\left(\frac{a^{-2\pi} - a^{-4\pi}}{1 - a^{-4\pi}}\right) = \pi\left(\frac{a^{2\pi} - 1}{a^{4\pi} - 1}\right).$$

S37. The expressions in this question are examples of Fourier sums, which are extensively used in higher engineering mathematics. Basically, the first expression tells us that a_n has been expressed as a sum of exponentials, weighted by the coefficients b_k:

$$a_n = \frac{1}{\sqrt{N}}\left(b_0 + b_1 e^{i\frac{2n\pi}{N}} + b_2 e^{i\frac{4n\pi}{N}} + b_3 e^{i\frac{6n\pi}{N}} + \cdots + b_{N-1}\, e^{i\frac{2(N-1)n\pi}{N}}\right).$$

Suppose we wish to find the value of b_3. If we multiply the above relation by $e^{-i\frac{6n\pi}{N}}$ on both sides, we obtain:

$$a_n e^{-i\frac{6n\pi}{N}} = \frac{1}{\sqrt{N}}\left(b_0 e^{-i\frac{6n\pi}{N}} + b_1 e^{-i\frac{4n\pi}{N}} + b_2 e^{-i\frac{2n\pi}{N}} + b_3 + \cdots + b_{N-1}\, e^{i\frac{2(N-4)n\pi}{N}}\right) \tag{1}$$

Note that there is no exponential term with b_3. By varying the value of n from 0 to $N - 1$ in (1), and adding all the N relations so obtained, we can get rid of all the other $b_i's$ except b_3. This is really the most crucial step in the solution to the problem and you must make sure that you understand this properly. We now have:

$$\begin{aligned}
\sum_{n=0}^{N-1} a_n e^{-i\frac{6n\pi}{N}} &= \frac{1}{\sqrt{N}}\left(\sum_{n=0}^{N-1} b_0 e^{-i\frac{6n\pi}{N}} + \cdots + \sum_{n=0}^{N-1} b_3 + \cdots + \sum_{n=0}^{N-1} b_{N-1} e^{i\frac{2(N-4)n\pi}{N}}\right) \\
&= \frac{1}{\sqrt{N}}(0 + \cdots + Nb_3 + \cdots + 0) \\
&= \sqrt{N} b_3 \\
\Rightarrow \quad b_3 &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a_n e^{-i\frac{6n\pi}{N}}
\end{aligned}$$

(a) Generalizing this from b_3 to b_n, we conclude that:

$$b_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k e^{-i\frac{2kn\pi}{N}}.$$

(b) We have:

$$\sum_{n=0}^{N-1} |a_n|^2 = \sum_{n=0}^{N-1} a_n \cdot \bar{a}_n = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{\left\{\left(\sum_{k=0}^{N-1} b_k e^{i\frac{2kn\pi}{N}}\right)\left(\sum_{j=0}^{N-1} \bar{b}_j e^{-i\frac{2jn\pi}{N}}\right)\right\}}_{P}$$

In the product P, we will have two kinds of terms:

(1) When $j = k$, then the exponentials cancel out and we are left with $b_k \cdot \bar{b}_j = b_k \cdot \bar{b}_k = |b_k|^2$.

(2) When $j \neq k$, we are left with the term:

$$b_k \bar{b}_j e^{i\frac{2(k-j)n\pi}{N}},$$

which when summed for $n = 0$ to $N - 1$, will give 0.

Thus, the right hand side becomes

$$\begin{aligned} RHS &= \frac{1}{N}\sum_{n=0}^{N-1}(|b_0|^2 + |b_1|^2 + \cdots + |b_{N-1}|^2) \\ &= \frac{1}{N}\cdot N\ (|b_0|^2 + |b_1|^2 + \cdots + |b_{N-1}|^2) \\ &= \sum_{n=0}^{N-1}|b_n|^2. \end{aligned}$$

This concludes our proof.

Permutations and Combinations

PART-A: Summary of Important Concepts

1. Fundamental Principle of Counting (FPC)

The entire subject of Permutations and Combinations rests on the foundation of the FPC. Starting from the FPC, one can derive so many non-trivial results that sometimes it may seem surprising to a curious student how a principle seemingly as simple as the FPC can lead to a mathematical world of such complexity and richness. Our advice to the student at this point is never to forget the FPC, primarily because it is the basis for the entire topic we are studying now, but also because many a times situations may arise where no standard results might apply, in which case the only resort will be to use the FPC and tackle the problem from first principles.

So what is the FPC? Well, the FPC tells us how to find the *number of ways* to accomplish a particular task, which is itself composed of a number of sub-tasks. The important thing to always remember is that what we are interested in is counting the *number of ways* to accomplish a task, we are not interested in particular ways of accomplishing that task. Lets understand this through an example.

Suppose that you have to travel from New Delhi to the Fiji Islands in the Pacific, via Singapore and Sydney. There are 4 flights available from New Delhi to Singapore, 3 flights from Singapore to Sydney, and 5 flights from Sydney to Fiji. The FPC tells us that that there are $4 \times 3 \times 5 = 60$ ways in which you can travel to Fiji. Note that we really do not care about how exactly you will finally end up travelling, what way you will choose out of the 60 possible ways. You might choose Air India, Singapore Airlines, and Qantas, or you might go with Jet, Qantas, Air Pacific, or any of the other possibilities. What we really *are* interested in, is the number of ways available before you, in this case: 60.

So what the FPC is all about is finding the number of ways of accomplishing a particular task. The general form of the FPC says that if a task T consists of a set of tasks $\{T_1, T_2, T_3, \ldots, T_k\}$, where T_1 can be done in n_1 ways, T_2 can be done in n_2 ways, and so on, the number of ways n in which we can accomplish T is

$$n = n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

To be able to apply the FPC, the sub-tasks must be independent of each other - in the sense that the number of ways in which you can accomplish a particular sub-task T_i is not dependent on how exactly you accomplish another sub-task T_j. Understanding this idea of independence of sub-tasks is of the utmost importance.

2. **Permutations:** The concept of permutations enables us to find the *number of ways* of *arranging* a set of objects, some of which may be identical. Any particular arrangement of the set of objects will be one permutation out of all the possible permutations. The important point is to associate permutations with arrangements. Here are some examples of problems on permutations:

- Finding the number of ways of seating n people on n chairs arranged in a row.
- Finding the number of ways of seating any r people out of n (where $r \le n$) on r chairs arranged in a row.

- Finding the number of arrangements (in a single line) of p identical red balls, q identical yellow balls and r identical green balls.
- Finding the number of possible arrangements of all the letters in ARRANGEMENTS.

The following table summarizes some basic results on permutations, all of which have proofs based on the FPC (it would be a good idea if you take a minute to use the FPC and prove these):

Problem	Result
Number of ways of arranging n distinct objects in a row	$n!$
Number of ways of arranging any r things out of n distinct objects in a row	$\frac{n}{(n-r)!} = {}^nP_r$
Number of ways of arranging n objects in a row, where n_1 objects are identical and of one n_2 kind, objects are identical and of another kind, and so on, such that $$n_1 + n_2 + \cdots = n$$	$\frac{n!}{n_1!n_2!....}$

3. **Combinations:** The concept of combinations enables us to find the *number of ways* of *selecting* a set of objects of a particular size, from a larger set. Any particular selection of objects from the larger set will be one particular combination out of all the possible combinations. Combinations should always be associated with selections. Here are some particular examples of problems on combinations:

- Finding the number of ways of selecting r people out of n - in other words, counting the *number of* groups of r people which can be formed out of a group of n people $(r \le n)$
- Finding the number of ways of selecting only 2 unit squares on a chessboard
- Finding the number of all possible 3-letter groups from the letters of SELECTIONS, so that the letters are distinct (For example, $\{S, E, L\}$ is a valid selection, while $\{E, L, E\}$ is not.)

Here are two basic results on combinations, and once again, it would be a good idea to prove them using the FPC:

Problem	Result
Number of ways of selecting r objects out of n distinct objects	$\frac{n!}{r!(n-r)!} = {}^nC_r$
Number of ways of forming a sub-group (of any non-zero size) of objects out of a set of n distinct objects	$2^n - 1$

We note that ${}^nC_r = \frac{{}^nP_r}{r!}$ (in fact, this is how the formula for ${}^nC_r$ is derived in the first place).

4. Deciding whether a scenario involves permutations or combinations: The most important thing to remember is that

- Permutations correspond to arrangements
- Combinations correspond to selections

One of the best examples to make this association clear is as follows. We have a squad of 15 cricket players. Suppose that we have to form a playing team of 11 players out of these 15. Now, two cases arise:

1. **Just forming the team:** If it is required only to decide which of the 11 out of the 15 players will be playing, then the problem is one of selection, that is, a combination problem. You only have to ascertain which 11 of the 15 players will play, which means that you have to select a playing team of 11 out of the total 15 possible players.
2. **Deciding the team *and* its batting order:** Now, in addition to deciding which of the 11 out of 15 players will play, you also have to decide the order in which they will bat. This is therefore, a problem of selection with arrangement, that is, it is a permutation problem.

   Whenever you are in confusion regarding whether to apply permutations or combinations, thinking and visualizing of this example should help you a lot. Let's try it with two simple examples:

Illustration 1: How many words of 4 letters exist?

Working: In a word, the arrangement of letters obviously matters. For example, *ABCD* is different from *ACBD*. This is a problem where we have to count the arrangements of 4 letters, that is, the permutations of 4 letters, out of the 26 possible letters in the English alphabet. The answer is therefore ${}^{26}P_4$.

Illustration 2: From a group of 26 people, you have to send an expedition of 4 people to Mt. Everest. In how many ways can you form your team?

Working: What matters here is the combination or group of 4 people, not any order or arrangement. Your task is restricted to selecting the 4 people – you do not have to arrange them in any certain fashion. Therefore, this is a combination problem and the answer is ${}^{26}C_4$.
Reflect on the operative words 'Arrangements' vs 'Selections' whenever you are faced with the confusion of 'Permutations' vs 'Combinations'.

5. Important results

5.1 Combinatorial relations: The following combinatorial relations have algebraic as well as logical justifications, and are used frequently:

1. ${}^nC_r = {}^nC_{n-r}$
2. ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$
3. ${}^nP_r = {}^{n-1}C_r + r \cdot {}^{n-1}P_{r-1}$
4. ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n-r+1}{r}\,{}^nC_{r-1}$
5. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

It is strongly urged that you work out the logical (not the algebraic!) justification for each of these relations. Some of them are explored in the solved examples.

5.2 Integer equations: The number of solutions to the integer equation $x_1 + x_2 + \cdots + x_n = r$ is given by ${}^{n+r-1}C_r$. The justification for this extremely important result is given in the solved examples of this chapter.

5.3 Number and sum of factors: Let N be any number whose prime factor representation is

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$$

The number of factors of N (including 1 and N) is

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1)$$

The sum of these factors is given by

$$S = \left(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right) \cdots \left(1 + p_n + p_n^2 + \cdots + p_n^{\alpha_n}\right)$$

The justification for these important results can be found in the solved examples.

5.4 Division into groups: A commonly encountered problem in this subject is to find the number of ways of dividing a set of objects (which may be distinct or identical) into a number of groups, wherein the ordering of the groups themselves might or might not be important. For example, you might be asked to find the number of ways of dividing a deck of cards equally into 4 groups, or equally among 4 players. What is *really* important is to keep in mind that these two situations are very different. In dividing a deck of cards equally into 4 groups, the ordering of the groups *does not* matter, whereas, in dividing the deck among 4 players, the ordering *does* matter, because which group of cards goes to which player carries significance.

It is concepts like these which you must understand carefully while dealing with any problem on division into groups. To understand the different possible situations and methods of tackling them, refer to the solved examples again.

5.5 Circular Permutations: The number of ways of arranging n distinct objects in a circular manner is $(n-1)!$. However, in some situations, the clockwise or anticlockwise order of the objects may not matter (as in necklaces). In such cases, the number of circular permutations reduces to $\frac{1}{2}(n-1)!$.

To understand how these results come about, and more applications of circular permutations, refer to the solved examples.

6. Binomial Theorem for positive integral index: As the name suggests, a binomial expression is any expression of the form $(x + y)$, that is, it involves two terms. The binomial theorem helps us to expand a binomial expression raised to the power n where n is an arbitrary positive integer (a more general binomial theorem also exists, but as of now we are restricted to studying only positive integer powers):

$$(x + y)^n = (x + y)(x + y)(x + y) \cdots (x + y) \quad (n \text{ times})$$

We need to find out the coefficient of $x^i y^j$, where $i + j = n$, so that we can write a general term of the expression as $x^r y^{n-r}$, such that $0 \le r \le n$.

To find out the coefficient of $x^r y^{n-r}$, or in other words, to find out the *number of times* the term $x^r y^{n-r}$ is generated in the expansion, we note that $x^r y^{n-r}$ will be formed whenever x is *contributed* by r of the binomial terms, while y is *contributed* by the remaining $n - r$ of the binomial terms. The number of ways to select r terms out of n is ${}^nC_r$, and thus we can say that the coefficient of $x^r y^{n-r}$ will be ${}^nC_r$, leading to the following expansion:

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r x^r y^{n-r} = {}^nC_0 x^0 y^n + {}^nC_1 x^1 y^{n-1} + {}^nC_2 x^2 y^{n-2} + \cdots + {}^nC_n x^n y^0$$

It is imperative to understand how the concept of combinations led us to the binomial theorem. The coefficients ${}^nC_i$ are called the binomial coefficients, and we have already seen many of their properties while studying permutations and combinations.

6.1 Important Results: Considering the expansion of $(x+y)^n$, we note the following important results:

(a) The binomial coefficients of terms equidistant from the beginning and the end are equal, because ${}^nC_r = {}^nC_{n-r}$

(b) ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

(c) There are $(n+1)$ terms in the expansion; if n is even, there will be an odd number of terms, and thus there will be only one middle term, which would be ${}^nC_{n/2}x^{n/2}y^{n/2}$. For example,

$$(x+y)^4 = x^4 + 4x^3y + \underbrace{6x^2y^2}_{\substack{\text{only one}\\ \text{middle term}}} + 4xy^3 + y^4$$

On the other hand, if n is odd, then there will be an even number of terms in the expansion, and thus there will be two middle terms, namely ${}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}}$ and ${}^nC_{\frac{n-1}{2}}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}}$. For example,

$$(x+y)^5 = x^5 + 5x^4y + \underbrace{10x^3y^2 + 10x^2y^3}_{\text{Two middle terms}} + 5xy^4 + y^5$$

(d) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$. We can prove this result by substituting $x = y = 1$ in the expansion.

(e) ${}^nC_0 - {}^nC_1 + {}^nC_2 - \cdots + (-1)^n\ {}^nC_n = 0$. To prove this, we can substitute $x = 1$ and $y = -1$ in the expansion.

(f) The greatest coefficient in the expansion will be the coefficient of the middle term(s):

n is odd: ${}^nC_{\frac{n-1}{2}},\ {}^nC_{\frac{n+1}{2}}$

n is even: ${}^nC_{\frac{n}{2}}$

(g) To find the greatest *term* in the expansion, the general idea is to consider the ratio of successive terms:

$$q = \frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^{n-r} y^r}{{}^nC_{r-1} x^{n-r+1} y^{r-1}} = \frac{(n-r+1)}{r} \cdot \frac{y}{x}$$

Starting from 0, we increase the value of r. As long as $q > 1$, we have $T_{r+1} > T_r$, and the sequence of terms is increasing. As soon as q becomes less than 1, the sequence starts to decrease. The point at which the switch occurs (from increasing to decreasing) corresponds to the greatest term.

6.2 Differentiation and Integration in Binomial Expansions: The techniques of Calculus enable us to sum a lot of series involving binomial coefficients. Suppose that we have to evaluate the sum S given by

$$S = {}^nC_1 + 2\ {}^nC_2 + 3\ {}^nC_3 + \cdots + n\ {}^nC_n$$

To avoid clutter, we'll write ${}^nC_r$ as simply C_r, where the upper index n should be understood to be present. Thus,

$$S = C_1 + 2C_2 + \cdots + {}^nC_n = \sum rC_r$$

This series can be generated using a manipulation involving differentiation, as follows. Consider the binomial expansion

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

If we differentiate both sides with respect to x, look at what we'll obtain:

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1}$$

Now, all that remains is to substitute $x = 1$, upon which we obtain:

$$n \cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \cdots + nC_n$$

This is what we were looking for. Thus, $S = n \cdot 2^{n-1}$. Had we substituted $x = -1$, we would've obtained:

$$0 = C_1 - 2C_2 + 3C_3 \cdots + (-1)^{n-1} nC_n$$

Thus, we have evaluated another interesting sum. Suppose that we now wish to evaluate S_1 given by

$$S_1 = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$$

The alert reader would immediately realize that integration needs to be applied here. Consider again the expansion

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

If we integrate this with respect to x, between some limits say a to b, we obtain

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_a^b = C_0x\Big|_a^b + C_1\left.\frac{x^2}{2}\right|_a^b + C_2\left.\frac{x^3}{3}\right|_a^b + \cdots + C_n\left.\frac{x^{n+1}}{n+1}\right|_a^b$$

To generate the sum S_1, a little thought will show that we need to use $a = 0$, $b = 1$, so that we obtain

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$$

Thus, S_1 equals $\frac{2^{n+1}-1}{n+1}$. Try some other values for a and b and hence generate other series on your own.

6.3 Miscellaneous Techniques: Not all questions can be subjected to the method(s) described earlier. For example, consider the sum S given by

$$S = C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n$$

Let us first go through a combinatorial approach, using the observation that $C_0 = C_n$, $C_1 = C_{n-1}$ and so on, so that S can be rewritten as

$$S = C_nC_1 + C_{n-1}C_2 + C_{n-2}C_3 + \cdots + C_1C_n$$

Consider a general term of this sum, which is of the form $C_{n-r}\,C_{r+1}$. We can think of this as the number of ways of selecting $(n - r)$ boys from a group of n boys and $(r + 1)$ girls from a group of n girls. The total number of people we are thus selecting is $(n-r)+(r+1)=(n+1)$. Therefore, S represents the *total* number of ways of selecting $(n + 1)$ people out of a group of $2n$, so that S is simply ${}^{2n}C_{n+1}$.

Now to a binomial approach. This will involve generating the general term $C_r\ C_{r+1}$ somehow, which is the same as $C_{n-r}\ C_{r+1}$. Consider the general expansion of $(1+x)^n$.

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \tag{1}$$

We have to have the terms C_nC_1, $C_{n-1}C_2$ and so on, which suggests that we write (1) twice, but in the second expansion we reverse the terms, multiply, and see what terms contain the (combinations of) coefficients we require.

$$(1+x)^n = C_0 \quad + C_1x \quad + C_2x^2 \quad + \ldots + C_nx^n$$

$$(1+x)^n = C_nx^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \ldots + C_1x + C_0$$

Multiplying, we find that on the left hand side we have $(1+x)^{2n}$, while on the right hand side, the terms containing the (combinations of) coefficients we want will always be of the form $(\quad)x^{n+1}$, that is, the power of x will be $(n + 1)$. No other terms will contain x^{n+1}; verify this for yourself. Thus, the sum $C_n\,C_1 + C_{n-1}C_2 + \ldots + C_1C_n$ is actually the *total* coefficient of x^{n+1} on the right hand side, and from the left hand side we know that the coefficient of x^{n+1} would be simply ${}^{2n}C_{n+1}$. Thus, $S = {}^{2n}C_{n+1}$

A very similar approach could have been:

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

$$\left(1 + \frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + \cdots + C_n\frac{1}{x^n}$$

Thus, $\quad S =$ Coefficient of x in $(1+x)^n(1+\frac{1}{x})^n$ = Coefficient of x in $\frac{(1+x)^{2n}}{x^n}$

$\quad =$ Coefficient of x^{n+1} in $(1+x)^{2n} = {}^{2n}C_{n+1}$

IMPORTANT IDEAS AND TIPS

1. *The Importance of The FPC*. The Fundamental Principle of Counting is the basis of the entire subject of Permutations and Combinations, and yet students frequently make the mistake of ignoring or underestimating it. Every formula that you've studied in this chapter derives from the FPC. In some situations, where no standard formula can be applied, you can proceed using the FPC. Therefore, it is important to understand what the FPC is, why it is true, and in particular, the independence of the sub-tasks which constitute the main task to which we are applying the FPC. Secondly, as mentioned earlier, we are interested in counting the number of ways of doing a task, rather than analyzing specific ways of doing it.
2. *Distinguishing Between Permutations and Combinations*. As most students would agree, the major pitfall in this chapter is deciding whether a given problem is one of permutations or combinations (or some other variant). We emphasize once again the following association: permutations correspond to arrangements, while combinations correspond to selections. Whenever in doubt about this, recall the cricket example we discussed.
3. *Tackling New Scenarios*. Another major stumbling block is trying to fit every given scenario into one of the standard cases / formulae you have studied (we have even seen students trying to apply the formulae for ${}^nC_r$ and ${}^nP_r$ to every problem they come across). Students tend to search for the closest match of the

given scenario to one of the few standard cases they have studied. Many times, however, the scenario will be completely new, and no standard formulae would apply. To decide how to proceed in a seemingly new situation, first write out a few test cases of whatever it is you are trying to count. Let's illustrate this with an example. Suppose that you want to count the number of ways you can fill up a soft-drinks crate (which can hold 24 bottles) with bottles of Sprite, Coke, Pepsi, and Dew. How would you proceed? Rather than trying to find the closest match from what you've studied, first write down a few test cases:

4	+	7	+	3	+	16	=	24
Sprite Bottles		Coke Bottles		pepsi Bottles		Dew Bottles		Total Bottles
6	+	0	+	16	+	2	=	24
Sprite Bottles		Coke Bottles		Pepsi Bottles		Dew Bottles		Total Bottles

Doing this exercise will give you a clearer picture of what you are trying to count, and hopefully shed more light on the right way to proceed. In this case, you see that if you represent the number of bottles of the different brands by x_{Sprite}, x_{Coke}, x_{Pepsi} and x_{Dew}, then what you are actually trying to count is the number of non-negative integer solutions to the following equation:

$$x_{\text{Sprite}} + x_{\text{Coke}} + x_{\text{Pepsi}} + x_{\text{Dew}} = 24$$

Once you've made this connection, finding the right answer is only a matter of calculation.

4. *Identical and Distinct Objects*. Suppose that you were asked to find out the number of groups of 4 letters you can form from the the letters of the word ARRANGEMENTS. There are 12 letters in this word, but the answer is not ${}^{12}C_4$. This is because the standard formulae for ${}^nC_r$ and ${}^nP_r$ hold only when the objects are all distinct. However, this simple fact is overlooked many times. In the problem just mentioned, we would proceed by splitting the scenario into a number of cases:

   Case 1: All letters in the group are distinct
   Case 2: Two letters are the same, the other two are distinct
   Case 3: Three letters are the same, the fourth one is different
   Case 4: All four letters are the same

   We observe that Case-3 and Case-4 are not possible. Therefore, we count the number of ways in which Case-1 and Case-2 can occur, and add to obtain our answer.

5. *Division Into Groups.* Another frequently encountered pitfall pertains to counting the number of ways of dividing a given set of objects into a number of groups. The problem occurs in distinguishing the following two cases:

   (a) the order of the groups matters, *i.e.*, groups can be assigned an order
   (b) the order of the groups does not matter, *i.e.*, groups cannot be assignment an order

   The best way to deal with any confusion related to this is to think of the situation of dividing a standard deck of 52 cards into 4 equal groups of 13 cards each (which we have already described in the theory and are repeating here again for the sake of emphasis):

   (a) if the resulting groups are to be distributed to 4 players, then the groups can be assigned an order, because which group goes to which player matters.
   (b) if all we need to do division (no distribution), then the groups cannot be assigned an order; all that matters is the cards in each group.

6. *Integer Equations*. Counting the number of non-negative solutions to an integer equation is a frequently encountered problem, and it occurs in direct and indirect forms. To be able to solve such problems correctly, you must understand the justification behind the general formula ${}^{n+r-1}C_r$ which tells us the number of non-negative solutions to the following equation:

$$x_1 + x_2 + \cdots + x_n = r$$

Refer to the solved examples for a proof of this result and carefully observe (in fact memorize) the solution.

7. *Logical Analysis of Binomial Expressions*. Consider the following binomial identity:

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$$

If asked to prove this, how would you proceed? It is likely that you would consider the binomial expansion of $(1+x)^n$ and substitute $x = 1$ as discussed in the theory. However, a much more instructive logical justification is possible and is described in the solved examples. It is important to appreciate that many binomial identities have such logical justifications, and to try and look for them. The binomial theorem itself can be justified logically, as is discussed in the preceding pages:

$$(x+y)^n = (x+y)(x+y)\cdots(x+y) \quad n \text{ times}$$

The coefficient of a term $x^r y^{n-r}$ corresponds to the number of times this term is generated when we expand the product on the right side. To form this term, r of the n brackets on the right side have to contribute x, and the remaining n-r of the n brackets have to contribute y. The number of ways of selecting r objects out of n is ${}^nC_r$ which means that the term $x^r y^{n-r}$ will be formed ${}^nC_r$ times; thus, the coefficient of the term $x^r y^{n-r}$ is ${}^nC_r$ and we can now write the binomial expansion as

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^r y^{n-r}$$

8. *Greatest Term and Greatest Coefficient.* In a binomial expansion, say $(x+y)^n$, do not confuse the greatest term and the greatest coefficient. The latter is simply the greatest among the binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \cdots, {}^nC_n$. The former depends on x and y as well.

Permutations and Combinations

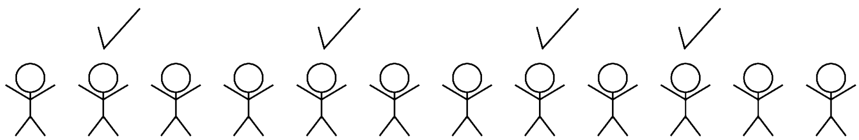
PART-B: Illustrative Examples

Example 1

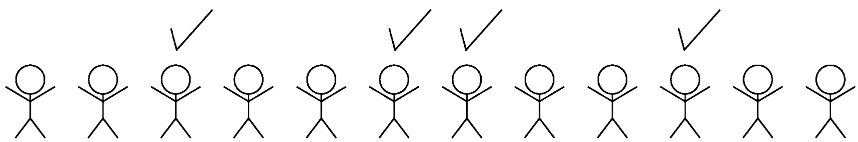
12 person are sitting in a row. In how many ways can we select 4 persons out of this group such that no two are sitting adjacent in the row?

(A) ${}^7C_4$ (B) ${}^8C_4$ (C) ${}^9C_4$ (D) ${}^{10}C_4$

Solution: To make the question more clear, here's a valid and a non-valid selection:



Valid



Non-valid

We have to find some (mathematical) way of specifying a valid-selection. An obvious method that strikes is this: In our row, we represent every unselected person by U and every selected person by S. Thus, the valid selection in the figure above becomes:

U S U U S UU S U S UU

To count the numbers of valid selections, we count the number of permutation of this string above, consisting of 8 'U' symbols and 4 'S' symbols, subject to the constraint that no two 'S' symbols are adjacent. To count such permutation, we first fix the 8 'U' symbols. There will then be 9 blank spaces generated for the 'S' symbols as shown below:

__U__U__U__U__U__U__U__U__

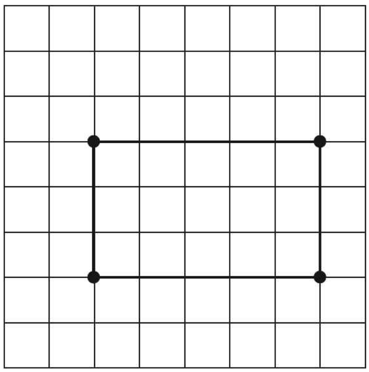
From these 9 blank spaces, any 4 spaces can be chosen and the 'S' symbols can be put there and it'll be guaranteed that no two 'S' symbols will be adjacent. Thus, our task now reduces to just selecting 4 blank spaces out of the 9 available to us, which can be done simply in ${}^{9}C_4$ ways. These are the required number of selections. The correct option is thus (C). ■

Example 2

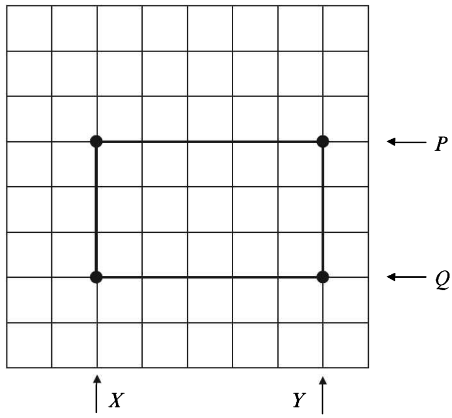
(a) The number of rectangles on a standard 8×8 chessboard is ____________

(b) The number of squares on a standard 8×8 chessboard is ____________

Solution: (a) Visualize any arbitrary rectangle on the chessboard, say the one depicted on the left in the figure below:



An arbitrary rectangle on the chessboard. How to specify this rectangle is explained in the figure on the right.



To mathematically characterize the rectangle that we selected, we specify the pair of vertical edges X and Y and the pair of horizontal edges P and Q. Doing so uniquely determines the rectangle.

As explained in the figure above, any rectangle that we select can be uniquely determined by specifying the pair of lines X and Y that make up the vertical edges of the rectangle and the pair of lines P and Q that make up the horizontal edges of the rectangle. On the chessboard, there are 9 vertical lines available to us from which we have to select 2. This can be done in ${}^{9}C_2$ ways. Similarly, 2 horizontal lines can be selected in ${}^{9}C_2$ ways. Thus, the total number of rectangles that can be formed is ${}^{9}C_2 \times {}^{9}C_2 = 1296$.

(b) To select a square, observe that the pair of lines X and Y must have the same spacing between X and Y as the pair of lines P and Q. Only then can the horizontal and vertical edges of the selected rectangle be of equal length (and thus, the selected rectangle is actually a square). In how many ways can we select a pair (X, Y) of lines which are spaced a unit distance apart ? The answer is obviously 8. Corresponding to each of these 8 pairs, we can select a pair (P, Q) of lines in 8 ways such that P and Q are a unit distance apart. Thus, the total number of unit squares is $8 \times 8 = 64$ (This is obvious otherwise also). Now we count the number of 2×2 squares. In how many ways can we select a pair (X, Y) of lines which are 2 units apart ? A little thought shows that it will be 7. Corresponding to each of these 7 pairs, we can select a pair (P, Q) of lines in 7 ways such that P and Q are 2 units apart. Thus, the total number

of 2 × 2 squares is 7 × 7 = 49. Reasoning this way, we find that the total number of 3 × 3 squares will be 6 × 6 = 36, the total number of 4 × 4 squares will be 5 × 5 = 25 and so on. Thus, the total number of all possible squares is

$$\underset{\substack{\uparrow\\1\times1\\\text{squares}}}{64} + \underset{\substack{\uparrow\\2\times2\\\text{squares}}}{49} + \underset{\substack{\uparrow\\3\times3\\\text{squares}}}{36} + \cdots\cdots\cdots + \underset{\substack{\uparrow\\7\times7\\\text{squares}}}{4} + \underset{\substack{\uparrow\\8\times8\\\text{squares}}}{1} = 204$$

■

Example 3

Consider an n-sided convex polygon.

(a) The number of ways in which a quadrilateral can be formed by joining the vertices of the polygon is

(A) $2\cdot{}^nC_3$ (B) $\dfrac{{}^nC_3 + {}^nC_5}{2}$ (C) ${}^nC_4$ (D) None of these

(b) The number of diagonals which can be formed in the polygon is

(A) $\dfrac{n(n-1)}{2}$ (B) $\dfrac{n(n-2)}{2}$ (C) $\dfrac{n(n-3)}{2}$ (D) $\dfrac{n(n-4)}{2}$

Solution: (a) Observe that a selection of any 4 points out of the n vertices of the quadrilateral will give rise to a unique quadrilateral (since the polygon is convex, the problem of our selection containing all or 3 collinear points does not exist). 4 points out of n can be selected in ${}^nC_4$ ways. Thus, we can have ${}^nC_4$ different quadrilaterals. The correct option is (C).

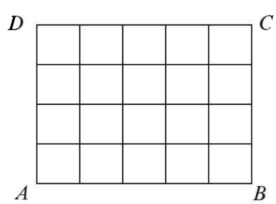
(b) To form a diagonal, we need 2 non-adjacent vertices (because 2 adjacent vertices will form a side of the polygon and not a diagonal). The total number of ways of selecting 2 vertices out of n is ${}^nC_2$. This number also contains the selections where the 2 vertices are adjacent. Those selections are simply n in number because the polygon has n sides. Thus, the total number of diagonals is

$${}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

The correct option is (C) again. ■

Example 4

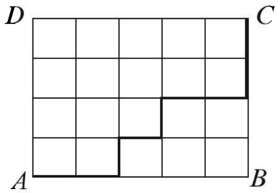
Consider a rectangular integral grid of size $m \times n$. For example, a 4×5 integral grid is drawn below:



A person has to travel from point $A(0, 0)$ to the diagonally opposite point $C(m, n)$. He moves one step at a time, towards the east or towards the north (that is, never moves towards the west or south at any time). How many distinct paths exist from the point A to the point C?

(A) $\dfrac{(m+n)!}{(m+1)!\times(n+1)!}$ (B) $\dfrac{(m+n-1)!}{m!\times n!}$ (C) $\dfrac{(m+n+1)!}{m!\times n!}$ (D) $\dfrac{(m+n)!}{m!\times n!}$

Solution: Let us draw a random path on our 4×5 grid in the figure above and think of some way to mathematically specify/describe this path:



A random path across the 4×5 grid that our travelling person can follow.
The question now is:
How do we mathematically characterize this path?

Suppose you had to describe this path to a person who cannot see the figure for some reason. If you use E for a step towards the east and N for a step towards the north, you'd tell that person that the travelling person took the following path:

'E E N E N E E N N'

This string that we just formed should immediately make you realise how to calculate the number of all the possible paths. We have 5 'E' steps and 4 'N' steps. Any permutation of these 9 steps gives rise to a different unique path. For example, the string **'E E E E E N N N N'** is the path that goes straight east from A to B and then straight north from B to C. Thus, any path can be uniquely characterised by a permutation of these 9 steps. The number of permutations of these 9 letters, 5 of which are 'E's and 4 are 'N's, is $\frac{9!}{5!4!}$. This is therefore the number of different paths that the travelling person can take from A to C. For an $m \times n$ grid we will have $(m + n)$ total steps, m of them being 'E' s and the remaining n being 'N' s. Thus, the number of possible paths is $\frac{(m+n)!}{m!\times n!}$. The correct option is (D). ■

Example 5

What is the value of the sum S given by the following?

$$S=\sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$$

(A) $\dfrac{3}{n+3}$ (B) $\dfrac{3}{n+6}$ (C) $\dfrac{6}{n+3}$ (D) $\dfrac{6}{n+6}$

Solution: We have

$$\frac{{}^nC_r}{{}^{r+3}C_r}=\frac{n!}{r!(n-r)!}\frac{r!3!}{(r+3)!}$$

$$=\frac{6\cdot n!}{(n-r)!(r+3)!}=\frac{6}{(n+1)(n+3)(n+3)}\cdot\frac{(n+3)!}{(r+3)!(n-r)!}$$

$$=\frac{6}{(n+1)(n+2)(n+3)}\,{}^{n+3}C_{r+3}$$

Thus,

$$S=\frac{6}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^r\;{}^{n+3}C_{r+3} \tag{1}$$

If we represent the summation expression in (1) by S_1, then

$$S_1=\sum_{r=0}^{n}(-1)^r\,{}^{n+3}C_{r+3}=\sum_{s=3}^{n+3}(-1)^{s-3}\,{}^{n+3}C_s\;=-\sum_{s=3}^{n+3}(-1)^s\,{}^{n+3}C_s$$

$$=-\left\{\underbrace{\sum_{s=0}^{n+3}(-1)^s\;{}^{n+3}C_s}_{T_1}-\underbrace{({}^{n+3}C_0-{}^{n+3}C_1+{}^{n+3}C_2)}_{T_2}\right\}$$

The term T_1 is 0 (why?), while

$$T_2=1-(n+3)+\frac{(n+3)(n+2)}{2}=\frac{(n+1)(n+2)}{2}$$

From (1),

$$S=\frac{3}{n+3}$$

The correct option is (A). ■

Example 6

If $\Sigma_{r=0}^{2n}a_r(x-2)^r=\Sigma_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$ for all $k\ge n$, then what is the value of $\dfrac{b_n}{{}^{2n+1}C_{n+1}}$?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2

Solution: To understand the provided equality better, we expand the sum S_1 on the left side:

$$S_1=a_0+a_1(x-2)+a_2(x-2)^2+\cdots+(x-2)^n+(x-2)^{n+1}+\cdots+(x-2)^{2n}$$

We have used the fact that $a_k=1$ for all $k\ge n$. Now, we express S_1 in terms of $t=(x-3)$ by using $(x-3)+1$ instead of $(x-2)$.

$$S_1=a_0+a_1(1+t)+a_2(1+t)^2+\cdots+(1+t)^n+(1+t)^{n+1}+\cdots+(1+t)^{2n}$$

Note that b_n is the coefficient of t^n on the right side. Comparing with the coefficient of t^n in S_1, we have:

$$b_n={}^nC_0+{}^{n+1}C_1+{}^{n+2}C_2+\cdots+{}^{2n}C_n$$

Using ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$ successively on the right side above, we can easily show that

$$b_n={}^{2n+1}C_{n+1}$$

Thus, the required answer is 1. The correct option is (C). ■

Example 7

Let n be a positive integer and

$$(1+x+x^2)^n=a_0+a_1x+\cdots+a_{2n}x^{2n}$$

What is the value of $\dfrac{a_0^2 - a_1^2 + \cdots + a_{2n}^2}{a_n}$?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2

Solution: To generate the term $S = a_0^2 - a_1^2 + \cdots + a_{2n}^2$, we use the multinomial expansion provided in the problem in the appropriate manner:

$$(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$$

$$\Rightarrow \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{2n}}{x^{2n}}$$

Multiplying these two expressions, we observe that S is the constant term in the product $(1+x+x^2)^n \, (1 - \frac{1}{x} + \frac{1}{x^2})^n$, *i.e.*, the coefficient of x^{2n} in $(1+x^2+x^4)^n$, which is immediately obvious to be a_n. Thus, $S = a_n$, and the required answer is 1. The correct option is (C). ■

SUBJECTIVE TYPE EXAMPLES

Example 8

Prove that ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$.

Solution: You can prove this assertion yourself very easily analytically. We will discuss a logical justification. The left hand side of this assertion says that we have a group of n people out of which we want to select a subset of r people; more precisely, we want to count the number of such r-subsets. Fix a particular person in this group of n people, say person X. Now, all the r-groups that we form will either contain X or not contain X. These are the only two options possible.

(i) To count the number of groups that contain X, we proceed as follows: we already have X; we need $(r-1)$ more people from amongst $(n-1)$ people still available for selection. Thus, such groups will be ${}^{n-1}C_{r-1}$ in number.
(ii) For the number of groups that do not contain X we need to select r people from amongst $(n-1)$ options available. Therefore, such groups are ${}^{n-1}C_r$ in number.

The total number of r-groups are hence ${}^{n-1}C_{r-1} + {}^{n-1}C_r$ in number, which is the same as ${}^nC_r$. ■

Example 9

Prove logically that ${}^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$.

Solution: The analytical justification is again very straightforward and is left to you as an exercise. The left hand side of this assertion says that we need to count the number of arrangements of n people, taken r at a time. We again fix a particular person, say person X. All the possible r-arrangements will either contain X or not contain X. These are the only two (mutually exclusive) cases possible.

(i) If we do not keep X in our permutation, we have r people to select from a potential group of $(n-1)$ people. The number of arrangements not containing X will therefore be ${}^{n-1}P_r$.
(ii) To count the number of permutations containing X, we first seat X in one of the r seats available. This can be done in r ways. The remaining $(r-1)$ seats can be filled by $(n-1)$ people in ${}^{n-1}P_{r-1}$ ways. Thus, the number of arrangements containing X is $r \cdot {}^{n-1}P_{r-1}$. These arguments prove that ${}^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$. ■

Example 10

(a) Prove that ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ (b) Prove that ${}^nC_r = \frac{n-r+1}{r}\,{}^nC_{r-1}$

Solution: (a) Let us consider this assertion in a particular example, say with $n = 6$ and $r = 4$. This will make things easier to understand. Our purpose is to select 4 people out of 6 people, say the set $\{A, B, C, D, E, F\}$. To select a group of 4, we can first select a single person: this can be done in 6 ways. The rest of the 3 people can now be selected in ${}^5C_3$ ways. The total number of groups possible would thus be $6 \times {}^5C_3$. But some careful thought will show that we have *overcounted*, doing the calculation this way.

Suppose that in our first step, we select A. While selecting the remaining 3 persons, we then select B, C, D thus forming the group $\{A, B, C, D\}$. But this same group would have been formed had we selected B in our first step and A, C, D in the second step, or C in the first step and A, B, D in the second step, or D in the first step and A, B, C in the second

step. We have thus counted the group $\{A, B, C, D\}$ 4 times in the figure $6 \times {}^5C_3$. The actual number of groups will hence be $\frac{6 \times {}^5C_3}{4}$.

We now generalise this: to select r people out of a group of n, we first select one person; this can be done in n ways. The remaining $(r-1)$ persons can be selected in $n \times {}^{n-1}C_{r-1}$ ways. The total number of r-groups thus becomes $n \times {}^{n-1}C_{r-1}$. However, as described earlier, in this figure each group has been counted r times. The actual number of r-groups is therefore:

$$ {}^nC_r = \frac{n \times {}^{n-1}C_{r-1}}{r} $$

(b) The logic for this part is similar to that of Part -(a) To select r people out of n, we first select $(r-1)$ people out of n. This can be done in ${}^nC_{r-1}$ ways. We now have $n-(r-1)=(n-r+1)$ persons remaining for selection out of which we have to choose 1 more person. This can therefore be done in $(n-r+1)$ ways. The total number of r-groups thus becomes $(n-r+1) \times {}^nC_{r-1}$. However, each r-group has again been counted r times in this figure (convice youself about this by thinking of a particular example). The actual number of r-groups is thus

$$ {}^nC_r = \frac{(n-r+1) \times {}^nC_{r-1}}{r} $$

■

Example 11

Prove logically that ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

Solution: Lets first interpret what the left side means. ${}^nC_0$ is the number of ways in which we can select *nothing* out of n things (there will obviously be only one such way: that we do nothing!). ${}^nC_1$ is the number of ways in which we can select 1 thing out of n. ${}^nC_r$ is the number of ways of selecting r things out of n. We want the value of the sum $\Sigma_{r=0}^{n} {}^nC_r$, which is the number of all groups possible of any size whatsoever. Thus, our selection could be any size from 0 to n (both inclusive); what we want is the total number of selections possible.

For example, consider the set $\{A, B, C\}$. The set of all possible selections that can be made from this set is $\{\phi, \{A\}, \{B\}, \{C\}, \{AB\}, \{AC\}, \{BC\}, \{ABC\}\}$. Thus, 8 total different selections are possible (note that $8 = 2^3$).

To count the total number of selections possible if we have n persons, we adopt an individual's perspective. An individual can either be or not be in our selection. Thus, we have two choices with respect to any individual; we either put him in our group or do not put him in our group. These two choices apply to every individual. Also, choosing or not choosing any individual is independent of choosing or not choosing another. Thus, the total number of ways in which an arbitrary number of individuals can be selected from n people is $\underbrace{2 \times 2 \times 2 \times \ldots\ldots\ldots \times 2}_{n \text{ times}} = 2^n$. This proves that:

$$ \sum_{i=0}^{n} {}^nC_i = 2^n. $$

■

Example 12

Prove logically that ${}^{n+m}C_r = {}^nC_0\,{}^mC_r + {}^nC_1\,{}^mC_{r-1} + {}^nC_2\,{}^nC_{r-2} + \cdots + {}^nC_r\,{}^mC_0$.

Solution: We interpret the left hand side as the number of ways of select r people out of a group of $(n + m)$ people.

Let this group of $(n + m)$ people consist of n boys and m girls. A group of r people can be made in the following ways:

No.	Group contains	No. of ways possible
1	0 boys, r girls	${}^nC_0 \times {}^mC_r$
2	1 boy, $(r-1)$ girls	${}^nC_1 \times {}^mC_{r-1}$
3	2 boys, $(r-2)$ girls	${}^nC_2 \times {}^mC_{r-2}$
.		
.		
.		
r	$(r-1)$ boys, 1 girl	${}^nC_{r-1} \times {}^mC_1$
$(r+1)$	r boys, 0 girls	${}^nC_r \times {}^mC_0$

This table is self-explanatory. The $(r + 1)$ types of groups that have been listed are mutually exclusive. Thus, the total number of r-groups is:

$${}^{n+m}C_r = {}^nC_0\,{}^mC_r + {}^nC_1\,{}^mC_{r-1} + \cdots + {}^nC_r\,{}^mC_0 \qquad \blacksquare$$

Example 13

Give a combinatorial (logical) justification for this assertion:

$${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+r}C_r = {}^{n+r+1}C_r$$

Solution: The right hand side tells us that we have to select r persons out of a group of $n+r+1$ persons. To do so, we consider any particular group of r persons from these $n+r+1$ persons. Specify these r persons by the symbols $A_1, A_2, \ldots, A_r$. Now, to count all the possible r-groups from this group of $n+r+1$, we consider the following mutually exclusive cases:

(1) **The r-group does not contain A_1:** Such r-groups can be formed in ${}^{n+r}C_r$ ways since we have to select r people out of $n + r$.

(2) **The r-group contains A_1 but not A_2:** We have to select $(r - 1)$ people out of $(n+r-1)$ because we already have selected A_1, so we need only $r-1$ more people and since we are not taking A_2, we have $(n+r-1)$ people to choose from. This can be done ${}^{n+r-1}C_{r-1}$ ways.

(3) **The r-group contains A_1, A_2 but not A_3:** We now have to select $(r - 2)$ people out of $(n+r-2)$. This can happen in ${}^{n+r-2}C_{r-2}$ ways.

Proceeding in this way, we arrive at the last two possible cases.

$\vdots$

(r) **The r-group contains $A_1, A_2, \ldots, A_{r-1}$ but not A_r:** We need to select only 1 person out of $(n + 1)$ available for selection. This can be done in ${}^{n+1}C_1$ ways.

(r + 1) The *r*-group contains $A_1, A_2, ..., A_r$: In this case, our *r*-group is already complete. We need not select any more person. This can be done in ${}^{n}C_0$ or equivalently 1 way.

Convince yourself that these (*r* + 1) cases cover all the possible cases that can arise in the formation of the *r*-groups. Also, all these cases are mutually exclusive. Thus, adding the number of possibilities of each case will give us the total number of *r*-groups possible, *i.e.*,

$${}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+r-1}C_{r-1} + {}^{n+r}C_r = {}^{n+r+1}C_r$$ ■

Example 14

Consider the integer equation $x_1 + x_2 = 4$ where $x_1, x_2 \in \mathbb{Z}$. The non-negative solutions to this equation can be listed down as $\{0,4\}$, $\{1,3\}$, $\{2,2\}$, $\{3,1\}$ and $\{4,0\}$. Thus, 5 non-negative integer solutions exist for this equation.

We would like to solve the general case. How many non-negative, integer solutions exist for the following equation?

$$x_1 + x_2 + \cdots + x_n = r$$

Solution: Many of you already know the answer to this general problem, but it is much more important to know the justification. Let us consider an arbitrary integer equation, say $x_1 + x_2 + x_3 = 8$. Consider any particular non-negative integer solution to this equation, say $\{2,3,3\}$. We somehow need to *tag* this solution in a new form; a form which is easily countable. This is how we do it. We break up the solution 2 + 3 + 3 = 8 as shown below:

$$11 \square\ 111 \square\ 111 = 8 \tag{1}$$

Similarly, 1 + 6 + 1=8 would be written as

$$1 \square\ 111111 \square 1 = 8 \tag{2}$$

and 0 + 1 + 7 = 8 would be written as

$$\square\ 1 \square 1111111 = 8 \tag{3}$$

and 0 + 0 + 8 = 8 would be written as

$$\square \square 11111111 = 8 \tag{4}$$

An alert reader must have realised the 'trick' by now. In each of (1), (2), (3) and (4), we have on the left hand side 8 '1' symbols and 2 '$\square$' symbols, in different orders. Any non-negative integer solution can thus be represented by a unique permutation of 8 '1' symbols and 2 '$\square$' symbols. Conversely, every permutation of 8 '1' symbols and 2 '$\square$' symbols represents a unique non-negative integer solution to the equation .

Thus, the set of non-negative integer solutions to the equation and the set of permutations of 8 '1' symbols and 2 '$\square$' symbols are in one-to-one-correspondence. To count the required number of solutions, we simply count the permutations of 8 '1' symbols and 2 '$\square$' symbols, which would be $\frac{(8+2)!}{8!2!} = \frac{10!}{8!2!} = 45$.

We now generalise this result. Any non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$ can be represented using *r* '1' symbols and *n* – 1 '$\square$' symbols. The total number of permutations of these symbols will be

$$\frac{(n+r-1)!}{r!(n-1)!} = {}^{n+r-1}C_r$$

and hence, this is the required number of solutions. ■

Example 15

(a) We have m apples, n oranges and p bananas. In how many ways can we make a non-zero selection of fruit from this assortment?
(b) How many factors does 1,44,000 have? In general, how many factors does a natural number N have?

Solution: These two seemingly unrelated questions have exactly the same approach to their solutions! Before reading the solution, can you imagine how?

(a) The most important point to realise in this question is the nature of objects to be selected. We have m apples. These m apples are exactly identical to each other. You cannot make out one apple from another. This means that if you have to choose r apples out of n, there's only one way of doing it: you just pick (any of the) r apples. It doesn't matter *which* apples you choose, because all the apples are identical. Thus, only 1 r-selection is possible. So how many total selections are possible? We either select 0 apples, 1 apple, 2 apples,..., r apples ,..., or m apples. Thus $(m + 1)$ ways exist to select apples. We therefore have $(m + 1)$ ways to select apples, $(n + 1)$ ways to select oranges and $(p + 1)$ ways to select bananas. The selection of a particular fruit is independent of the selection of another fruit. Hence, we have $(m + 1)(n + 1)(p + 1)$ ways to select a group of fruit.

But we have overlooked something! In the $(m + 1)$ ways of selecting apples, there's also one way in which we select no apple. Similarly, in the $(n + 1)$ ways of selecting oranges, there's one in which we select no orange, and in the $(p + 1)$ ways of selecting bananas, there's one in which we select no banana. Thus, in the product $(m + 1)(n + 1)(p + 1)$, there'll be one case involving no fruit of any type. We have to exclude this case if we want a non-zero selection of fruit. Therefore, the number of ways of making a non-zero selection is $(m + 1)(n + 1)(p + 1) - 1$.

(b) Lets consider a smaller number first. Take 60, for example, and list down all its factors (including 1 and 60):

$$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

From elementary mathematics, you know that any number can be factorized into a product of primes. For example, 60 can be written in its prime factorization from as:

$$\begin{aligned} 60 &= 2 \times 2 \times 3 \times 5 \\ &= 2^2 \cdot 3^1 \cdot 5^1 \end{aligned}$$

Such a representation exists for every natural number N. Can we somehow use this representation to find the number of factors? Consider any factor of 60, say 12, The prime factorization form of 12 is $2^2 \cdot 3^1$. Similarly, this representation for 15, for example, is $3^1 \cdot 5^1$ and for 30 would be $2^1 \cdot 3^1 \cdot 5^1$. With this discussion in our mind, we rephrase our problem: We have 60 whose prime representation is $2^2 \cdot 3^1 \cdot 5^1$. Thus, we have 2 twos, 1 three and 1 five with us. You could imagine that we have 2 apples, 1 orange and 1 banana.

To form a factor of 60, what we have to do is to make a selection of prime factors from amongst the available prime factors. For 60, we have 2 twos (apples), 1 three (orange) and 1 five (banana). In how many ways can we make our selection?

The last part tells is that we can do it in $(m+1)(n+1)(p+1)$ ways or $(2+1)(1+1)(1+1) = 12$ in this particular case. We also allow no selection of any prime factor, this corresponds to

the factor 1 of 60. For the general case of a natural number N whose prime representation is of the form $p_1^{\alpha_1} p_2^{\alpha_2}, \ldots, p_n^{\alpha_n}$, the number of factors (including 1 and N) is:

$$(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_n + 1)$$

For example, 1,44,000 can be represented as

$$1,44,000 = 2^7 \cdot 3^2 \cdot 5^3.$$

The required number of factors is

$$(7+1)(2+1)(3+1) = 96.$$

■

Example 16

Consider a standard pack of 52 cards.

(a) In how many ways can this pack be divided
  (i) equally into 4 sets?
  (ii) equally among 4 players?
(b) In how many ways can this pack be
  (i) divided into 4 sets of 9, 13, 14 and 16 cards each?
  (ii) distributed among 4 players with 9, 13, 14 and 16 cards?
(c) In how many ways can this pack be
  (i) divided into 4 sets of 10, 14, 14 and 14 cards each?
  (ii) distributed among 4 players with 10, 14, 14 and 14 cards?
  (iii) distributed among 4 players with 12, 12, 14 and 14 cards?

Solution: For this question, you must understand that the division of a set into a number of subsets is different from the division of a set into a number of subsets and then distribution of those subsets to different persons. For example, suppose you are alone by yourself and you are dividing your deck of cards into 4 sets. What you'll do is do the division of the deck and keep the 4 sets in front of you. Which set lies where in front of you doesn't matter. Your only objective was to divide the pack into 4 sets which you did. Precisely speaking, the order of the groups doesn't matter.

On the other hand, suppose you were playing a game of *Bluff* with 3 other people. Before starting the game, you divide the deck into 4 sets of 13 cards each. But now you have to give each set of 13 cards to a different person. Where a set of cards goes matters. In other words, the order of the groups matters.

As another example, suppose you had to divide a class into 3 sub-groups. You do your job-the order of the groups doesn't matter, you just have to divide them. But suppose you had to divide the class into 3 sub-groups and place each sub-group in a different bogey of a roller coaster. In this case, the order of the groups matters because which sub-group sits in which bogey matters.

(a) (i) We want to divide the pack into 4 equal sets of 13 cards each. We proceed as follows. First select a group of 13 cards from the 52 cards; call this group A. This can be done in ${}^{52}C_{13}$ ways. Group B of another 13 cards can now be selected from the remaining 39 cards in ${}^{39}C_{13}$ ways. Group C of another 13 cards from the remaining 26 can now be selected in ${}^{26}C_{13}$ ways. This automatically leaves group D of the last 13 cards.
Thus, the division can be carried out in

$${}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \text{ ways}$$

However, notice that the order of the groups doesn't matter, as discussed earlier. This means that even if our order of group selection was *BACD* or *ADBC* (or for that matter any other permutation of *ABCD*), such a selection would essentially be the same as *ABCD*. The number of permutations of *ABCD* is 4! = 24. Thus, the actual number of ways of division is:

$$\frac{1}{24}\left({}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13}\right) = \frac{1}{24}\left(\frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!}\right) = \frac{52!}{(13!)^4 \cdot 4!}$$

(ii) If we had to distribute the cards equally among 4 players, the order of group selection would have mattered. For example, *ABCD* would be different from *BACD*. Thus, we don't divide by 4! in this case. The number of possible distributions is $\frac{52!}{(13!)^4}$.

(b) (i) For this part, observe that the group sizes are all unequal. This is a somewhat different situation than the previous part where the group sizes were equal. We'll soon understand why. We follow a similar sequence of steps as described earlier.

(i) Select group *A* of 9 cards from the deck of 52: ${}^{52}C_9$ ways.
(ii) Select group *B* of 13 cards from the remaining 43: ${}^{43}C_{13}$ ways.
(iii) Select group *C* of 14 cards from the remaining 30: ${}^{30}C_{14}$ ways.
(iv) This automatically leaves group *D* of the remaining 16 cards.

The number of ways for this division is:

$$^{52}C_9 \times {}^{43}C_{13} \times {}^{30}C_{14}$$

$$= \frac{52!}{9!\,42!} \times \frac{43!}{13!\,30!} \times \frac{30!}{14!\,16!} = \frac{52!}{9!\,13!\,14!\,16!}\cdot$$

On a side note, you may note in general that

$$^{m+n+p+q}C_m \times {}^{n+p+q}C_n \times {}^{p+q}C_p \times {}^{q}C_q = \frac{(m+n+p+q)}{m!\,n!\,p!\,q!}.$$

Why don't we divide by 4! here as we did earlier, even when the order of groups doesn't matter? This is because the group sizes are unequal. A particular selection in a particular order, say *ABCD*, will never be repeated in any other order of selection. Our order of selection (in terms of group size) is 9, 13, 14 and 16 cards. Carefully think about it; a particular selection like *ABCD* will be done only once (and not 4! times as in the previous part due to equal group sizes).

(ii) If we had to distribute the 4 sets to 4 players, the situation changes because which subset of cards goes to which player matters. A particular group of sets of cards, say *ABCD*, can be distributed among the 4 players in 4! ways. Thus, now the number of distributions will be

$$\begin{pmatrix}\text{no. of ways of division of the} \\ \text{deck as specified into the 4 sets}\end{pmatrix} \times \begin{pmatrix}\text{no. of ways of distribution} \\ \text{of those sets to the 4 players}\end{pmatrix}$$

$$= \left(\frac{52!}{9!\ 13!\ 14!\ 16!}\right) \times (4!)$$

(c) (i) In this part, the situation is a hybrid of the previous two parts. 3 of the 4 groups are equal in size while the 4th is different. We first select 10 cards out of 52. This can be done in ${}^{52}C_{10}$ ways. The remaining 42 cards can be divided into 3 equal groups (by the logic of part -(a)) in

$$\frac{42!}{(14!)^3} \times \frac{1}{3!} \text{ ways}$$

Division by 3! is required since the order of groups doesn't matter. Thus, a particular order of 14-card groups, say *ABC*, is, for example, the same as *ACB*. Now, the required number of ways is

$$^{52}C_{10} \times \frac{42!}{(14!)^3} \times \frac{1}{3!} = \frac{52!}{10!(14!)^3} \times \frac{1}{3!} \tag{1}$$

(ii) For the second question where we have 4 players and we want to give them 10, 14, 14 and 14 cards, we already have obtained the number of possible division in (1).

To distribute the subsets, each division of 4 sets can be given to the 4 players in 4! ways. Thus, the number of distributions of the 4 subsets is:

$$= \frac{52!}{10!(14!)^3} \times \frac{1}{3!} \times 4! = \frac{4 \times 52!}{10!(14!)^3}.$$

We can also look at this current problem in the following way. We want to distribute the 52 cards to the 4 players in sets of 10, 14, 14 and 14. We first decide which guy to give the set of 10 cards. This can be done in 4 ways. Now we choose 10 cards for him (in possibly ${}^{52}C_{10}$ ways). The other three players will now get the remaining 42 cards equally. We can select 14 cards for one of them in ${}^{42}C_{14}$ ways. The third one can get another 14 cards in ${}^{28}C_{14}$ ways and finally the fourth one gets the remaining 14 cards. Thus, the number of possible ways of distribution is:

$$4 \times {}^{52}C_{10} \times {}^{42}C_{14} \times {}^{28}C_{14} = \frac{4 \times 52!}{10!\,(14!)^3}.$$

(iii) Finally, we now have two sets of one size and two of another size. We first just carry out the division into subsets, without assigning any subset to any player. This can be done in:

$$\underset{\substack{\nearrow \\ \text{Group A of} \\ \text{12 cards}}}{{}^{52}C_{12}} \times \underset{\substack{\nearrow \\ \text{Group B of} \\ \text{12 cards}}}{{}^{40}C_{12}} \times \underset{\substack{\nearrow \\ \text{Group C of} \\ \text{14 cards}}}{{}^{28}C_{14}} \times \underset{\substack{\nearrow \\ \text{Group D of} \\ \text{14 cards}}}{{}^{14}C_{14}} \times \underset{\substack{\downarrow \\ \text{Groups A and} \\ \text{B are of the} \\ \text{same size}}}{\frac{1}{2!}} \times \underset{\substack{\downarrow \\ \text{Groups C and} \\ \text{D are of the} \\ \text{same size}}}{\frac{1}{2!}}$$

$$= \frac{52!}{(12!)^2(14!)^2} \times \frac{1}{(2!)^2}.$$

Once the division of the deck into the 4 subsets has been accomplished, we assign a subset to each of the 4 players. This can be done in 4! ways. Therefore, the possible number of ways of distribution of the cards to the 4 players is:

$$\frac{52!}{(12!)^2(14!)^2} \times \frac{1}{(2!)^2} \times 4!.$$

■

Example 17

In this example, we talk about circular permutations, *i.e.*, arrangements in a circular fashion.

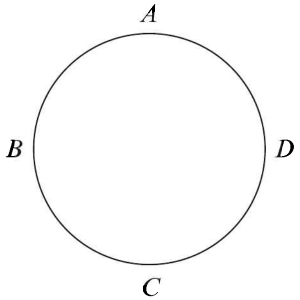
(a) In how many ways can n people be seated around a circular table?

(b) We have a group of 5 men and 5 women. In how many ways can we seat this group around a circular table such that:

(i) all the 5 women sit together? (ii) no two women sit together?

(c) In how many ways can a necklace be formed from n different beads?

Solution: Circular permutations are somewhat different from linear permutations. Lets understand why. Consider the letters A, B, C and D. The 4 linear permutations $ABCD$, $BCDA$, $CDAB$, and $DABC$ correspond to a single circular permutation as shown in the following figure:



In a circular permutation, only the relative ordering of the symbols matters. The 4 linear permutations $ABCD$, $BCDA$, $CDAB$, and $DABC$ all have the same relative ordering so that they correspond to a single circular permutation.

This means that the number of circular permutations of $ABCD$ is only $\dfrac{4!}{4} = 3! = 6$.

(a) We have n people, say $P_1, P_2, \ldots, P_n$. As just described, n linear permutations of these people will correspond to a single circular permutation as depicted in the figure below:

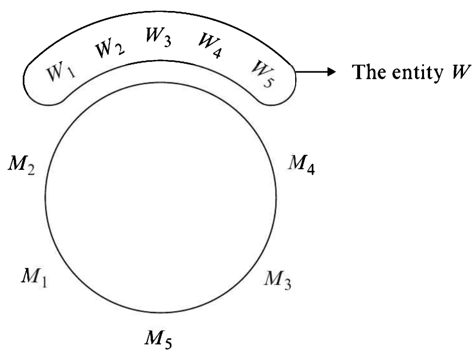
$$\left.\begin{array}{l} P_1P_2\ldots P_n \\ P_2P_3\ldots P_nP_1 \\ P_3P_4\ldots P_1P_2 \\ \vdots \\ P_nP_1\ldots P_{n-2}P_{n-1} \end{array}\right\} \qquad P_1P_2\ldots P_n$$

n linear permutations correspond to $\rightarrow$ a single circular permutation because of the same relative ordering.

Thus, the number of circular permutations is $\frac{n!}{n} = (n-1)!$. We can also arrive at this number in another way. We take a particular person, say P_1, and seat him anywhere on the table. Once P_1's seat becomes fixed, the rest of the $(n - 1)$ seats bear a fixed relation to P_1's. In other words, once P_1's seat becomes fixed, we can treat the $(n - 1)$ seats left as a linear row of $(n - 1)$ seats. Thus, the remaining $(n - 1)$ people can be seated in $(n - 1)!$ ways.

(b) Let the 5 women be represented by W_1, W_2, W_3, W_4, and W_5 and the 5 men by M_1, M_2, M_3, M_4, and M_5. Since we need all the women to sit together, we first treat all the 5 women as a single entity W. Now, the 5 men and the entity W can be seated around the table in $((5+1)-1)!$ ways, *i.e.*, 120 ways.

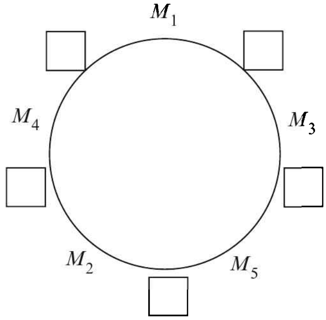
(i) Once the 5 men and the entity W have been seated, we now permute the 5 women inside the entity W. This can be done in $5! = 120$ ways. The total number of ways is thus $120 \times 120 = 14400$.



(i) First, seat the 5 men and the entity W.

(ii) Then permute the women inside the entity W.

(ii) Since we want no two women to sit together, we first seat all the 5 men, which can be done in (5 – 1)! = 24 ways. Seating the 5 men first creates 5 non-adjacent seats where the women can then be seated in 5! = 120 ways.

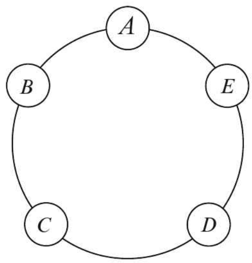


(i) First, seat the 5 men, 4! ways are possible.

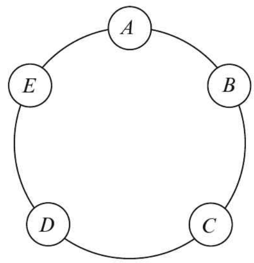
(ii) Then seat the 5 women in the 5 spaces created as shown in the figure 5! ways are possible.

The total number of ways is thus 5! × 4!.

(c) The situation of a necklace is slightly different than that of seating people around a circular table. The reason is as follows. Suppose we have 5 beads *A*, *B*, *C*, *D* and *E*. Consider two circular necklaces of these 5 beads:



Necklace-1



Necklace-2

Are these two necklaces different? No, because a necklace can be worn from both ways. Necklace-2 is the same as necklace-1 if I look into it from the other side of the page. In other words, for a necklace, a clockwise permutation and its corresponding anti-clockwise permutation are identical. Thus, the number of circular permutations would reduce by a factor of two, *i.e.*, the number of different necklaces possible is $\frac{1}{2}(n-1)!$. ■

Permutations and Combinations

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. How many distinct throws are possible with a throw of n dice which are identical to each other, *i.e.*, indistinguishable among themselves?

(A) ${}^{n+3}C_3$ (B) ${}^{n+4}C_4$ (C) ${}^{n+5}C_5$ (D) ${}^{n+6}C_6$ (E) None of these

P2. A composition of a natural number N is a sequence of non-zero integers $\{a_1, a_2, ..., a_k\}$ which add up to N. How many compositions of N exist?

(A) 2^{N-2} (B) 2^{N-1} (C) 2^N (D) 2^{N+1} (E) None of these

P3. Consider n points in a plane such that no three of them are collinear. These n points are joined in all possible ways by straight lines. Of these straight lines so formed, no two are parallel and no three are concurrent. What is the number of the points of intersection of these lines exclusive of the original n points?

(A) $\dfrac{(n^2-1)(n^2-4)}{8}$ (C) $\dfrac{n(n-1)(n-2)(n+3)}{6}$

(B) $\dfrac{(n^2-1)(n^2+2n)}{8}$ (D) $\dfrac{n(n-1)(n-2)(n-3)}{8}$

P4. The number of 4 letters strings which can be formed from the letters of the work INEFFECTIVE is

(A) 1392 (B) 1422 (C) 1452 (D) 1482

P5. We have $3n$ objects, of which n are identical and the rest are all different. In how many ways can we select n objects from this group ?

(A) ${}^{2n}C_{n-1} + 2^{2n}$ (C) $\dfrac{1}{2}\cdot {}^{2n}C_{n-1} + 2^{2n+1}$

(B) $\dfrac{1}{2}\cdot {}^{2n}C_n + 2^{2n-1}$ (D) $\dfrac{1}{2}\cdot {}^{2n}C_n + 2^{2n+1}$

P6. n persons are seated around a circular table. In how many ways can we select 3 persons such that no 2 of them are adjacent to each other on the table?

(A) $\dfrac{1}{3}n(n-1)(n-2)$ (B) $\dfrac{1}{4}n(n-3)(n-1)$ (C) $\dfrac{1}{6}n(n-4)(n-5)$

(D) $\dfrac{1}{8}n(n-1)(n-3)$ (E) None of these

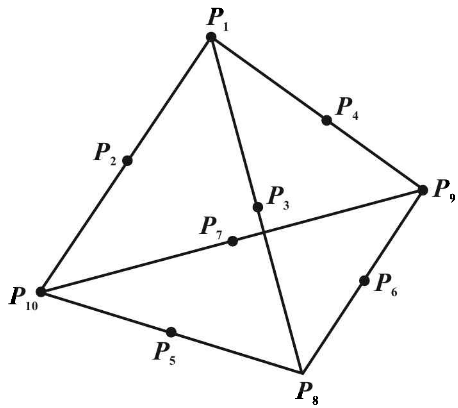
P7. We have 21 identical balls available with us which we need to be distributed amongst 3 boys A, B and C such that A always gets an even number of balls. The number of possible ways of doing this is:

(A) 112 (B) 120 (C) 126 (D) 132

P8. What is the sum of all the five-digit numbers that can be formed using the digits 1, 2, 3, 4 and 5, if no digit is repeated?

(A) 3999900 (B) 3999930 (C) 3999960 (D) 3999990

P9. As shown in the figure below, points $P_1, P_2, ..., P_{10}$ are either the vertices or the mid-points of the edges of a tetrahedron.



For $1 < i < j < k \le 10$, how many groups of four coplanar points (P_1, P_i, P_j, P_k) exist?

(A) 31 (B) 33 (C) 35 (D) 37

P10. Consider the following two sets of real numbers:

$$A = \{a_1, a_2, ..., a_{100}\}$$
$$B = \{b_1, b_2, ..., b_{50}\},\ b_1 \le b_2 \le ... \le b_{50}$$

Consider an onto mapping f from A to B such that

$$f(a_1) \le f(a_2) \le .... \le f(a_{100}).$$

How many such mappings exist?

(A) ${}^{99}C_{49}$ (B) ${}^{99}C_{48}$ (C) ${}^{100}C_{49}$ (D) ${}^{100}C_{48}$

P11. What is the sum S given by the following?

$$S = 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n$$

(A) $n^2 \cdot 2^{n-1}$ (B) $n(n+1) \cdot 2^{n-1}$ (C) $n^2 \cdot 2^{n-2}$ (D) $n(n+1) \cdot 2^{n-2}$

P12. What is the value of ${}^mC_1{}^nC_m - {}^mC_2{}^{2n}C_m + {}^mC_3{}^{3n}C_m - \cdots + (-1)^{m-1}\ {}^mC_m{}^{nm}C_m$?

(A) $(-1)^m n^m$ (B) $(-1)^{m+1} n^m$ (C) $(-1)^m n^{m+1}$ (D) $(-1)^{m+1} n^{m+1}$

SUBJECTIVE TYPE EXAMPLES

P13. Prove logically the following assertion:

$$\sum_{k=0}^{n} 2^k \cdot {}^nC_k = 3^n$$

P14. Prove logically the following assertion:

$${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^rC_r = {}^{n+1}C_{r+1}$$

P15. Give combinatorial arguments to prove that

$$\sum_{r=1}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}$$

P16. Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

P17. Find the sum of the divisors of 120. Generalise the result for an arbitrary natural number N.

P18. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. In how many ways can we place the balls into the boxes so that no box remains empty, if
(a) balls and boxes are all different.
(b) balls are identical but boxes are different.
(c) balls are different but boxes are identical.
(d) balls as well as boxes are identical.
(e) balls as well as boxes are identical, but the boxes are kept in a row.

P19. Find the total possible number of 5 letter words such that they are alphabetically first in the sequence of words formed by all their permutations, in the following two cases:
(a) repetition of letters is not allowed
(b) repetition of letters is allowed

P20. Evaluate the following sums:

(a) $S_1 = \dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \cdots$ (b) $S_2 = \dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \cdots$

P21. Using a binomial approach, find the sum S given by

$$S = {}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots + {}^{n+r}C_r$$

P22. Find the value of $\Sigma_{r=1}^{k} (-3)^{r-1}\ {}^{3n}C_{2r-1}$, where $k = (3n)/2$ and n is an even positive integer.

P23. Find the sum S of the following series (C_r denotes ${}^nC_r$):

$$S = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \frac{C_4}{4} + \cdots + (-1)^{n+1}\frac{C_n}{n}$$

P24. For any positive integers m, n (with $n \geq m$), let $\binom{n}{m} = {}^nC_m$. Prove that:

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1}.$$

Using this result, find the value of

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n-m+1)\binom{m}{m}.$$

P25. If $\Sigma_{r=0}^{n} \frac{r}{{}^nC_r} = a$, find the value of $\Sigma_{0 \leq i < j \leq n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$

P26. You are given

$$S_n = 1 + q + q^2 + \cdots + q^n$$

and

$$\lambda_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \cdots + \left(\frac{q+1}{2}\right)^n, q \neq 1.$$

Find the value of

$${}^{n+1}C_1 + {}^{n+1}C_2 S_1 + {}^{n+1}C_3 S_2 + \cdots + {}^{n+1}C_{n+1} S_n.$$

Permutations and Combinations

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. The important point to be realised here is that the dice are totally identical. Suppose we had just 2 dice, say die A and die B. Suppose that, upon throwing these dice, we get a 'two' on A and a 'three' on B. This case would be the same as the one where we get a 'three' on A and a 'two' on B because we cannot distinguish between A and B. What we are concerned with is only what numbers show up on the top of the dice. We are not concerned with which die shows what number. This means that if we have n dice and we throw them, we are only concerned with how many 'ones', 'twos', 'threes' etc show on the top faces of the dice; we are not at all interested in which die throws up what number. If we denote the number of 'ones' we get by x_1, number of 'twos' we get by x_2 and so on, we will have

$$x_1 + x_2 + \cdots + x_6 = n$$

Thus, the total number of distinct throws will be simply the number of non-negative solutions to this integer equation. This number will be ${}^{n+6-1}C_{6-1} = {}^{n+5}C_5$, and so the correct option is (C).

S2. Let us make the question more clear by taking a particular example for N, say $N = 4$. As described in the question, the compositions of $N = 4$ will be

$$\{4\}, \{1, 3\}, \{2,2\}, \{3,1\}, \{1,1,2\}, \{1,2,1\}, \{2,1,1\} \text{ and } \{1,1,1,1\},$$

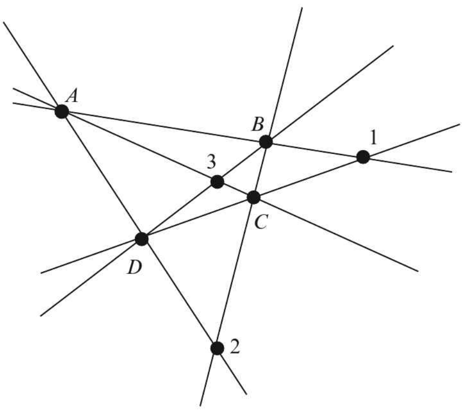
which are 8 in number. Observe carefully the compositions listed out. How can we characterize each of these compositions mathematically? Recall the problem of finding the number of non-negative solutions of the integer equation $x_1 + x_2 + \cdots + x_n = r$ where each solution corresponded to a unique permutation of r '1' symbols and $n - 1$ '$\square$' symbols. Can we do something like that here? In other words, can we represent each composition in another form whose permutations are easier to count? It turns out that we can, as follows:

$$\begin{aligned}
\{4\} &\equiv 1 \quad 1 \quad 1 \quad 1 \\
\{1, 3\} &= 1+1 \quad 1 \quad 1 \\
\{3, 1\} &= 1 \quad 1 \quad 1+1 \\
\{2, 2\} &= 1 \quad 1+1 \quad 1 \\
\{1, 1, 2\} &= 1+1+1 \quad 1 \\
\{1, 2, 1\} &= 1+1 \quad 1+1 \\
\{2, 1, 1\} &= 1 \quad 1+1+1 \\
\{1, 1, 1, 1\} &= 1+1+1+1
\end{aligned}$$

On the right hand side, we have 4 '1's and thus 3 blank spaces between the 4 '1's. We can insert '+' signs in these blank spaces; each different arrangement of '+' signs in these blank spaces will correspond to a different composition. To count the number of these arrangements, we proceed as follows. For each blank space, we have 2 options, we can either insert or not insert a '+' sign into that space. There are 3 blank spaces, so the total number of all arrangements of '+' signs in the 3 blank spaces is $2 \times 2 \times 2 = 8$ (which is the number of compositions we already listed out).

In the general case, we will have $(N-1)$ blank spaces and 2 options for each such blank space. Thus, the total number of ways in which we can arrange '+' signs in these blank spaces, and therefore, the total number of compositions, will be $\underbrace{2\times2\times2\times........2}_{(N-1)\text{ times}} = 2^{N-1}$. The correct option is (B).

S3. To gain more insight into the problem, let us consider the case when $n = 4$:



We originally have 4 points in the plane labelled as A, B, C and D. When we draw all the straight lines possible (by joining every possible pair of points), we see that 3 new intersection points are generated, labelled as 1, 2 and 3. As we now discuss the general case, refer to this figure for help.

Since we have n points, the number of straight lines that can be generated is equal to the number of pairs of points that can be selected from n points, which is equal to ${}^nC_2$. Every pair of straight lines so generated will intersect at some point. Thus, the total number of intersection points if we count this way should be equal to the number of pairs of straight lines possible from the ${}^nC_2$ lines, which is ${}^{({}^nC_2)}C_2$.

However, observe that in this number, the original intersection points have also been counted, and that too multiple times. Lets determine how many times a particular point, say A has been counted. $(n-1)$ lines pass through A. Thus ${}^{n-1}C_2$ pairs of straight lines are possible from these $(n-1)$ lines. This means that A has been counted ${}^{n-1}C_2$ times, which further means that the n original points have been counted $n \times {}^{n-1}C_2$ times. To calculate the new intersection points, we subtract this number from the (apparent) number of intersection points we obtained earlier. Thus, the actual number of new intersection points is

$$
{}^{({}^nC_2)}C_2 - n \times {}^{n-1}C_2 = \frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} - 1\right)}{2} - \frac{n \times (n-1)(n-2)}{2}
$$

$$
= \frac{n(n-1)(n^2-n-2)}{8} - \frac{n(n-1)(n-2)}{2} = \frac{n(n-1)\{(n^2-n-2)-4(n-2)\}}{8}
$$

$$
= \frac{n(n-1)(n^2-5n+6)}{8} = \frac{n(n-1)(n-2)(n-3)}{8}
$$

Verify that this formula works for $n = 4$. Thus, the correct option is (D).

S4. Observe that some letters repeat more than once:

Letter	E	F	I	C	T	V	N
Frequency	3	2	2	1	1	1	1

This means that our string of 4 letters could contain repeated letters. It's thus obvious that we cannot straightaway use ${}^nP_r$ to evaluate the number of strings. In such a case, we divide the *types* of strings that we can form into different mutually exclusive cases:

Case 1: All 4 letters are different

In this case, we have 7 letters to choose from and we have to arrange them in 4 places. The number of such strings will be ${}^7P_4 = 840$.

Case 2: 2 letters are same, the other 2 are different

We have 3 types of letters (E, F and I) that can be repeated twice. Thus, the twice-repeated letter can be selected in 3 ways. Once that is done, the rest of the 2 letters can be selected from amongst the remaining 6 options in ${}^6C_2 = 15$ ways. Once we have formed the combination of the 4 letters, we can permute them in $\frac{4!}{2!} = 12$ ways. The '2!' occurs in the denominator because of the twice-repeated letter. Thus, the total number of such strings will be $3 \times 15 \times 12 = 540$.

Case 3: 2 letters are the same, the other 2 are also the same

For example, the string 'EFEF' will be such a string. There are 2 letters now that we want to be twice-repeated; observe that there are only 3 types of letters (E, F and I) that can be twice repeated. Thus, the letters that will occur in the string can be selected in ${}^3C_2 = 3$ ways. Once that is done, we can permute the 4 letters in $\frac{4!}{2!2!} = 6$ ways. The total number of such strings is therefore $3 \times 6 = 18$.

Case 4: 3 letters are the same, the 4th is different

Only 'E' can be repeated thrice so that E must occur in this string. The 4th letter can be selected in 6 ways. Then, the combination so formed can be permuted in $\frac{4!}{3!} = 4$ ways. The number of such strings is $6 \times 4 = 24$.

Verify that these four cases are mutually exclusive and they exhaust all the possibilities. The total number of strings is 840 + 540 + 18 + 24 = 1422. The correct option is (B).

S5. To select n objects, we select k objects from the identical ones and the remaining $(n-k)$ from the different ones (k will vary from 0 to n). As discussed somewhere earlier, from the group containing identical objects, there will always be only 1 way of selection. From the group of $2n$ non-identical objects, $(n-k)$ objects can be selected in ${}^{2n}C_{n-k}$ ways. Thus, the group of n objects containing k identical objects and the remaining $(n-k)$ as non-identical objects can be formed in $1 \times {}^{2n}C_{n-k} = {}^{2n}C_{n-k}$ ways. The total number of ways is S, where:

$$S = \sum_{k=0}^{n} {}^{2n}C_{n-k} = {}^{2n}C_n + {}^{2n}C_{n-1} + \cdots + {}^{2n}C_0$$

A closed-form expression for S can be obtained by the following manipulation:

$$2S = 2({}^{2n}C_n + {}^{2n}C_{n-1} + \cdots + {}^{2n}C_0)$$

$$= \left\{ \begin{array}{c} {}^{2n}C_n + {}^{2n}C_{n-1} + \cdots + {}^{2n}C_0 \\ + \\ {}^{2n}C_n + {}^{2n}C_{n-1} + \cdots + {}^{2n}C_0 \end{array} \right\}$$

$$= \left\{ \begin{matrix} {}^{2n}C_n + {}^{2n}C_{n-1} + \cdots + {}^{2n}C_0 \\ + \\ {}^{2n}C_n + {}^{2n}C_{n+1} + \cdots + {}^{2n}C_{2n} \end{matrix} \right\} \left(\begin{matrix} \text{For all the terms in the} \\ \text{lower row, we can use} \\ {}^{2n}C_r = {}^{2n}C_{2n-r} \end{matrix} \right)$$

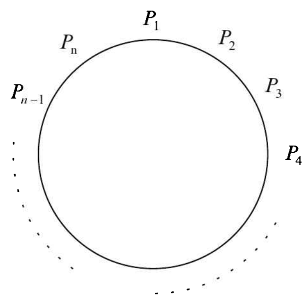
$$= {}^{2n}C_n + ({}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \cdots + {}^{2n}C_{2n})$$

$$= {}^{2n}C_n + 2^{2n}$$

$$\Rightarrow \quad S = \frac{1}{2} \cdot {}^{2n}C_n + 2^{2n-1}$$

The correct option is (D).

S6. We will solve this problem by first considering the total number of selections possible and then subtracting from this, the number of cases where 2 persons or all the 3 persons are adjacent. Observe the following figure for reference.



(i) The total number of ways of selecting 3 persons out of these n is ${}^nC_3$.

(ii) Let us now count the cases where only 2 of the 3 persons are adjacent. 2 adjacent people can be selected as follows:

$$\{P_1P_2\} \text{ or } \{P_2P_3\} \text{ or } \{P_3P_4\} \text{ or ... or } \{P_nP_1\}$$

Thus, there are n ways in which 2 adjacent persons can be selected. Once these 2 have been selected (for example, say we select $\{P_2 P_3\}$), we can select the third non-adjacent person in $(n - 4)$ ways (since for example, if we selected $\{P_2 P_3\}$, we can not select P_1 or P_4). Thus the cases where only 2 of the 3 people are adjacent are $n(n - 4)$ in number.

(iii) We now count the cases where all 3 persons are adjacent. 3 persons can be adjacent in the following manner:

$$\{P_1P_2P_3\} \text{ or } \{P_2P_3P_4\} \text{ or... or } \{P_nP_1P_2\}$$

Thus, observe that there are n ways of selecting 3 persons who are adjacent.

Finally, the number of ways to select 3 persons such that no 2 are adjacent is

$$\begin{aligned} {}^nC_3 - n(n-4) - n &= \frac{n(n-1)(n-2)}{6} - n(n-4) - n \\ &= \frac{1}{6} n(n-4)(n-5) \end{aligned}$$

The correct option is (C).

S7. The only constraint is that A should get an even number of balls. There's no constraint on the minimum number of balls a boy should get. This means that a boy can also not be given any ball. We can represent

the number of balls given to A by $2x$ since A must get an even number of balls. If we represent the number of balls given to B and C by y and z, we should have

$$2x + y + z = 21 \qquad (1)$$

This means that to find the number of distributions possible, we find the number of non-negative integer solutions to the equation (1). Note that x can take a maximum value of 10 and a minimum value of 0. We rearrange (1) so that we get an integer equation with y and z as variables, treating x as a constant

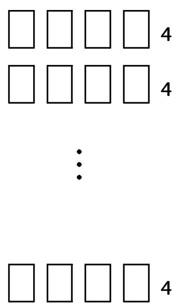
$$y + z = 21 - 2x \qquad (2)$$

The number of non-negative integer solutions of (2) is

$$^{21-2x+2-1}C_1 = {}^{22-2x}C_1 = 22 - 2x$$

We now add the number of solutions so obtained for all the possible values of x. The total number of solutions is therefore $\sum_{x=0}^{10}(22-2x) = 132.$ The correct option is (D).

S8. This problem can be solved very easily if we view it from an individual digit's perspective. Suppose that we only consider the digit '4'. How many numbers will there be with '4' in the units place?



There are 24 numbers with 4 in the units place because the remaining four digits can be permuted among the remaining 4 places in 4! = 24 ways.

From these 24 numbers, what is the total contribution of the digit '4' to the sum we are required to calculate? Since '4' is at the *units* place and it occurs 24 times, its contribution will be 4 × 24 = 96.

Similarly, there will be 24 numbers where '4' is at the *tens* place. The total contribution of '4' from these 24 numbers will be 4 × 240 = 960.

Proceeding in this way, we see that the total contribution of the digit '4' from all the 120 numbers that can be possibly formed is:

$$\begin{aligned} &4(24 + 240 + 2400 + 24000 + 240000) \\ &= 4 \times 24 \times (1 + 10 + 100 + 1000 + 10000) \\ &= 4 \times 24 \times 11111 \end{aligned}$$

This is the contribution to the sum from only the digit '4'. To calculate the entire sum S, we calculate the contributions from all the five digits. Thus, the sum is

$$\begin{aligned} S &= (1 + 2 + 3 + 4 + 5) \times 24 \times 11111 \\ &= 3999960 \end{aligned}$$

The correct option is (C).

S9. Observe that P_1 is a vertex for three of the four faces of the tetrahedron. For each of these faces, there are 5 points other than P_1 from which we can select 3 points in $^5C_3 = 10$ ways. These three points will obviously

be coplanar with P_1. Also, note that the groups $\{P_1, P_3, P_8, P_7\}$, $\{P_1, P_2, P_{10}, P_6\}$ and $\{P_1, P_4, P_9, P_5\}$ are also groups of coplanar points. The total number of groups is thus $10 \times 3 + 3 = 33$. The correct option is (B).

S10. All you have to do is to partition the ordered sequence $A = \{a_1, a_2, ..., a_{100}\}$ into 50 groups, with numbers in the ith group mapped to b_i. It is easy to see that this partioning can be done in ${}^{99}C_{49}$ ways, because there are 99 places between two successive terms in A, and you have to select 49 of these 99 places along which A will be partitioned. The correct option is (A).

S11. We have to plan an approach wherein we are able to generate r^2 with C_r. We can generate one r with every C_r using differentiation:

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Differentiating both sides with respect to x, we have

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1}$$

Now we have reached the stage where we have an r with every C_r. We need to think how to get the other r. If we differentiate once again, we'll have $r(r-1)$ with every C_r instead of r^2 (understand this point carefully). To 'make-up' for the power that falls one short of the required value, we simply multiply by x on both sides of the relation above to obtain:

$$nx(1+x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + \cdots + nC_nx^n$$

It should be evident now that the next step is differentiation again:

$$n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1} = C_1 + 2^2 \cdot C_2x + 3^2 \cdot C_3x^2 + \cdots + n^2 \cdot C_nx^{n-1}$$

Now we simply substitute $x = 1$ to obtain:

$$n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1} = C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n$$

The required sum S is thus:

$$S = n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1} = n \cdot 2^{n-2}\{(n-1)+2\} = n(n+1) \cdot 2^{n-2}$$

The correct option is (D).

S12. This series sum which we denote by S, might seem a bit complicated, until you observe each term separately, and realize that each term can be written as one of the coefficients in some binomial expansion series. The general term T_r of the given series is $(-1)^{r-1}\ {}^mC_r\ {}^{nr}C_m,\ r = 1, ..., m$. Now, ${}^{nr}C_m$ is the coefficient of x^m in $(1+x)^{nr}$. Thus,

$$T_r = (-1)^{r-1}\ {}^mC_r \times \{\text{Coeff. of } x^m \text{ in } (1+x)^{nr}\}$$

$$= -\{\text{ Coeff. of } x^m \text{ in } (-1)^r\ {}^mC_r\ (1+x)^{nr}\}$$

Thus,

$$S = \sum_{r=1}^{m} T_r = -\text{Coeff. of } x^m \text{ in } (1-(1+x)^n)^m \quad \text{(how?)}$$

$$= (-1)^{m+1} \times \text{Coeff. of } x^m \text{ in } (C_1x + C_2x^2 + \cdots + C_nx^n)^m$$

$$= (-1)^{m+1} \times (C_1)^m = (-1)^{m+1} n^m$$

The correct option is (B).

SUBJECTIVE TYPE EXAMPLES

S13. Suppose that we have to form a string of length n, consisting of only letters from the set $\{A, B, C\}$. Thus, we have 3 options to fill any particular place in the string: We fill that place with either A, B or C. Thus the total number of different strings of length n would be:

$$\underbrace{3\times 3\times 3\times\cdots\times 3}_{n \text{ times}} = 3^n.$$

We now approach the task of formation of these strings from a different perspective. Suppose that our string contains a total of r 'A' s. How many such strings will exist? We first select r places out of n which we will fill with 'A'. This can be done in ${}^nC_r$ ways. For each of the remaining $(n-r)$ places, we have two options; we either fill it with 'B' or 'C'. Thus, the number of strings containing r 'A' s will be ${}^nC_r \cdot 2^{n-r}$. Now, we vary r from 0 to n and thus get the total number of strings as

$$\sum_{r=0}^{n} {}^nC_r \cdot 2^{n-r} = \sum_{k=0}^{n} {}^nC_{n-k} \cdot 2^k \quad (\text{where } k = n-r)$$

$$= \sum_{k=0}^{n} {}^nC_k \cdot 2^k \quad (\text{since } {}^nC_{n-k} = {}^nC_k)$$

Thus, $\displaystyle\sum_{k=0}^{n} {}^nC_k \cdot 2^k = 3^n.$

S14. The right hand side says that we have to select $(r+1)$ people out of a group of $(n+1)$. To do so, we list down the following (mutually exclusive) cases which exhaust all the possible cases:

(1) The group contains A_1 : ${}^nC_r$ ways (since we have to select r people more apart from A_1 from n that are available for selection).

(2) The group does not contain A_1, but contains A_2 : ${}^{n-1}C_r$ ways (since we have to select r people more apart from A_2 from $(n-1)$ that are available for selection).

(3) The group does not contain A_1 and A_2 but contains A_3 : ${}^{n-2}C_r$ ways (since we have to select r people more apart from A_3 from $(n-2)$ that are available for selection).

$\vdots$

$(n-r)$ The group does not contain $A_1, A_2, \ldots, A_{n-r-1}$, but contains A_{n-r} : ${}^{r+1}C_r$ ways (since we have to select r people more apart from A_{n-r} from the $(r+1)$ that are available for selection.

$(n-r+1)$ The group does not contain $A_1, A_2, \ldots, A_{n-r}$ but contains A_{n-r+1} : ${}^rC_r$ ways (since we have to select r more people apart from A_{n-r+1} from the remaining r that are available for selection).

Convince yourself that all these cases are mutually exclusive and exhaust all the possibilities. Thus, the total number of $(r + 1)$ – groups from $(n + 1)$ people is

$$^{n+1}C_{r+1} = {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^{r}C_r.$$

S15. Let us first interpret what the left hand side of this assertion says. Suppose we have a group of n people. We select a sub-group of size r from the group of n people. This can be done in ^nC_r ways. Once the sub-group has been formed, we select a leader of that sub-group, and send that sub-group on an excursion. The leader can be selected in r ways. Thus, the total number of different ways in which an r - group can be formed with a unique leader is $r \times {}^nC_r$. Now, r can take any integer value from 1 to n, *i.e.*, $1 \le r \le n$. Thus, the total number of all possible sub-groups, each sub-group being assigned a unique leader, will be $\sum_{r=0}^{n} r \cdot {}^nC_r$ which is the left hand side of our assertion.

To prove this equal to the right hand side, we count the sub-groups from a different angle. We count all those sub-groups in which a particular person, say A, is the leader. Since A is the leader, A is fixed in our sub-group. For each of the remaining $(n - 1)$ people, we have two options. We either put the person in the group led by A or we don't. Thus, the total number of sub-groups in which A is the leader will be

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{(n-1) \text{ times}} = 2^{n-1}$$

Since any of the n persons can be the leader, and under each person's leadership, 2^{n-1} groups can be formed, the total number of sub-groups, each sub-group under some unique person's leadership, is $n \cdot 2^{n-1}$. This proves our assertion that

$$\sum_{r=1}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}$$

S16. We can equivalently show that $\frac{(n!)!}{(n!)^{(n-1)!}}$ is an integer. To get more insight into the problem, let us take $n = 6$. Thus, we need to show that $\frac{(6!)!}{(6!)^{5!}}$ is an integer. (You must appreciate the magnitude of the numbers we are dealing with here!). Consider the following sequence of symbols:

$$a_1 a_1 a_1 a_1 a_1 a_1 a_2 a_2 a_2 a_2 a_2 a_2 \quad \ldots \quad a_{120} a_{120} a_{120} a_{120} a_{120} a_{120}$$

This is a sequence of symbols whose length is 720, because each symbol a_i occurs 6 times in this sequence and there are in all 120 symbols, and thus the total number of symbols is $6 \times 120 = 720$. Let us find the number of permutations of this string. Since there are 120 *types* of symbols, each symbol being repeated 6 times, we will have to divide by 6! a total of 120 times, *i.e.*, the number of permutations of this string will be

$$\frac{(720)!}{\underbrace{6! \times 6! \times 6! \times \cdots \times 6!}_{120 \text{ times}}} = \frac{(6!)!}{(6!)^{5!}}$$

Since the number of permutations of any string must obviously be an integer, the term $= \frac{(6!)!}{(6!)^{5!}}$ must be an integer! We can now easily generalise this result.

S17. The divisors of 120 are listed out below:

$$\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$$

The sum of these divisors is 360. We have to determine an elegant way to deduce this sum because we cannot repeat everytime the procedure of listing down all the factors and summing them. For this purpose, we make use of the prime factorization form:

$$120 = 2^3 \cdot 3^1 \cdot 5^1$$

The sum of the divisors will be

$$S = \sum_{\substack{0 \le i \le 3 \\ 0 \le j \le 1 \\ 0 \le k \le 1}} 2^i \cdot 3^j \cdot 5^k$$

This notation is simply a shorthand which implies that we vary the integer indices i, j and k (in their respective allowed ranges) and this way we will have listed down all the factors and hence evaluated the required sum. To generate the expression for the sum S, we can use the following method:

$$S = (1 + 2^1 + 2^2 + 2^3)(1 + 3^1)(1 + 5^1)$$

Did you realize the trick? Writing S this way generates all the possible factors. You are urged to convince yourself about this by expanding this expression and observing that all possible factors will be generated. Thus, S can now be simply evaluated as follows:

$$S = (1 + 2^1 + 2^2 + 2^3)\ (1 + 3^1)\ (1 + 5^1)\ = 15 \times 4 \times 6\ = 360$$

This is the same result that we got earlier by explicit summation.
To do the general case, assume that the prime factorization form of N is

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$$

where the α_i' are all positive integers and the p_i's are all primes. The sum S_f of all the factors will be

$$S_f = \left(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right) \ldots \left(1 + p_n + p_n^2 + \cdots + p_n^{\alpha_n}\right)$$

S18. One of the constraints that should always be satisfied is that no box should remain empty. Thus, each box should get at least one ball. This means that the distribution of balls can have the configuration (1, 1, 3) or (1, 2, 2). Only in these two configurations does no box remain empty.

(a) In this case, the balls and the boxes are all different. If you observe carefully, you will note that this case is equivalent to dealing cards to players. Here, we are dealing balls (all different) into boxes (all different). In the card game, we were dealing cards (all different) to players (all different). Suppose we distribute the balls in the configuration (3, 1, 1). We first divide the group of balls into this configuration. This can be done in ${}^5C_3$ ways since we just need to select a group of 3 balls and

our division will be accomplished. Once the division of the group of balls into 3-subgroups in the configuration (3, 1, 1) has been done, we can permute the 3 sub-groups among the 3 *different* boxes in 3! ways.Thus,the number of ways to achieve the (3, 1, 1) configuration is ${}^5C_3 \times 3! = 60.$

We now find the number of ways to achieve the (1, 2, 2) configuration. We first select 2 balls out of the 5 which can be done in ${}^5C_2$ ways. We then select 2 balls from the remaining 3, which can be done in ${}^3C_2$ ways. Thus simple division into the configuration (2, 2, 1) can be achieved in $\frac{{}^5C_2 \times {}^3C_2}{2!}$ ways. Division by 2! is required since two subgroups are of the same size and right now we are just dividing into sub groups so the order to the sub-groups doesn't matter. Once division has been accomplished, we permute the 3 subgroups so formed amongst the 3 different boxes in 3! ways. Thus, the number of ways to achieve the (2, 2, 1) configuration is $\frac{{}^5C_2 \times {}^5C_2}{2!} \times 3! = 90.$

The total number of ways is 60 + 90 = 150. To make things clearer, let us list down in detail the various configurations possible for the 3 different boxes, *A*, *B* and *C*:

Box A	Box B	Box C	Number of Ways
1	1	3	20
1	3	1	20
3	1	1	20
1	2	2	30
2	1	2	30
2	2	1	30
			Total=150

(b) The balls are now identical so it doesn't matter which ball goes into which box. What matters is only the configuration of the distribution. By simple enumeration, only 6 configurations exist for this case, as shown below. The notation [*a*, *b*, *c*] implies that Box A has *a* balls, Box B has *b* balls and Box C has *c* balls.

$$[1,1,3] \qquad [1,3,1] \qquad [3,1,1]$$

$$[1,2,2] \qquad [2,1,2] \qquad [2,2,1]$$

Thus, 6 possible ways exist for this case.

(c) The boxes are identical. This means that it does not matter which sub-group of balls you put in which box. What matters is only the division of the group of balls. This case is akin to the one where you have to divide a deck of cards into sub-groups (you aren't required to distribute those sub-groups to players). For the configuration (1, 1, 3), the number of ways of division is ${}^5C_3 = 10$ (we just choose 3 balls out the 5 and the division is automatically accomplished. For the configuration (1, 2, 2), the number of ways of division is $\frac{{}^5C_2 \times {}^3C_2}{2!} = 15$ (division by 2! is required since two sub-groups are of the same size, and here the order of the group doesn't matter). Thus, the total number of ways is 10 + 15 = 25.

(d) This case is quite straightforward. The balls are identical. The boxes are identical too. The only 2 possible configurations are (1, 1, 3) and (1, 2, 2). There can be no permutation of these configurations since the boxes are indistinguishable. Thus, only 2 ways of division exist.

(e) If we keep the boxes in a row, we have inherently ordered them and made them non-identical, since the boxes can be numbered now. Therefore, in this case, the balls are identical but the boxes are different so this question becomes the same as the one in part(b).

S19. (a) This is a simple matter of selecting a group of 5 letters from 26, which can be done in ${}^{26}C_5$ ways.

(b) Consider any particular word which satisfies the given constraint, say AAMNN. If we number the alphabets sequentially from 1 to 26, then

$$\text{AAMNN} \equiv 1,\ 1,\ 13,\ 14,\ 14$$

If we consider the start point as 1, and express the difference between each pair of successive letters, we have

$$0,\quad 0,\quad 12,\quad 1,\quad 0,$$

Diff. b/w start point and A

Diff. b/w A and A

Diff. b/w A and M

etc

Denote the 5 differences by d_i, $i = 1$ to 5. Note that when the given constraint is satisfied, then

$$d_i \geq 0, \qquad (d_i)_{\max} = 25, \qquad (d_1 + d_2 + d_3 + d_4 + d_5)_{\max} = 25.$$

The required number of permutations is simply the number of non-negative integer solutions to

$$d_1 + d_2 + d_3 + d_4 + d_5 \leq 25, \tag{1}$$

which is

$$\sum_{j=0}^{25} {}^{j+4}C_4. \tag{2}$$

This answer is arrived at as follows: in (1), the left side is smaller than or equal to 25, which means that it can take any value from 0 to 25. Thus, we have to calculate the number of non-negative integer solutions to the integer equation,

$$d_1 + d_2 + d_3 + d_4 + d_5 = r,$$

where r can vary from 0 to 25. That is why we have a summation expression in (2), with the summation index going from 0 to 25.

S20. The first sum contains only the even-numbered binomial coefficients, while the second contains only odd-numbered ones. Recall that we have already evaluated the sum S given by

$$S = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

Note that S is the sum of S_1 and S_2, *i.e.*,

$$S_1 + S_2 = \frac{2^{n+1} - 1}{n+1}$$

Thus, if we determine S_1, S_2 is automatically determined, and vice-versa. Let us try to determine S_1 first.

(a) Consider again the general expansion

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

Integrating with respect to x, we have (we have not yet decided the limits)

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_a^b = C_0 x\big|_a^b + C_1 \left.\frac{x^2}{2}\right|_a^b + C_2 \left.\frac{x^3}{3}\right|_a^b + \cdots + C_n \left.\frac{x^{n+1}}{n+1}\right|_a^b$$

Since we are trying to determine S_1, which contains only the even-numbered terms, we have to choose the limits of integration such that the odd-numbered terms vanish. This is easily achievable by setting $a = -1$ and $b = 1$ (understand this carefully). Thus, we have:

$$\frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \cdots\right),$$

which implies that

$$S_1 = \frac{2^n}{n+1}.$$

(b) S_2 is now simply given by

$$S_2 = S - S_1 = \frac{2^{n+1}-1}{n+1} - \frac{2^n}{n+1} = \frac{2^n - 1}{n+1}.$$

S21. We note that

$$^{n+r}C_r = \text{Coeff. of } x^n \text{ in } (1+x)^{n+r}$$

$$\Rightarrow \sum {}^{n+r}C_r = \sum(\text{Coeff. of } x^n \text{ in } (1+x)^{n+r})$$

$$= \text{Coeff. of } x^n \text{ in} \sum (1+x)^{n+r}$$

Thus,

$$S = \text{Coeff. of } x^n \text{ in } [(1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+r}]$$

$$= \text{Coeff. of } x^n \text{ in } (1+x)^n \ \{1 + (1+x) + (1+x)^2 + \cdots + (1+x)^r]$$

$$= \text{Coeff. of } x^n \text{ in } \frac{(1+x)^n\{(1+x)^{r+1} - 1\}}{x}$$

$$= \text{Coeff. of } x^{n+1} \text{ in } \{(1+x)^{n+r+1} - (1+x)^n\}$$

$$= {}^{n+r+1}C_r$$

$$\Rightarrow \quad S = {}^{n+r+1}C_r$$

This can also be proved logically.

S22. Denoting ${}^{3n}C_r$ by simply C_r, the given series sum S can we written as:

$$S = C_1 - 3C_3 + 3^2 C_5 - 3^3 C_7 + \cdots$$

The alternating + and – signs suggest that we have to use a substitution of the form $i\lambda$ in the appropriate binomial expansion. This will be done as follows:

$$(1+x)^{3n} = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$\Rightarrow \quad \frac{(1+x)^{3n}}{x} = \left(\frac{C_0}{x} + C_2 x + C_4 x^3 + \cdots\right) + (C_1 + C_3 x^2 + C_5 x^4 + \cdots)$$

Substituting $x = \sqrt{3}\, i$, we see that the right hand side has a real part equal to S, while the left hand side's real part is 0. Thus, $S = 0$.

S23. We will use integration to obtain the appropriate expression from the following standard binomial expression:

$$(1-x)^n = 1 - C_1 x + C_2 x^2 - \cdots + (-1)^n C_n x^n$$

$$\Rightarrow \quad C_1 - C_2 x + C_3 x^2 - \cdots + (-1)^{n+1} C_n x^{n-1} = \frac{1-(1-x)^n}{x}$$

Now, if we integrate both sides with respect to x between the limits 0 and 1, the LHS becomes S (verify). Thus,

$$S = \int_0^1 \frac{1-(1-x)^n}{x}\, dx.$$

Substituting $(1-x) \to t$, we have

$$S = \int_1^0 \frac{1-t^n}{1-t}(-dt)$$

$$= \int_0^1 (1 + t + t^2 + \cdots + t^{n-1})\, dt = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

S24. We write out the LHS of the first relation in reverse:

$$\text{LHS} = {}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + {}^{m+3}C_m + \cdots + {}^nC_m$$

Now, we write the first term ${}^mC_m$ as ${}^{m+1}C_{m+1}$ and make use of the general relation ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$:

$$\text{LHS} = \underbrace{\left({}^{m+1}C_{m+1} + {}^{m+1}C_m\right)}_{\swarrow} + {}^{m+2}C_m + {}^{m+3}C_m + \cdots + {}^nC_m$$

$$= \underbrace{\left({}^{m+2}C_{m+1} + {}^{m+2}C_m\right)}_{\swarrow} + {}^{m+3}C_m + \cdots + {}^nC_m$$

$$= ({}^{m+3}C_{m+1} + {}^{m+3}C_m) + \cdots + {}^nC_m$$

$$\vdots$$

$$= {}^{n+1}C_{m+1}$$

To prove the second relation, we write its LHS in the form of an expanded expression, a 'triangle' of terms:

$$\text{LHS} = \sum \left\{ \begin{array}{ccccccc} {}^{n}C_m & & & & & & \\ {}^{n-1}C_m & {}^{n-1}C_m & & & & & \\ {}^{n-2}C_m & {}^{n-2}C_m & {}^{n-2}C_m & & & & \\ {}^{n-3}C_m & {}^{n-3}C_m & {}^{n-3}C_m & {}^{n-3}C_m & & & \\ & \vdots & & & & & \\ \underline{{}^{m}C_m} & \underline{{}^{m}C_m} & \underline{{}^{m}C_m} & \underline{{}^{m}C_m} & \cdots & \underline{{}^{m}C_m} \\ \downarrow\Sigma & \downarrow\Sigma & \downarrow\Sigma & \downarrow\Sigma & \cdots & \downarrow\Sigma \\ \boxed{{}^{n+1}C_{m+1}} & \boxed{{}^{n}C_{m+1}} & \boxed{{}^{n-1}C_{m+1}} & \boxed{{}^{n-2}C_{m+1}} & \cdots & \boxed{{}^{m+1}C_{m+1}} \end{array} \right.$$

We first summed the terms column-wise (making use of the result of the previous part) and obtained the terms which are boxed. In the second step, we sum the boxed terms and obtain the required sum as ${}^{n+2}C_{m+2}$.

S25. To get a better understanding about the nature of the sum S we intend to calculate, it is best to write S out in full. To do that, we treat j as our index, while i varies according to j. For example, when $j = 1$, then $i = 0$, while for $j = 4$, $i = 0, 1, 2, 3$, etc. Thus,

$$S = \left\{ \begin{array}{ll} \left(\dfrac{0}{C_0} + \dfrac{1}{C_1}\right) & (j = 1, i = 0) \\ \quad + & \\ \left(\dfrac{0}{C_0} + \dfrac{2}{C_2}\right) + \left(\dfrac{1}{C_1} + \dfrac{2}{C_2}\right) & (j = 2, i = 0, 1) \\ \quad + & \\ \left(\dfrac{0}{C_0} + \dfrac{3}{C_3}\right) + \left(\dfrac{1}{C_1} + \dfrac{3}{C_3}\right) + \left(\dfrac{2}{C_2} + \dfrac{3}{C_3}\right) & (j = 3, i = 0, 1, 2) \\ \quad + & \\ \quad \vdots & \end{array} \right.$$

$$= n\left(\frac{0}{C_0} + \frac{1}{C_1} + \frac{2}{C_2} + \cdots + \frac{n}{C_n}\right) = an$$

S26. We have $S_n = \dfrac{1-q^{n+1}}{1-q}$, while $\lambda_n = \dfrac{2^{n+1}-(1+q)^{n+1}}{2^n(1-q)}$. Now,

$$S = {}^{n+1}C_1 + {}^{n+1}C_2 S_1 + {}^{n+1}C_3\, S_2 + \cdots = {}^{n+1}C_1 + {}^{n+1}C_2\left(\frac{1-q^2}{1-q}\right) + {}^{n+1}C_3\left(\frac{1-q^3}{1-q}\right) + \cdots$$

This can be split into two simpler series:

$$S = \frac{1}{1-q}\left({}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1}\right) - \frac{1}{1-q}\left({}^{n+1}C_1\, q + {}^{n+1}C_2\, q^2 + \cdots + {}^{n+1}C_{n+1}\, q^{n+1}\right)$$

$$= \frac{1}{1-q}(2^{n+1}-1) - \frac{1}{1-q}((1+q)^{n+1}-1) = \frac{2^{n+1}-(1+q)^{n+1}}{(1-q)}$$

Thus, $S = 2^n \lambda_n$.

Probability

PART-A: Summary of Important Concepts

1. Basic Concepts

In this section, we will summarize some important concepts and terminology used in this chapter.

1.1 Outcomes

Outcomes are possible results of a *random experiment*. For example, in the tossing of a coin, obtaining 'Heads' is a possible outcome. In the rolling of a die, obtaining a '3' is a possible outcome.

1.2 Equally likely

Two outcomes A and B of an experiment can be said to be equally likely when there is no evident reason to favor A over B or vice versa. To make the idea more concrete, you can say that as you repeat the experiment an indefinitely large number of times, the relative occurrence of A and B will be equal. For example, if you toss a fair coin an indefinitely large number of times, the relative occurrence of both Heads and Tails will be 50%. Similarly, if a die is rolled an indefinitely large number of times, each of the six faces will have a relative occurrence of $\frac{1}{6}$.

1.3 Events and Sample Space

An event is a set of outcomes. Thus, an event can be viewed as a subset of the universe of all outcomes of the experiment, which is termed as the *sample space* of the experiment. For example, in drawing a card from a well-shuffled deck of 52 cards at random, the sample space is of size 52. The event E defined as

E: The card drawn is red

is a set of 26 outcomes. The event F defined as

F: The card drawn is a king

is a set of 4 outcomes. The event G defined as

G: The card drawn is the Ace of Spades

is a set of only 1 outcome, and is thus an *elementary* event, whereas E and F are *compound* events.

1.4 Probability

If all outcomes in an experiment are equally likely (like in tossing a fair coin, rolling a fair die, drawing a card at random from a well-shuffled deck), then the probability of occurrence of an event E is simply

$$P(E) = \frac{\text{No. of outcomes favorable to } E}{\text{Total No. of outcomes}}$$

Note that $P(E) \in [0, 1]$.

1.5 Events as Sets

Since events are sets of outcomes, set operations can be defined for events. Let E and F be two events associated with a random experiment. We will denote the number of elements in a set S by $\#S$. In general, we have:

$$\#(E \cup F) = \#E + \#F - \#(E \cap F).$$

In case of *mutually exclusive* events (for which no outcome is common), this reduces to:

$$\#(E \cup F) = \#E + \#F.$$

1.6 Conditional Probability

If E and F are two events, then the probability of E occurring given that F has already occurred is termed the conditional probability of E given F, and is written as $P(E/F)$. We have:

$$P(E/F) = \frac{P(E \cap F)}{P(F)}.$$

1.7 Independent Events

If E and F are independent events, then:

$$P(E/F) = P(E) \quad \text{and} \quad P(F/E) = P(F),$$

which implies that:

$$P(E \cap F) = P(E) \cdot P(F).$$

This basically means that the probability of E occurring is not affected by the occurrence or non-occurrence of F, and vice versa.

1.8 Difference between mutually exclusive and independent events

If E and F are two (non-null) events, then

- **If A and B are two non-impossible mutually exclusive events, then they are not independent**.

  Reason: This should be obvious. If A and B are ME events, then the occurrence of one *rules out, i.e., affects*, the occurrence of the other which means that they are not independent. Mathematically speaking, we have

  $$P(A) \neq 0, P(B) \neq 0, \text{but } P(A \cap B) = 0 \text{ because } A \cap B = \phi.$$

  $$\text{Thus, } P(A \cap B) \neq P(A) \cdot P(B).$$

- **If A and B are two non-impossible independent events, they are not mutually exclusive.**

  Reason: This is again very simple to understand. If A and B are independent events, then obviously the occurrence of one *does not rule out* the occurrence of the other, which means that they are not ME. Mathematically, since $P(A) \neq 0$, $P(B) \neq 0$, we have

  $$P(A \cap B) = P(A) \cdot P(B) \neq 0$$

  which means $A \cap B$ is not ϕ.

1.9 Examples

We will highlight all these concepts through three simple scenarios.

Scenario 1:

Suppose that you have with you a standard deck of 52 cards, and you shuffle it really well. You now draw a card at random from this deck. It is obvious that for a well-shuffled deck, you are equally likely to draw any of the 52 cards. For example, if E is the event that the card drawn is the King of Hearts while F is the event that the card drawn is the seven of Diamonds, we have

$$P(E) = P(F)$$

Since there are 52 cards and each is equally likely to turn up, and also, one of the 52 cards must turn up, the numerical value of the probability of any card being drawn is therefore $\frac{1}{52}$. For example,

$$P(E) = P(F) = \frac{1}{52}$$

Now, let G be the event that a card of Spades is drawn. Since G can occur in 13 ways (there are 13 cards in any suit), or in other words, the number of cases favorable to G is 13 out of the total 52 possibilities, we must have

$$P(G) = \frac{13}{52} = \frac{1}{4}$$

This value of $P(G)$ should have been obvious directly, since there are only 4 suits in the deck, and any suit is equally likely to turn up.

Going further, let H be the event that a black card is drawn. We have (# represents 'Number of'),

$$\#\text{Black cards} = \#\text{ Spades} + \#\text{ Clubs}$$

$$\Rightarrow \quad \frac{\#\text{Black cards}}{\#\text{All cards}} = \frac{\#\text{ Spades}}{\#\text{All cards}} + \frac{\#\text{ Clubs}}{\#\text{All cards}}$$

$$\Rightarrow \quad \frac{26}{52} = \frac{13}{52} + \frac{13}{52}$$

We see that $P(H) = \frac{26}{52} = \frac{1}{2}$, since there are 26 cases favorable to H. Again, the value of $P(H)$ should have been obvious directly, since there are only 2 colors in the deck (red, black), and each color is equally likely to turn up.

Scenario 2:

Again, the experiment involves drawing a card at random from a well-shuffled deck of 52 cards, and two events are defined as follows:

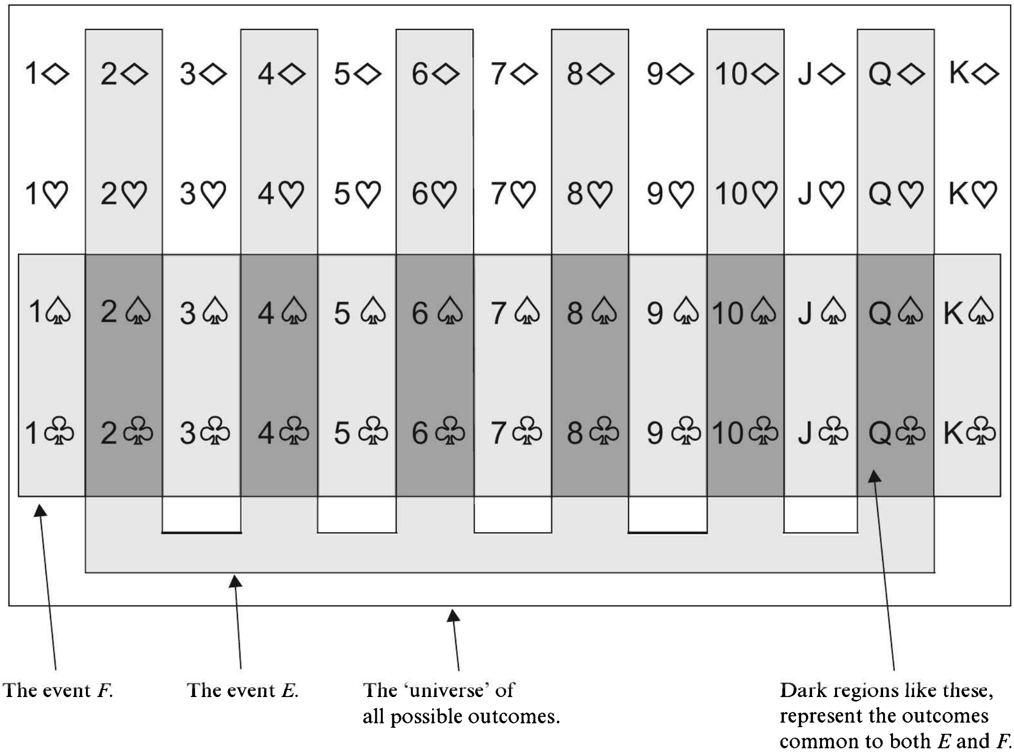
E: The card drawn is even

F: The card drawn is black

The number of outcomes in E is 24 (why?). The number of outcomes in F is 26. However, if we now consider the event G which consists of all the outcomes in E and F, we find that the relation

$$\#G = \#E + \#F$$

is not satisfied. Why? Because the events E and F have some outcomes in common. Let us understand this visually. We let the large rectangle below represent all the 52 possible outcomes. The events E and F are then ***subsets*** of this large rectangle as shown.



Note that E and F have some elements in common, that is, they are not mutually exclusive. As stated earlier, the event G is defined to consist of all the outcomes in E and F, which means that G is the event that the card drawn is *either* even or black (or both). How many elements are there in G? Note that if we write:

$$\#G = \#E + \#F \qquad \text{(× not correct)}$$

the outcomes common to E and F (represented by the dark regions in the figure above) are counted twice on the right side. The actual number of outcomes in G will thus be obtained by subtracting from the RHS the number of these common outcomes, and thus we can write:

$$\#G = \#E + \#F - \#EF \qquad \text{(✓ correct)}$$

where $\#EF$ represents the number of common outcomes. In this case, EF would represent the event that the card drawn is *both* even and black. There are 12 such cards, as is evident from the figure, and thus,

$$\#G = \underset{\#E}{24} + \underset{\#F}{26} - \underset{\#(EF)}{12}$$

$$= 38$$

We note that event G which has been defined as the event consisting of all the outcomes in E and F is technically termed as the *union* of the events E and F, and this fact is symbolically written as

$$\boxed{G = E \cup F} \qquad \text{Union of events}$$

You can think of the union of two events to be the event which can be said to occur when *either* of the two given events (or both) occurs. Analogously, the event H defined to consist of all outcomes common to E and F is technically termed as the *intersection* of the events E and F, and this fact is symbolically written as

$$\boxed{H = E \cap F} \qquad \text{Intersection of events}$$

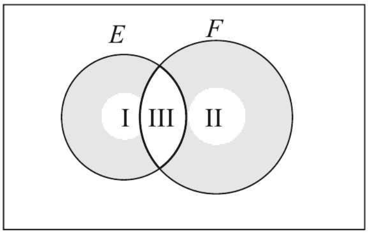
You can think of the intersection of two events to be the event which can be said to occur when *both* the two given events occur *simultaneously*. Finally, as we've seen above, we always have

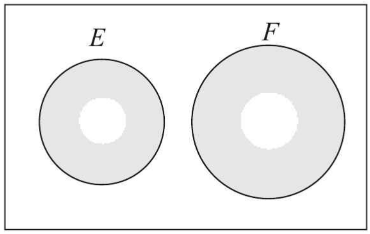
$$\boxed{\#(E \cup F) = \#E + \#F - \#(E \cap F)}$$

If E and F have no outcomes in common, that is, if E and F are mutually exclusive $(\#(E \cap F) = 0)$, this relation reduces to

$$\boxed{\#(E \cap F) = \#E + \#F}$$

Here's a visual depiction:





E and F are not mutually exclusive

$\# E \cup F$	=	$\#$ E	+	$\#$ F	−	$\#$ $(E \cap F)$
Shaded region		Complete cricle (I)		Complete cricle (II)		Region (III)

E and F are mutually exclusive since their intersection is null, *i.e.*, they have no outcomes in common:

$\# E \cup F \;=\; \# E + \# F$

In terms of probabilities, we will have:

$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)} \qquad \text{General relation}$$

$$\boxed{P(E \cup F) = P(E) + P(F)}$$

The particular case of E and F being mutually exclusive.

Scenario 3:

In the experiment of drawing one card at random from a standard well-shuffled deck of 52 cards, consider the event E defined as:

E: The card drawn is a King

Obviously, we have

$$P(E) = \frac{\#\text{Kings}}{\#\text{All cards}} = \frac{4}{52} = \frac{1}{13}$$

However, if you are now asked to find the probability of event E occurring *given that* the card drawn has a number greater than 10, what would you say? To answer correctly, you must understand that now the situation is entirely different from earlier. Now we already *know* that the card drawn has a number greater than 10, which means that the number of possibilities for the card has reduced;

earlier, there were 52 possibilities, but now the number of possible cards are the 4 Jacks, the 4 Queens and the 4 Kings, which means that now there are only 12 possibilities for the card. Of these, the favorable possibilities are 4; thus,

$$P\begin{Bmatrix}\text{Event } E \text{ given that the card drawn} \\ \text{has a number greater that 10}\end{Bmatrix} = \frac{\#\text{ Kings}}{\#\text{ Reduced possibilities}} = \frac{4}{12} = \frac{1}{3}$$

Let us denote by F the event that the card drawn has a number greater than 10. The probability just calculated above is then written in standard notation as $P(E/F)$ which is read as $P(E$ given $F)$, that is, the probability of event E occurring given that F has occurred. Let us consider another example. In the random experiment of throwing two dice, let events G and H be defined as

G: The sum of the two numbers on top is 6

H: One of the numbers on top is 4

First, let us find the probability of G occurring. This is

$$P(G) = \frac{\#\text{favorable possibilities}}{\#\text{total possibilites}} \equiv \frac{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)}{6\times 6} = \frac{5}{36}$$

Now, suppose we are given that H has already occurred, meaning we now know that one of the numbers on top is 4. What is the probability of G now when we already possess this information? The number of total possibilities now is 11; we list them explicitly:

$$\begin{Bmatrix}(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (1,4) & (2,4) & (3,4) & & (5,4) & (6,4)\end{Bmatrix}$$

Out of these, the favorable possibilities are only 2, namely: (4, 2) and (2, 4). Thus

$$P(G/H) = \frac{\#\text{favorable possibilities}}{\#\text{reduced possibilities}} = \frac{2}{11}$$

In both the preceeding examples, what happens is that with the possession of some information, the number of possibilities reduce, *i.e.*, the sample space reduces. In the first example, with the information that F has occurred, the number of possibilities reduces from 52 to 12. We then looked for favorable cases only within this reduced sample space of 12 outcomes. Similarly, in the second example, when we possess the information that H has occurred, the number of possibilities reduces from 36 to 11; it is in this reduced sample space of outcomes that we then looked for the favorable possibilities.

As we know, probabilities of the form $P(A/B)$ are called *conditional* probabilities. $P(A/B)$ is said to be the conditional probability of A given that B has occurred.

An important, related issue needs to be considered at this point. Let E and F be two events. Let the probability of E occurring be $P(E)$. Now, if we come to know that F has occurred, will the probability of E occurring always increase, like it did in the last two examples, or can it decrease too? Can it remain the same? It turns out that all the three are possible.

We have already seen an example where the knowledge that F has occurred increases the probability of E occuring. Let us now consider the case where the information that F has occurred *decreases* the probability of E occurring. Let E and F be, in the card drawing experiment of the first example, two events defined as follows:

E: The card drawn has a number which is a multiple of four

F: The card drawn has a number greater than eight

The original probability of E occurring is:

$$P(E)=\frac{\#\text{favourable possibilities}}{\#\text{total possibilites}}\equiv\frac{\{4,8,12\}\text{ per suit}\times 4\text{ suits}}{52}=\frac{12}{52}=\frac{3}{13}.$$

But, when F has already occurred, the probability of E occurring is:

$$P(E/F)=\frac{\#\text{favourable possibilities}}{\#\text{ reduced possibilites}}=\frac{4\text{ Queens}}{\{9,10,J,Q,K\}\text{ per suit }\times\ 4\text{suits}}=\frac{4}{20}=\frac{1}{5},$$

which is lesser than the original probability of E occurring. This should be intuitively obvious: In the first case, there are 3 multiples of four per suit of 13 cards. In the second, when we are told that the number of the card is greater than eight; the multiple of four now possible is only 1 per suit which is the Queen. Thus the favorable possibilities decrease to one-third. The total possibilities also decrease *i.e.*, from 13 to 5 per suit, but it can be easily appreciated that the percentage reduction in favorable possibilities is greater than the percentage reduction in total possibilities.

Finally, let us consider the third case: for two events E and F, can $P(E/F)$ be the same as $P(E)$? This is equivalent to saying that the probability of E occurring is *not affected* by the occurrence or non-occurrence of F. As we said earlier, this is possible, and such events are called independent events:

If $P(E/F) = P(E)$ $\Rightarrow$ E and F are *independent* events

Let us see an example. In the card-drawing experiment, let events E and F be defined as

E: The card has a number greater than eight

F: The card is black

The reader may observe that $P(E/F)$ is the same as $P(E)$. Why? Because, the knowledge that the card is black *does not change* the number of cards *per suit* that are greater than eight. Stated explicitly,

$$P(E)=\frac{\{9,10,J,Q,K\}\text{ per suit}\times 4\text{ suits}}{52}=\frac{20}{52}=\frac{5}{13}$$

$$\text{and, }P(E/F)=\frac{\{9,10,J,Q,K\}\text{ per suit}\times 2\text{ suits}}{26}=\frac{5}{13}\begin{Bmatrix}\text{because there are}\\\text{only two black suits}\end{Bmatrix}$$

There is one important point you must notice and appreciate:

If $P(E / F) = P(E)$ $\rightarrow$ this means that E and F are independent events

$\rightarrow$ this should also mean that $P(F/E)$ should be the same as $P(F)$

Let us verify this in the example above. We have,

$$P(F)=\frac{\#\text{ black cards}}{\#\text{ total cards}}=\frac{26}{52}=\frac{1}{2}$$

$$\text{and, }P(F/E)=\frac{\#\text{ black cards greater than eight}}{\#\text{ total cards greater than eight}}$$

$$=\frac{\{9,10,J,Q,K\}\text{ of Spades and of Clubs}}{\{9,10,J,Q,K\}\text{ per suit}\times 4\text{ suits}}=\frac{10}{20}=\frac{1}{2}$$

which confirms our assertion. Thus, if A and B are independent events, we have

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

Note that the favorable cases while calculating $P(A/B)$ are those cases in A that are common to B; the total cases are all cases in B. Thus

$$P(A/B) = \frac{\#\text{ favorable cases}}{\#\text{ total cases}} = \frac{\#(A \cap B)}{\#(B)}$$

If the entire sample space of the experiment consists of N outcomes, we can write the above relation as

$$P(A/B) = \frac{\#(A \cap B)/N}{\#(B)/N} = \frac{P(A \cap B)}{P(B)}$$

$$\text{Similarly, } P(B/A) = \frac{P(A \cap B)}{P(A)}$$

This means that if A and B are independent, then

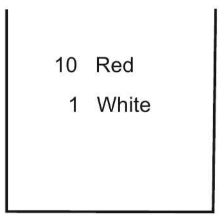
$$P(A/B) = P(A) = \frac{P(A \cap B)}{P(B)} \Rightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Thus, the probability of the intersection of two independent events is the product of the individual probabilities. We note that two events A and B from two different sample spaces will always be independent. For example, in an experiment of throwing a coin and a die simultaneously, any outcome of the coin throw is independent of any outcome of the die roll.

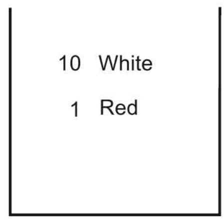
2. Inverse Probability and Baye's Theorem

Calculating inverse probabilities using Baye's Theorem is one of the most important requirements of this chapter. Many students tend to memorize the formula for the theorem without clearly understanding its basis. Therefore, we have chosen to describe this theorem in some detail, and also highlight the use of *probability trees* to solve all inverse probability problems.

Suppose that you have two bags α and β, one containing 10 red and 1 white balls, and the second containing 10 white and 1 red balls.

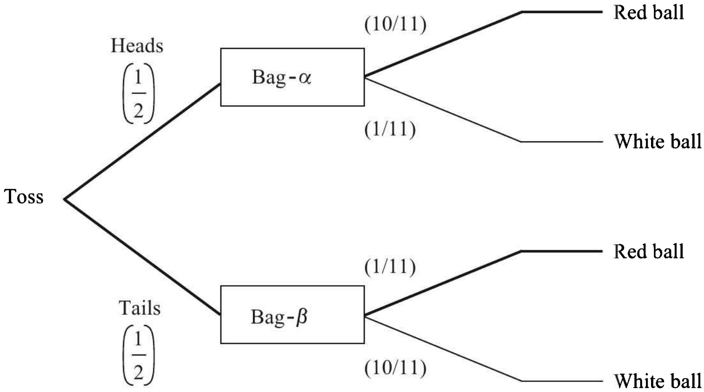


Bag-α



Bag-β

You play a game with your friend. The friend tosses a fair coin, and without telling you the outcome, if he gets *Heads*, he withdraws a ball from Bag-α while if he gets *Tails*, he withdraws a ball from Bag-β, with you looking away all the time. After doing this once, he has a red ball in his hand. Which bag do think is the more likely one from which this ball was drawn? Intuition immediately tells us that it should be Bag-α, since it has a large number of red balls. What we need to do now is quantify the *inverse probability* of the ball being drawn from Bag-α and Bag-β, the term 'inverse' being used since you are trying to find the probability of an event that has already taken place, using information from a subsequent event. Let us draw a tree diagram highlighting the various possible actions your friend can take (the brackets show the probabilities of the corresponding paths):



Now comes the crucial part. Note that the total probability of selecting a red ball is the sum of the probabilities of the two darkened paths (one through Bag-α, one through Bag-β). This is:

$$P(\text{Red Ball}) = \frac{1}{2} \times \frac{10}{11} + \frac{1}{2} \times \left(\frac{1}{11}\right) = \frac{1}{2}.$$

We further observe that the probability of selecting a red ball through Bag-α corresponds to the upper path only, and it equals:

$$P(\text{Red Ball from Bag-}\alpha) = \frac{1}{2} \times \frac{10}{11} = \frac{5}{11}.$$

Similarly,

$$P(\text{Red Ball from Bag-}\beta) = \frac{1}{2} \times \frac{1}{11} = \frac{1}{22}.$$

Finally, it might be intuitively clear to you:

$$P\begin{Bmatrix}\text{Bag selected was Bag-}\alpha \\ \text{given that ball is red}\end{Bmatrix} = \frac{P(\text{Red ball from Bag-}\alpha)}{P(\text{Red ball})} = \frac{5/22}{1/2} = \frac{10}{11},$$

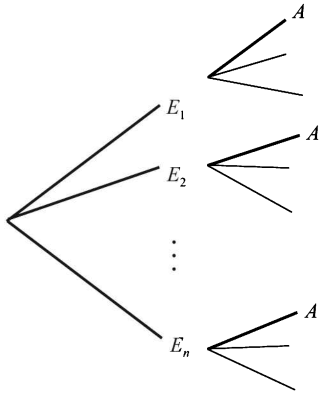
while $$P\begin{Bmatrix}\text{Bag selected was Bag-}\beta \\ \text{given that ball is red}\end{Bmatrix} = \frac{P(\text{Red ball from Bag-}\beta)}{P(\text{Red ball})} = \frac{1/22}{1/2} = \frac{1}{11}.$$

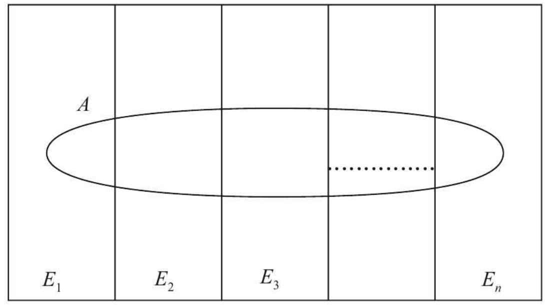
Note what a huge difference there is between the two probabilities, which was expected. Also expected is the fact that the two probabilities sum to 1.

This, then, is the essence of calculating inverse probabilities. We are given the information that an event E has occurred. This event E can occur through n paths: $\text{Path}_1, \text{Path}_2, \ldots, \text{Path}_n$. We want to find the probability that E occurred through some particular path, say Path_i, which is

$$P(E \text{ occurred through Path}_i) = \frac{P(\text{Path}_i)}{P(\text{Path}_1) + P(\text{Path}_2) + \ldots + P(\text{Path}_n)}$$

Let us write this in standard terminology, which will give us the Baye's theorem. Suppose that the sample space consists of n mutually exclusive events $E_1, E_2, .., E_n$. Now, an event A occurs, which could have resulted from any of the events E_i (for example, think of A as obtaining a red ball in the previous example, while E_1 and E_2 are selecting Bag-α and Bag-β respectively). We intend to find $P(E_i/A)$, *i.e.*, the probability that E_i occurred given that A has occurred. There are now two ways to do the visualisation:





The left hand tree we have already explained. The right hand side shows that A is an event that has occurred, which must have been a result of one of the E_i occurring (*i.e.*, one of the E_i's must occur for A to occur)

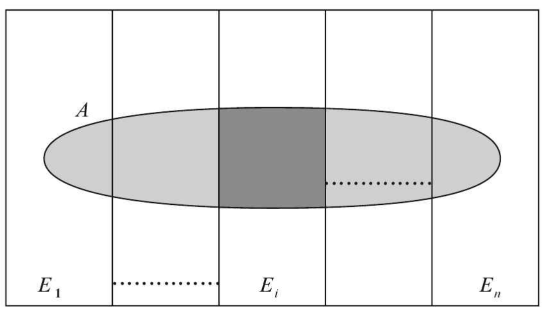
From the tree, evaluating $P(E_i/A)$ has already been explained:

$$P(E_i/A) = \frac{P(\text{Path to } A \text{ through } E_i)}{\sum_{j=1}^{n} P(\text{Path to } A \text{ through } E_j)}$$

$$= \frac{P(E_i \text{ occurs and } \textbf{then } A \text{ occurs})}{\sum_{j=1}^{n} P(E_j \text{ occurs and then } A \text{ occurs})}$$

$$= \frac{P(E_i \text{ occurs}) \times P\{A \text{ occurs given that } E_i \text{ has occured}\}}{\sum_{j=1}^{n} P(E_j \text{ occurs}) \times P\{A \text{ occurs given that } E_j \text{ has occured}\}} = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}$$

The same relation follows from the second figure:



$$P(E_i/A) = \frac{P(\text{Darkly shaded region})}{P(\text{Total shaded region})}$$

$$= \frac{P(A \cap E_i)}{\sum_{j=1}^{n} P(A \cap E_j)} = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}$$

Thus, the famous Baye's theorem is:

$$\boxed{P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}}$$

The name 'inverse' stems from the fact that this relation gives us $P(E_i/A)$ in terms of $P(A/E_j)$. The theorem is also known as a theorem on the probability of causes. In the examples on Baye's Theorem in the next section, we'll be using tree diagrams to calculate inverse probabilities.

IMPORTANT IDEAS AND TIPS

1. ***Concept of Probability:*** At this stage, the best way to think of probability is in terms of relative frequency. Consider a typical example: a bag contains 3 identical red, 4 identical blue and 5 identical green balls. A pair of balls is drawn. The probability that the pair is red comes out to be $\frac{^3C_2}{^{12}C_2}$ or $\frac{1}{22}$, (which we approximate as around 5%). How do we interpret this result? We think of a large number of such bags (say 1 million), each bag containing 3 identical red, 4 identical blue and 5 identical green balls. For each bag, a pair of balls is drawn. Then, from approximately 5% of the bags, a red pair would have been drawn. This is the concept of relative frequency and it helps give a physical significance to the more abstract concept of probability.
2. ***Equal Likelihood:*** The formula that we apply to calculate probability, $\frac{\#\text{favourable outcomes}}{\#\text{total outcomes}}$, is valid only when all the cases have equal likelihood of occurrence. This very important point is overlooked a lot of times. For example, if a rolling die is not fair, then you cannot assign a probability of $\frac{1}{6}$ for each face showing up. Sometimes, the way you count the total cases and favorable cases can lead to a mistake. Consider a random experiment involving the rolling of two dice simultaneously. Suppose you have to evaluate the probability of getting a total of less than 6. What is the mistake with the following argument?

   'There are a total of 11 possible cases, namely {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, out of which 4 are favorable, namely {2, 3, 4, 5}, and thus the required probability is $\frac{4}{11}$.'

   The mistake is that the different cases do not have equal likelihood of occurrence. For example, the sum 6 is more likely to occur than the sum 2 (why?). The correct way to solve this problem would be to consider the 36 equally likely outcomes (x, y) where x and y can take integer values from 1 to 6, and then consider those outcomes from this set of 36 outcomes which lead to a sum of less than 6. You can verify that there will be 10 such favorable outcomes. And now, it would be correct to apply the formula $\frac{\#\text{favourable outcomes}}{\#\text{total outcomes}}$ to obtain the required probability as $\frac{10}{36}$.
3. *Terminology:* Being intimately familiar with the terminology of probability helps a lot in thinking with more clarity. In particular, always think of outcomes as the most elementary results of an experiment, events as sets of outcomes, the sample space as the set of all possible outcomes, and events as subsets of the sample space.
4. *Identical Objects:* Suppose that you had a bag containing 10 identical red balls and 3 identical blue balls. You are asked two questions:

   (a) What is the number of ways in which two balls can be drawn from the bag?
   (b) What is the probability that if you draw two balls, both turn out to be red?

   The answer to the first question is that there are only 3 possible ways: {Red, Red}, {Red, Blue} and {Blue, Blue}, due to the identical nature of the balls. Now, for the second question, you might take the following line of reasoning. 'If there are only 3 possible ways to draw two balls (i.e, only 3 total cases), and there is only one favorable case, namely {Red, Red}, the required probability should be $\frac{1}{3}$'. However, once again we encounter a situation where the different outcomes are not equally likely. The outcome {Red, Red} is much more likely than the outcome {Blue, Blue}, since there are so many

more red balls than blue. So the answer $\frac{1}{3}$ is incorrect. The correct way to solve this problem would be as follows: we think of all the balls as different for a moment. Then, the number of possible pairs of balls is ${}^{13}C_2$, whereas the number of possible pairs of red balls is ${}^{10}C_2$, and so the required probability is $\frac{{}^{10}C_2}{{}^{13}C_2}$ or $\frac{15}{23}$ (approximately 58%), which is much higher than the $\frac{1}{3}$ or 33% probability we calculated first. Observe that even though the red balls are identical, and the blue balls are identical, we thought of them as distinct, to be able to calculate the required probability. This is because probability depends on the numbers of objects. If instead of 10 red and 3 blue balls, we had 10 red and 30 blue balls, the number of ways of forming pairs would still be 3, but the likelihood of each way changes in such a way that the probability of obtaining a red pair would go down significantly (you can verify that it will come down to about 10%).

5. ***Baye's Theorem:*** One of the most interesting (and also confusing) aspects of probability is the calculation of inverse probabilities through Baye's theorem. Generally, Baye's theorem is remembered as a formula, and whenever students encounter an inverse probability problem, they try to apply that formula without analyzing that problem in more depth. In our opinion, you should always draw a probability tree corresponding to the situation described in any such problem. This will always give you more insight into the problem than a direct application of the formula, and it may even prevent you from obtaining wrong results. For details on how to draw probability trees for inverse probability problems, refer to some of the examples discussed in this chapter.

Probability

PART-B: Illustrative Examples

Example 1

Two persons X and Y are playing a game: They throw a coin alternately until one of them gets Heads and wins. What is the probability of winning of the person who makes the first throw?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$ (E) None of these

Solution: Suppose that X makes the first throw. Let us calculate the probability of X winning the game. Let H_x, T_x denote a Heads and a Tails respectively obtained by X. A similar notation follows for Y. Now, X will win the game in the following (mutually exclusive) sequences of tosses:

Sequence	***P*(Sequence)**
H_x	$\frac{1}{2}$
$T_xT_yH_x$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{2^3}$
$T_xT_yT_xT_yH_x$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{2^5}$
$T_xT_yT_xT_yT_xT_yH_x$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{2^7}$
$\vdots$	$\vdots$

Thus, the probability of X winning the game is

$$P(X \text{ wins}) = \frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots\infty \text{ (a GP) } = \frac{2}{3}$$

This means that one who makes the first throw has twice the chance $(\frac{2}{3})$ of winning than the other $(\frac{1}{3})$. The correct option is (B). ■

Example 2

Consider the sequence of numbers 1, 2, 3,..., 13. A person chooses three numbers at random from this sequence. What is the probability that the three numbers form an AP?

(A) $\frac{18}{143}$ (B) $\frac{21}{157}$ (C) $\frac{24}{163}$ (D) $\frac{29}{180}$ (E) None of these

Solution: Note that APs can be formed with varying common differences (CD). (1, 2, 3) and (3, 4, 5) are examples of APs with CD 1. (2, 5, 8) is an AP of CD 3. The maximum CD possible is 6, in the AP (1, 7, 13). Let us count all such APs in a table. Verify the column on the right.

CD	No. of APs
1	11
2	9
3	7
4	5
5	3
6	1
	Total=36

Thus, the total number of APs possible from this set is 36. Also, from this set of 13 numbers, a selection of 3 numbers can be made in ${}^{13}C_3 = 286$ ways. Therefore, the probability that three numbers picked at random from this set form an AP is $\frac{36}{286} = \frac{18}{143}$. The correct option is (A). ■

Example 3

A pair of dice is rolled until a sum of either 5 or 7 is obtained. What is the probability that 5 comes before 7?

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{2}{5}$ (D) $\frac{1}{3}$ (E) None of these

Solution: A sum of 5 can be obtained in 4 ways, namely

$$\{(1, 4)\ (2, 3)\ (3, 2)\ (4, 1)\}$$

from a total number of 36 ways of throwing a pair of dice. If we let E denote the event of obtaining a sum of 5, we have

$$P(E) = \frac{4}{36} = \frac{1}{9}$$

Similarly, let the event F be that of obtaining a sum of 7; this can happen in 6 ways, namely

$$\{(1, 6)\ (2, 5)\ (3, 4)\ (4, 3)\ (5, 2)\ (6, 1)\}$$

so that

$$P(F) = \frac{6}{36} = \frac{1}{6}$$

Finally, if we let G be the event of obtaining neither a 5 or a 7, we have:

$$P(G) = 1 - P(E) - P(F)$$
$$= 1 - \frac{1}{9} - \frac{1}{6} = \frac{13}{18}$$

Now, we want a sum of 5 to come before a sum of 7. Think about how this can happen. Every time you roll the pair of dice, you should either get a sum of 5 or you should get neither a sum of 5 nor 7. Therefore, the following (mutually exclusive) sequences of throws lead us to a sum of 5 before a sum of 7. The first column shows the various sequences, and the second column shows the probability of occurrence of these sequences:

Sequence	**P(Sequence)**
E	$\frac{1}{9}$
G, then E	$\frac{13}{18} \times \frac{1}{9}$ (Successive Rolls are independent events!)
G, then G, then E	$\frac{13}{18} \times \frac{13}{18} \times \frac{1}{9}$
G, then G, then G, then E	$\frac{13}{18} \times \frac{13}{18} \times \frac{13}{18} \times \frac{1}{9}$
$\vdots$	$\vdots$

The required probability is obtained by adding the terms in the right column.

$$P\text{ (5 before 7)} = \frac{1}{9} + \frac{13}{18} \cdot \frac{1}{9} + \left(\frac{13}{18}\right)^2 \cdot \frac{1}{9} + \left(\frac{13}{18}\right)^3 \cdot \frac{1}{9} + \cdots\infty$$
$$= \frac{1}{9}\left(1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2 + \left(\frac{13}{18}\right)^3 + \cdots\infty\right)$$
$$= \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}} = \frac{2}{5}$$

Thus, the correct option is (C). In passing, note that the probability of obtaining a 7 before 5 is simply

$$P\text{ (7 before 5)} = 1 - P\text{(5 before 7)} = \frac{3}{5}.$$

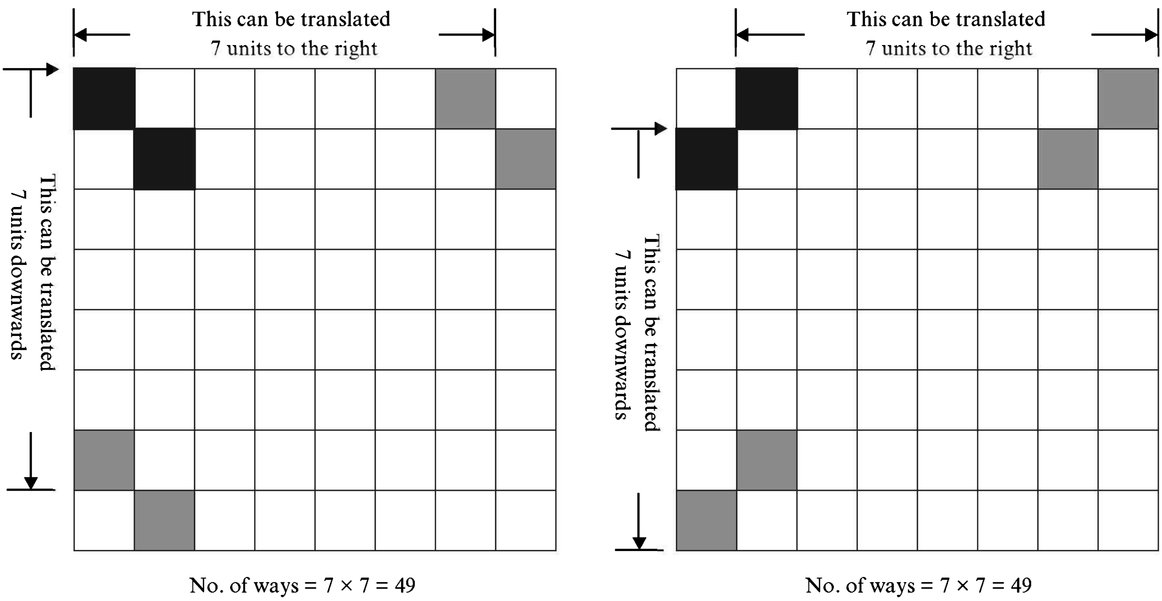
Can you appreciate why obtaining a 7 before 5 is more likely than obtaining a 5 before 7? ■

Example 4

Two unit squares are chosen at random from a standard chessboard. What is the probability that the two squares have exactly one corner in common?

(A) $\frac{5}{144}$ (B) $\frac{7}{144}$ (C) $\frac{11}{144}$ (D) $\frac{13}{144}$

Solution: The total number of ways of selecting two squares from a standard chessboard is ${}^{64}C_2 = 2016$. Now, let us find the number of ways of selecting a pair with exactly one corner in common. For that, consider the following figure.



It is immediately evident that the number of favorable ways is $49 + 49 = 98$. The required probability is $\frac{98}{2016} = \frac{7}{144}$. The correct option is (B). ■

Example 5

Three standard dice are rolled together. What is the probability that the sum of the numbers appearing on the dice belongs to the set {3, 4, 5, 6, 7, 8}?

(A) $\frac{17}{72}$ (B) $\frac{1}{4}$ (C) $\frac{19}{72}$ (D) $\frac{5}{18}$ (E) None of these

Solution: The sample space consists of $6 \times 6 \times 6 = 216$ outcomes. Now, let us find the number of ways in which a sum of n can be obtained. This is the number of solutions to the non-negative integr equation

$$x_1 + x_2 + x_3 = n, \text{ subject to } 1 \le x_i \le 6$$

Since each x_i will at least be 1, we can equivalently find the number of solutions to

$$y_1 + y_2 + y_3 = n - 3, \quad y_i = x_i - 1$$

which, in this particular case, is straightforward to obtain by the relevant relation we use to find the number of non-negative solutions to an integer equation; the reason why that formula is applicable directly to the second equation, but not to the first is very important to understand. Can you see why it is so?

The required number of solutions to the second equation is

$$^{(n-3)+3-1}C_2 = {}^{n-1}C_2 = \frac{(n-1)(n-2)}{2}.$$

Thus, the probability that the sum is n is

$$P(\text{Sum} = n) = \frac{(n-1)(n-2)/2}{216} = \frac{n^2 - 3n + 2}{432}.$$

The total probability is obtained by summing the right hand side for $n = 3$ to $n = 8$; thus,

$$P(\text{Sum} \in \{3, 4, 5, 6, 7, 8\})$$

$$= \frac{\sum_{n=3}^{8}(n^2 - 3n + 2)}{432} = \frac{\sum_{n=3}^{8} n^2 - 3\sum_{n=3}^{8} n + 2}{432}$$

$$= \frac{199 - 3 \times 33 + 2}{432} = \frac{102}{432} = \frac{17}{72}.$$

The correct option is (A). ■

Example 6

If $6n$ balls numbered $0, 1, 2, \ldots, 6n-1$ are placed in a bag and three are drawn at random without replacement, the probability that the sum of the three numbers on the balls is $6n$ will be

(A) $\dfrac{3n}{(6n-1)(6n-2)}$ (C) $\dfrac{4n}{(6n-1)(6n+2)}$ (E) None of these

(B) $\dfrac{2n}{(6n-1)(6n+1)}$ (D) $\dfrac{n}{(6n-1)(6n-2)}$

Solution: The important thing to realise here is that we cannot use the integer equation

$$x_1 + x_2 + x_3 = 6n \tag{1}$$

to count the number of solutions in this case. This is because this equation also counts those solution in which the variables might have repeated values. For example, $x_1 = x_2 = x_3 = 2n$ is

a possible solution to (1), but it is clearly not admissible in the present case. What we then do is count the number of solutions explicitly, using the lowest number as our *anchor*.

Number on Ball 1	Numbers on the Other Two Balls	Number of Solutions
0	$1, 6n-1$ $2, 6n-2$ $3, 6n-3$ $\vdots$ $3n-1, 3n+1$	$3n-1$
1	$2, 6n-3$ $3, 6n-4$ $\vdots$ $3n-2, 3n+1$ $3n-1, 3n$	$3n-2$
2	$3, 6n-5$ $4, 6n-4$ $\vdots$ $3n-2, 3n$ $3n-1, 3n-1$	$3n-4$
3	$4, 6n-7$ $5, 6n-8$ $\vdots$ $3n-2, 3n-1$	$3n-5$
$\vdots$ $2n-1$	$2n, 2n+1$	$\vdots$ 1

Go through this table thoroughly. In particular, notice carefully the terms in the last column. We thus have the total number of favorable solutions as

$$\begin{aligned}&\{(3n-1)+(3n-2)\}+\{(3n-4)+(3n-5)\}+\cdots+\{(5+4)\}+\{(2+1)\}\\ &= (6n-3)+(6n-9)+\cdots+9+3 \quad \text{(an AP)}\\ &= 3n^2\end{aligned}$$

The total number of ways of choosing 3 balls out of $6n$ is ${}^{6n}C_3$. The required probability p is therefore

$$p = \frac{3n^2}{{}^{6n}C_3} = \frac{3n}{(6n-1)\,(6n-2)}.$$

The correct option is (A). ■

Example 7

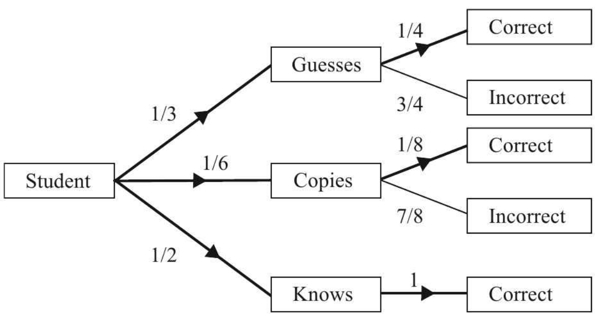
In an objective type examination, each question has four options, of which only one is correct. A student taking the examination either guesses or copies or knows the answer to any question. The probability that he guesses is $\frac{1}{3}$ and that the he copies is $\frac{1}{6}$. The probability that he answers correctly if he copies is $\frac{1}{8}$. If an answer is given to be correct, what is the probability that the student knew the answer?

(A) $\dfrac{19}{26}$ (B) $\dfrac{4}{7}$ (C) $\dfrac{24}{29}$ (D) $\dfrac{27}{34}$ (E) None of these.

Solution: We see that the student can make three moves, with associated probabilities as:

$$P(\text{makes a guess}) = \frac{1}{3}, \quad P(\text{copies}) = \frac{1}{6}, \quad P(\text{knows}) = 1 - \left(\frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2}$$

Now, if he copies, we know that the probability of his answer being correct is $\frac{1}{8}$. Also, if he guesses, he'll select any of the four options with equal likelihood, which means that the probability of his answer being correct is $\frac{1}{4}$. Thus, we have the following probability tree:



We see that we have three paths leading to the correct answer, and we want to find the probability of the bottomost path, that is the probability that the student knew the answer given that he answered it correctly. This probability is now given by:

$$P(\text{knew/answered correctly}) = \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}.$$

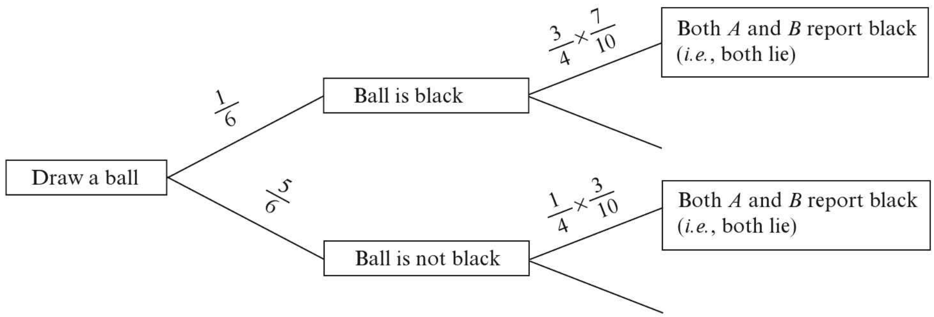
The correct option is (C). Can you appreciate why this value is so high (greater than 80%)? ■

Example 8

A speaks truth 3 times out of 4 while *B* does so 7 times out of 10. A ball is drawn at random from a bag containing one black ball and five other balls of different colors. Both *A* and *B* report that a black ball has been drawn from the bag. What is the probability of their assertion being true?

(A) $\dfrac{1}{2}$ (B) $\dfrac{7}{12}$ (C) $\dfrac{2}{3}$ (D) $\dfrac{3}{4}$ (E) None of these

Solution: To slightly rephrase the problem, we've to find the probability of the ball actually being black given that both A and B report it to be black. The probability tree can easily be drawn, as shown below (partially shown here, only the paths relevant to us have been completed):



We want to find the probability of the upper path which is:

$$P\{(\text{Ball is black}) \mid (\text{Both } A \,\&\, B \text{ report it to be black})\}$$

$$= \frac{\frac{1}{6}\times\left(\frac{3}{4}\times\frac{7}{10}\right)}{\frac{1}{6}\times\left(\frac{3}{4}\times\frac{7}{10}\right)+\frac{5}{6}\times\left(\frac{1}{4}\times\frac{3}{10}\right)} = \frac{7}{12}$$

Let us now rewrite the solution in standard terminology. Let us define the following events:

B: Black ball is drawn from the bag

X: Both A and B report black.

Thus, observe that we need to find $P(B/X)$ which is by the Baye's theorem (in fact compare this relation with the tree diagram in the figure above and see how the two correspond):

$$P(B/X) = \frac{P(B)P(X/B)}{P(X)} = \frac{P(B)P(X/B)}{P(B)P(X/B)+P(\bar{B})P(X/\bar{B})}$$

$$= \frac{\frac{1}{6}\times\left(\frac{3}{4}\times\frac{7}{10}\right)}{\frac{1}{6}\times\left(\frac{3}{4}\times\frac{7}{10}\right)+\frac{5}{6}\times\left(\frac{1}{4}\times\frac{3}{10}\right)} = \frac{7}{12}.$$

The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

Example 9

A fair coin is tossed 10 times. Find the probability of obtaining:

(a) exactly 6 Heads (b) at the most 6 Heads (c) at least 6 Heads.

Solution: **(a)** Consider any arbitrary sequence of 10 tosses that contains exactly 6 Heads. For example, consider

{H T H H H T T H H T}

Any such sequence has a probability of occurrence equal to $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\cdots$ 10 times $=\frac{1}{2^{10}}$. Thus, what we need to do is count the number of sequences with exactly 6 Heads. The required probability will then be $\frac{1}{2^{20}}$ $\times$(Number of sequences). Counting such sequences is straightforward, and the answer is that there are ${}^{10}C_6$ such sequences. Thus,

$$P(\text{exactly 6 Heads}) = {}^{10}C_6 \times \frac{1}{2^{10}} = \frac{210}{1024} = \frac{105}{512}.$$

In general, we see that

$$P(\text{exactly } n \text{ Heads}) = \frac{{}^{10}C_n}{2^{10}}.$$

(b) For this question, we consider all the possible cases and add their respective probabilities like this:

$$\begin{aligned} P(\text{at most 6 Heads}) &= P(0 \text{ Head}) + P(1 \text{ Head}) + P(2 \text{ Heads}) + P(3 \text{ Heads}) \\ &\quad + P(4 \text{ Heads}) + P(5 \text{ Heads}) + P(6 \text{ Heads}) \\ &= \frac{{}^{10}C_0}{2^{10}} + \frac{{}^{10}C_1}{2^{10}} + \frac{{}^{10}C_2}{2^{10}} + \frac{{}^{10}C_3}{2^{10}} + \frac{{}^{10}C_4}{2^{10}} + \frac{{}^{10}C_5}{2^{10}} + \frac{{}^{10}C_6}{2^{10}} \\ &= \frac{1+10+45+120+210+252+210}{1024} = \frac{53}{64}. \end{aligned}$$

(c) Similarly, we have:

$$\begin{aligned} P(\text{at least 6 Heads}) &= P(6 \text{ Heads}) + P(7 \text{ Heads}) + P(8 \text{ Heads}) \\ &\quad + P(9 \text{ Heads}) + P(10 \text{ Heads}) \\ &= \frac{{}^{10}C_6}{2^{10}} + \frac{{}^{10}C_7}{2^{10}} + \frac{{}^{10}C_8}{2^{10}} + \frac{{}^{10}C_9}{2^{10}} + \frac{{}^{10}C_{10}}{2^{10}} \\ &= \frac{210+120+45+10+1}{2^{10}} = \frac{193}{512} \end{aligned}$$

■

Example 10

For two events A and B, we are given the following information:

$$P(A) = \frac{1}{4}, \quad P(A/B) = \frac{1}{4} \text{ and } P(B/A) = \frac{1}{2}$$

(a) Determine whether the two events are ME (mutually exclusive) or independent, or neither.
(b) Find $P(\overline{B}/\overline{A})$. The notation $\overline{A}$ stands for the complementary event of event A, *i.e.*, $\overline{A}$ occurs if A does not occur.

Solution: (a) Since $P(A)$ and $P(A/B)$ are the same, this means that A and B are independent events. Hence they are obviously not ME. We have:

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{(independence of the two events)}$$

$$\Rightarrow \quad P(B) = \frac{1}{2}.$$

(b) Now, if A and B are independent events, then so are $\overline{A}$ and $\overline{B}$ (and in fact (A and $\overline{B}$) and ($\overline{A}$ and B). This fact should be intuitively obvious but let us justify it with some rigor. We have:

$$\overline{A} \cap \overline{B} = (\overline{A \cup B}) \quad \text{(Justify this using a Venn diagram)}$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cap B)$$

$$= 1 - \{P(A) + P(B) - P(A \cap B)\}$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B) \quad (A, B \text{ are independent})$$

$$= (1 - P(A))\,(1 - P(B)) = P(\overline{A}) \cdot P(\overline{B})$$

$$\Rightarrow \quad \overline{A} \text{ and } \overline{B} \text{ are independent.}$$

Similar proofs follow for the other two pairs. Returning to the question, we see that since $\overline{A}$ and $\overline{B}$ are independent, we have:

$$P(\overline{B}/\overline{A}) = P(\overline{B}) = 1 - P(B) = \frac{1}{2}.$$

■

Example 11

An urn contains m white and n black balls. A ball is drawn at random and is put into the urn along with k additional balls of the same color as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn is now white?

Solution: Two cases are possible; the brackets represent the corresponding probabilities:

Stage I	Stage II	Total Probability
Case-I: White ball is drawn: $\left(\dfrac{n}{m+n}\right)$ k additional white balls are put, so now we have $(m + k)$ white balls and n black balls.	White ball is drawn: $\left(\dfrac{m+k}{m+n+k}\right)$	$\left(\dfrac{n}{m+n}\cdot\dfrac{m}{m+n+k}\right)$
Case-II: Black ball is drawn: $\left(\dfrac{n}{m+n}\right)$ k additional black balls are put so now we have m white and $(n + k)$ black balls.	White ball is drawn: $\left(\dfrac{m}{m+n+k}\right)$	$\left(\dfrac{n}{m+n}\cdot\dfrac{m}{m+n+k}\right)$

Thus, the probability of a white ball being finally drawn is:

$$P(\text{white}) = \frac{m}{m+n}\cdot\frac{m+k}{m+n+k} + \frac{n}{m+n}\cdot\frac{m}{m+n+k}$$

$$= \frac{m(m+k+n)}{(m+n)\,(m+n+k)} = \frac{m}{m+n}.$$

Is it surprising that the final result is independent of k? ■

Example 12

A player tosses a fair coin and scores 1 point for Heads and 2 points for Tails. He keeps playing until his score reaches or passes n. If P_n denotes the probability of getting a score of exactly n at some point, find the value of P_n in terms of n.

Solution: First of all, note that $P_1 = \frac{1}{2}$ and $P_2 = \frac{3}{4}$ (why?). Both these values satisfy the general relation which we have to prove to be true. Now, the form of the problem suggests that a recursion can be set up. To write the recursion, note that a score of exactly n can be reached in two mutually exclusive ways (the corresponding probabilities are also mentioned):

(1) A score of exactly $(n - 1)$, then Heads: $P_{n-1} \times \dfrac{1}{2}$

(2) A score of exactly $(n - 2)$, then Tails: $P_{n-2} \times \dfrac{1}{2}$

Thus, we have

$$P_n = \frac{1}{2}(P_{n-1} + P_{n-2}).$$

This recursion relation, upon some rearrangement, gives:

$$\begin{aligned}
P_n + \frac{1}{2}P_{n-1} &= P_{n-1} + \frac{1}{2}P_{n-2} = P_{n-2} + \frac{1}{2}P_{n-3} \\
&\;\vdots \\
&= P_2 + \frac{1}{2}P_1 = 1 \\
\Rightarrow \quad P_n + \frac{1}{2}P_{n-1} &= 1 \\
\Rightarrow \quad P_n - \frac{2}{3} = \frac{1}{3} - \frac{1}{2}P_{n-1} &= \frac{-1}{2}\left(P_{n-1} - \frac{2}{3}\right)
\end{aligned}$$

In the last step, why we did the indicated manipulation is very important to understand. What we have done is express $(P_n - \frac{2}{3})$ in terms of $(P_{n-1} - \frac{2}{3})$, which effectively means expressing the nth 'term' in terms of the $(n-1)$th 'term'. Why does this help us? It is useful because we can now apply it repeatedly till we exhaust the possibility of repetition:

$$\begin{aligned}
P_n - \frac{2}{3} = \frac{-1}{2}\left(P_{n-1} - \frac{2}{3}\right) &= \left(\frac{-1}{2}\right)^2\left(P_{n-2} - \frac{2}{3}\right) = \left(\frac{-1}{2}\right)^3\left(P_{n-3} - \frac{2}{3}\right) \\
&\;\vdots \\
&= \left(\frac{-1}{2}\right)^{n-1}\left(P_1 - \frac{2}{3}\right) \\
&= \frac{-1}{6}\cdot\left(\frac{-1}{2}\right)^{n-1} = \frac{(-1)^n}{3\cdot 2^n} \\
\Rightarrow \quad P_n &= \frac{1}{3}\left[2 + (-1)^n \frac{1}{2^n}\right]
\end{aligned}$$

You are urged to reread the solution until you fully understand how the recursion was setup and solved. ■

Example 13

If n different things are distributed among x boys and y girls, find the probability that the number of things received by the girls is even.

Solution: One individual thing can be assigned in a total of $(x + y)$ ways, so that n different things can be distributed in $(x + y)^n$ ways. Now, suppose r things are given to girls and the rest $(n - r)$ are given to the boys. The number of ways W_r of doing this is

$$W_r = \underset{\substack{\text{select } r \text{ things} \\ \text{out of } n}}{{}^nC_r} \times \underset{\substack{\text{distribute the} \\ \text{remaining } (n-r) \\ \text{things among the boys}}}{x^{n-r}} \times \underset{\substack{\text{distribute the} \\ r \text{ things among} \\ \text{the girls}}}{y^r}$$

What we need to do is evaluate $W_0 + W_2 + W_4 + \cdots$. This can be done as follows:

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + \cdots + {}^nC_r\, x^{n-r} y^r + \cdots$$

$$(x-y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + \cdots + (-1)^r\, {}^nC_r\, x^{n-r} y^r + \cdots$$

Adding the two, we are left with only the even terms:

$$\frac{1}{2}\left\{(x+y)^n + (x-y)^n\right\} = {}^nC_0 x^n + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \cdots$$

Thus, the required probability is:

$$\frac{W_0 + W_2 + W_4 + \cdots}{(x+y)^n} = \frac{1}{2}\left[\frac{(x+y)^n + (x-y)^n}{(x+y)^n}\right].$$

■

Example 14

For a student to qualify he must pass at least two out of three examinations. The probability that he will pass the first examination is p. If he fails in one of the examination, then the probability of his passing in the next examination is $\frac{p}{2}$, otherwise it remains the same. Find the probability that he will qualify.

Solution: Let us denote by E_i the event that the student passes the ith examination, and by E, the event that he qualifies. Thus, E can happen in four possible mutually exclusive ways (sequences):

$$E_1\, E_2\, \bar{E}_3,\ E_1\, \bar{E}_2\, E_3,\ \bar{E}_1\, E_2\, E_3,\ E_1\, E_2\, E_3.$$

We have

$$P(E_1\, E_2\, \bar{E}_3) = p \times p \times (1-p) = p^2(1-p)$$

$$P(E_1\, \bar{E}_2\, E_3) = p \times (1-p) \times \frac{p}{2} = \frac{p^2}{2}(1-p)$$

$$P(\bar{E}_1\, E_2\, E_3) = (1-p) \times \frac{p}{2} \times p = \frac{p^2}{2}(1-p)$$

$$P(E_1\, E_2\, E_3) = p \times p \times p = p^3$$

Thus, $P(E)$ is obtained by adding the four probabilities, which gives:

$$P(E) = 2p^2 - p^3.$$

You may observe that this expression is always positive, since $p < 1$. ■

Example 15

Two numbers are selected at random without replacement from the numbers 1, 2,..., n. Find the probability that the difference between the first and the second is not less than m, where $0 < m < n$.

Solution: The total number of ways of choosing two numbers out of n (order thers) is $n(n-1)$. Let us denote by x and y the two chosen numbers, so that x represents the larger number. We need to find the

number of ways in which the inequality $x - y \geq m$ can be satisfied. It should be obvious that for $x \leq m$, no such y will exist. For $x > m$, the following values of y will satisfy the inequality:

$$y = 1, 2, \ldots, x - m.$$

For example, if $x = m + 4$, then y can take the values 1, 2, 3, 4. Thus, we see that for the inequality to be satisfied, if some $x > m$ is chosen, y can take on $x - m$ values, ranging from 1 to $x - m$. Clearly, the number of ways this can be done is:

$$\sum_{x=m+1}^{n} (x-m) = 1 + 2 + \cdots + (n-m) = \frac{(n-m)(n-m+1)}{2}$$

Thus, the required probability p is:

$$p = \frac{(n-m)\,(n-m+1)}{2n(n-1)}.$$

■

Example 16

A man takes a step forward with probability 0.4 and backward with probability 0.6. What is the probability that at the end of 11 steps, he will be one step away from the starting point?

Solution: Visualize the situation. To be just one step away from the starting point after 11 steps, there are only two cases possible:

(a) He has taken 6 steps forward and 5 backward in some order.

OR

(b) He has taken 5 steps forward and 6 backward in some order.

If we let a step forward denote success and a step backward denote failure (and denote the respective probabilities by s and f), we have:

$$s = 0.4, \quad f = 0.6$$

so that,

$$P\begin{Bmatrix}\text{one step away}\\ \text{after 11 steps}\end{Bmatrix} = P\begin{Bmatrix}\text{6 successes,}\\ \text{5 failures}\end{Bmatrix} + P\begin{Bmatrix}\text{5 successes,}\\ \text{6 failures}\end{Bmatrix}$$

$$= {}^{11}C_6\, s^6 f^5 + {}^{11}C_5\, s^5 f^6 = {}^{11}C_6\, s^5 f^5 (s+f)$$

$$= \frac{11!}{6!\;5!}(0.4)^5(0.6)^5 = 0.37.$$

■

Example 17

X and Y are playing a tournament, consisting of matches. The first one to win $(n+1)$ matches wins. The probabilities of their winning a match are p and q respectively. Find their respective probabilities of winning the tournament.

Solution: To make things easier to understand, let us write down some cases explicitly. We will calculate the probability of X winning the tournament:

Case in Which X Wins the Tournament	P(Case)
• X wins the first $(n + 1)$ matches straight.	p^{n+1}
• X wins n matches out of the first $(n + 1)$ and then the $(n + 2)$nd match.	$\underbrace{({}^{n+1}C_n\, p^n q)}_{\text{first}(n+1)\text{matches}} \cdot \underbrace{p}_{(n+2)\text{nd match}}$
• X wins n matches out of the first $(n + 2)$, and then wins the $(n + 3)$rd match ⋮	$\underbrace{({}^{n+2}C_n\, p^n q^2)}_{\text{first}(n+2)\text{matches}} \cdot \underbrace{p}_{(n+3)\text{rd match}}$
• X wins n matches out of the first $(n + r)$ and then wins the $(n + r + 1)$st match ⋮	$\underbrace{({}^{n+r}C_n\, p^n q^r)}_{\text{first}(n+r)\text{matches}} \cdot \underbrace{p}_{(n+r+1)\text{st match}}$
• X wins n matches out of the first $2n$, and then wins the $(2n + 1)$th match	$\underbrace{({}^{n+n}C_n\, p^n q^n)}_{n+n\text{ matches}} \cdot \underbrace{p}_{(2n+1)\text{st match}}$

Thus,

$$P\{X \text{ wins the tournament}\} = \sum_{r=0}^{n} {}^{n+r}C_n\, p^{n+1} q^r.$$

Similarly,

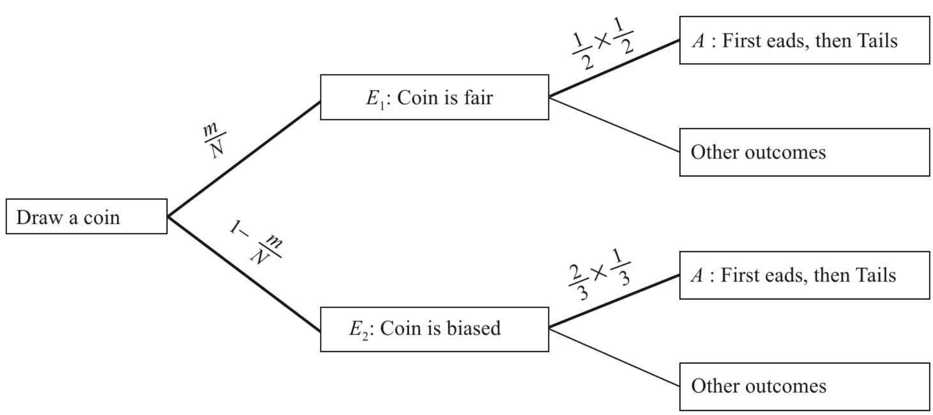
$$P\{Y \text{ wins the tournament}\} = \sum_{r=0}^{n} {}^{n+r}C_n\, q^{n+1} p^r.$$

■

Example 18

A box contains N coins, m of which are fair and the rest are biased. The probability of getting Heads when a biased coin is tossed is $\frac{2}{3}$. A coin is drawn at random from the box and tossed twice. The first time it shows Heads and the second time it shows Tails. What is the probability that the coin drawn is fair?

Solution: The probability tree for this problem is as follows:



The required probability p is:

$$p = \frac{\frac{m}{N}\times\left(\frac{1}{2}\times\frac{1}{2}\right)}{\frac{m}{N}\times\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(1-\frac{m}{N}\right)\times\left(\frac{2}{3}\times\frac{1}{3}\right)}$$

$$= \frac{\frac{m}{4N}}{\frac{m}{4N}+\frac{2N-2m}{9N}} = \frac{9m}{m+8N}.$$

In the terminology of Baye's theorem, the expression for p will be:

$$p = P(E_1/A) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)}.$$

■

Probability

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. A fair coin is independently flipped n times, k time by X and $n-k$ times by Y. What is the probability that X and Y obtain the same number of Heads?

(A) $\frac{{}^{n-1}C_k}{2^n}$ (B) $\frac{{}^{n}C_k}{2^n}$ (C) $\frac{{}^{n+1}C_k}{2^n}$ (D) $\frac{{}^{n}C_{k-1}}{2^n}$ (E) None of these

P2. Consider n independent flips of a fair coin. A *changeover* is said to occur if successive outcomes are different. What is the probability that there are exactly k changeovers?

(A) $\frac{{}^{n}C_{k-1}}{2^{n-1}}$ (B) $\frac{{}^{n}C_k}{2^{n-1}}$ (C) $\frac{{}^{n-1}C_k}{2^{n-1}}$ (D) $\frac{{}^{n-1}C_{k-1}}{2^{n-1}}$ (E) None of these

P3. An experiment has 16 equally likely outcomes. Let A and B be two non-empty independent events of the experiment such that the number of outcomes in B is twice that in A. The number of outcomes in A can be

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

P4. A deck of 52 cards is randomly divided into 4 piles of 13 cards each. Which among the following is the closest in value to the probability that the 4 aces are in different piles (in terms of percentages)?

(A) 10% (B) 12% (C) 15% (D) 18% (E) 20%

P5. Eight players $P_1, P_2, ..., P_8$ play a knock-out tournament. It is known that whenever the players P_i and P_j play, the player P_i will win if $i < j$. Assuming that the players are paired at random in each round, what is the probability that the player P_4 reaches the final?

(A) $\frac{1}{11}$ (B) $\frac{4}{35}$ (C) $\frac{1}{7}$ (D) $\frac{7}{36}$ (E) None of these

P6. Two non-negative integers a and b are chosen at random from the set of non-negative integers. In terms of percentages, what is the probability of a^2+b^2 being divisible by 10?

(A) 16% (B) 18% (C) 20% (D) 24% (E) None of these

P7. A card from a pack of 52 cards is lost. From the remaining cards of the pack two cards are drawn at random and are found to be Spades. What is the probability that the missing card was a Spade?

(A) $\frac{1}{5}$ (B) $\frac{11}{50}$ (C) $\frac{6}{25}$ (D) $\frac{1}{4}$ (E) None of these

P8. A bag contains 12 black and 6 white balls. 6 balls are drawn at random one by one without replacement, of which at least 4 balls are white. What is the probability that in the next two draws, exactly one white ball is drawn?

(A) $\frac{87}{476}$ (B) $\frac{95}{492}$ (C) $\frac{95}{492}$ (D) $\frac{99}{500}$ (E) None of these

SUBJECTIVE TYPE EXAMPLES

P9. An experiment E has r outcomes $X_i, i = 1, 2, ..., r$, such that $P(X_i) = p_i$ and $\Sigma_{i=1}^{r} p_i = 1$. E is performed n times. Find $P(X_1 \equiv x_1, X_2 \equiv x_2, ..., X_r \equiv x_r)$, that is, the probability that X_i occurs x_i times. $(\Sigma_{i=1}^{r} x_i = n)$.

P10. A and B play a series of independent games, and their respective probabilities of winning are p and $1-p$ respectively. The overall winner is the first one to win a total of 4 games.

(a) Find the probability P that a total of 7 games are played.
(b) For what value of p is P maximum?

P11. Consider the set $S = \{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., \frac{n}{n}\}$. Let a number X be chosen from S such that:

$$P\left(X = \frac{r}{n}\right) = \frac{kr(n-r)}{n^3}.$$

Assuming $n \to \infty$, evaluate:

(a) k (b) $P\left(X > \frac{3}{4}\right)$ (c) $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

P12. A certain communication system transmits 0 and 1 with probabilities α and $1-\alpha$ respectively. Due to noise in the channel, a transmitted 0 can be converted into 1 with probability p, while a transmitted 1 can be converted into 0 with probability q. Find the probability that

(a) a received 0 was transmitted as a 0.
(b) a received 1 was transmitted as a 1.
(c) the received input is incorrect.

P13. For an airplane, the probability of failure of an engine is $1-p$ and independent from engine to engine. For the airplane to make a successful flight, at least half of its engines must remain operative. For what values of p is a 4-engine plane preferable to a 2-engine plane?

P14. Let p, q be positive integers and r, s, t be prime numbers such that $\text{LCM}(n-r) = r^2s^2t^4$. Find the probability that the greatest common divisor of p and q is rt.

P15. A coin has probability p of showing Heads when tossed. It is tossed n times. Let P_n denote the probability that no two (or more) consecutive Heads occur. Prove that $P_1 = 1$, $P_2 = 1 - p^2$ and $P_n = (1-p)P_{n-1} + p(1-p)P_{n-2}$ for all $n \geq 3$.

P16. Consider the sample space $S = \{(x, y) \mid x, y \in \mathbb{Z}\}$, such that $|x| + |y| \leq 100$. All values in S are equally likely. Define the events A_k, $k \in \{1, 2, ..., 100\}$ as

$$A_k = \{(x, y) \mid x, y \in \mathbb{Z} \text{ and } |x| + |y| \leq k\}$$

For two positive integer p, q such that $p < q < 100$, find the probability $P(A_p / A_q)$.

P17. Sixteen players $S_1, S_2, ..., S_{16}$ play in a tournament. They are divided into eight pairs at random and from each pair, a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength.

(a) Find the probability that the player S_1 is among the eight winners.
(b) Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.

P18. Consider a 3 × 3 matrix A with its entries (equally likely) from the set {0, 1, 2,..., 9}. Define two events E_1 and E_2 as follows:

$$E_1 : \{a_{ij} = 0 \quad \forall\, i < j\}$$
$$E_2 : \{\,|A| \neq 0\,\}$$

Evaluate

(a) $P(E_1)$ (b) $P(E_2/E_1)$

P19. A tosses a fair coin $(n + 1)$ times whereas B tosses a fair coin n times. What is the probability that A gets more Heads than B?

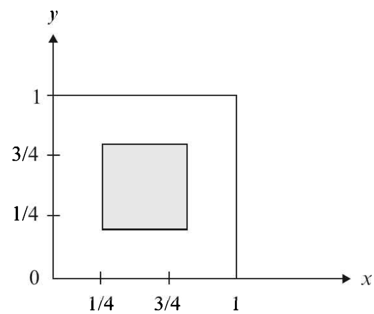
P20. A and B are shooting a target. Suppose that A can hit the target with probability p and B with probability q, independently. They both take one shot each at the target simultaneously.

(a) Given that exactly one shot hit the target, what is the probability that it was B's shot?
(b) Given that the target is hit, what is the probability that B hit it?

P21. This problem is based on continuous sample spaces. In a discrete problem, the probability of an event E occurring is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

in the case of equal likelihood of all events. Now lets understand continuous sample spaces. Suppose that the sample space is the region $R : \{(x, y) \big| 0 \le x, y \le 1\}$ while the event E is given $E : \{(x, y) \big| \frac{1}{4} \le x, y \le \frac{3}{4}\}$.



Now, we pick a point in R at random. The probability that E will occur is:

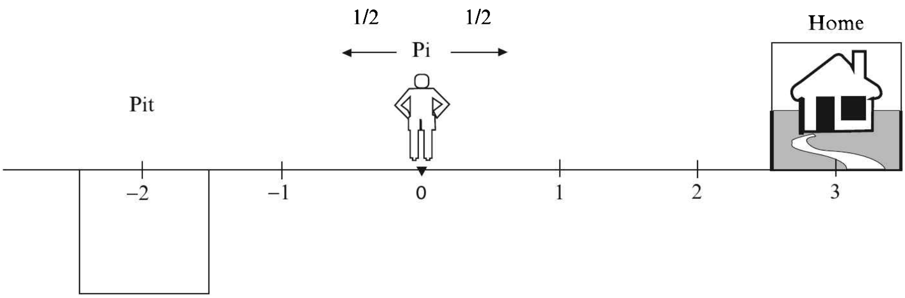
$$P(E) = \frac{\text{area of favorable region}}{\text{total area of sample space}} = \frac{1/4}{1} = \frac{1}{4}.$$

Based on this, do the following problems.

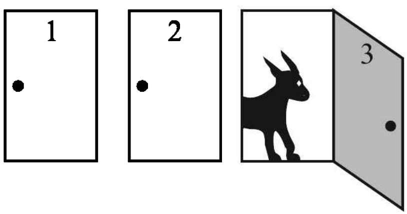
(a) Two points are picked at random on a unit scale, so that the scale is divided into three segments. Find the probability that the three segments can form a triangle.
(b) On an (infinite) sheet of paper, parallel lines are drawn at spacing of one unit. A needle of unit length is dropped randomly onto this sheet. Find the probability that the needle will cross a line.

(c) A circle of radius 1 is randomly placed in a 15×36 rectangle $ABCD$ so that it lies completely within the rectangle. Given that the probability that the circle will not touch the diagonal AC is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

P22. A drunk mathematician called Pi is performing a random walk along the number line as follows. His original position is $x = 0$. He takes a step forward or a step backward with equal probability. Thus, for example, if he is at $x = 1$, he can go to $x = 2$ or $x = 0$ with equal probability. His home is at $x = 3$, while there is a pit at $x = -2$. What is the probability that he'll reach home before falling into the pit?



P23. *Monty–Hall Problem:* This problem is one of the most well known problems in probability. The version of the problem discussed here and its solution have been taken from *Wikipedia.* Suppose you are in a game show, and you're given the choice of three doors. Behind one door is a new car; behind the others, goats.The car and the goats were placed randomly behind the doors before the show. The rules of the game show are as follows: After you've chosen a door, the door remains closed for the time being. The game show host, Monty Hall, who knows what is behind the doors, now has to open one of the two remaining doors, and the door he opens must have a goat behind it. If both remaining doors have goats behind them, he chooses one randomly. After Monty Hall opens a door with a goat, he will ask you to decide whether you want to stay with your first choice or to switch to the last remaining door. Imagine that you choose Door 1 and the host opens Door 3, which has a goat. He then asks you 'Do you want to switch to Door 2?' Is it to your advantage to change your choice?



In search of a new car, the player picks door 1. The game host then opens door 3 to reveal a goat and offers to let the player pick door 2 instead of door 1.

Probability

PART-D: Solutions to Advanced Problems

S1. Let P_i be the event that both X and Y obtain i heads. Then,

$$P_i = \frac{{}^kC_i\ {}^{n-k}C_i}{2^n}.$$

The required probability P is obtained by summing the p_i's upto the appropriate index.

$$P = \sum P_i = \frac{1}{2^n}\sum {}^kC_i{}^{n-k}C_i = \frac{{}^nC_k}{2^n}.$$

The correct option is (B).

S2. A transition can be of two types: a changeover or a non-changeover, both with respective probabilities $\frac{1}{2}$. There are $n-1$ transitions; the probability of k of them being changeovers is simply $\frac{{}^{n-1}C_k}{2^{n-1}}$. The correct option is (C).

S3. Let the number of outcomes in A and B be n and $2n$ respectively, and in $A \cap B$ be x.

$$P(A \cap B) = P(A)P(B) \quad \Rightarrow \quad \frac{x}{16} = \frac{n}{16} \cdot \frac{2n}{16} \quad \Rightarrow \quad n^2 = 8x$$

Since both n, x are integers and $n \le 8$, the possible values for n are 4 and 8. The correct options are (A) and (C).

S4. To count the number of favorable cases, remove the aces from the pile, divide the 48 remaining cards into 4 equal group of 12 cards each, and assign each group to one of the 4 aces. Thus,

$$\#\,(\text{Favourable cases}) = \underbrace{\left(\frac{48!}{(12!)^4\,4!}\right)}_{\substack{\text{Dividing 48 cards} \\ \text{into 4 equal groups.}}} \quad \underbrace{(4!)}_{\substack{\text{Assigning each} \\ \text{group to one of} \\ \text{the 4 aces.}}} = \frac{48!}{(12!)^4}$$

$$\#\,(\text{Total cases}) = \frac{52!}{(13!)^4\,4!}.$$

The required probability is:

$$P = \frac{48!}{(12!)^4} \times \frac{(13!)^4\,4!}{52!} = \frac{2197}{20825} \approx 10.5\ \%.$$

Among the options given, the closest value is 10%. Therefore, the correct option is (A).

S5. This is a simple yet very interesting question. We make the observation that for P_4 to reach the finals, he must be paired with one of $\{P_5, P_6, P_7, P_8\}$ in the first round, and one of these players, say P_i must reach the second round, so that P_4 can be paired with P_i in the second round. This means, for example, that the following pairings will not do:

$$\begin{pmatrix} P_1 \\ P_5 \end{pmatrix}\begin{pmatrix} P_2 \\ P_6 \end{pmatrix}\begin{pmatrix} P_3 \\ P_7 \end{pmatrix}\begin{pmatrix} P_4 \\ P_8 \end{pmatrix} \qquad \begin{pmatrix} P_1 \\ P_4 \end{pmatrix}\begin{pmatrix} P_2 \\ P_5 \end{pmatrix}\begin{pmatrix} P_3 \\ P_8 \end{pmatrix}\begin{pmatrix} P_6 \\ P_7 \end{pmatrix}$$

P_4 cannot win in the second round. P_4 loses in the first round itself.

Therefore, we need to ensure the following:

(a) In the first round, P_4 must be paired with one of $\{P_5, P_6, P_7, P_8\}$, and two of the remaining three players from this set must be paired with each other (call this pair P). There are $4 \times 3 \times 3 = 36$ ways to do this, while the total number of pairings is $\frac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!} = 105$.

(b) In the second round, P_4 must be paired with the winner of pair P. There is only 1 way to do this, while the total number of pairings possible in this round is 3.

The required probability is $\frac{36}{105} \times \frac{1}{3} = \frac{4}{35}$. The correct option is (B).

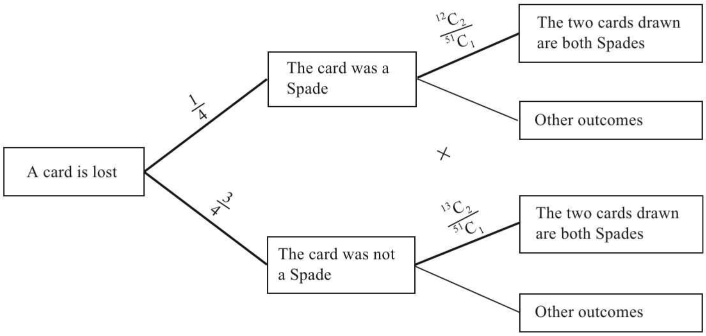
S6. $a^2 + b^2 = c$ (say), will be divisible by 10 only if the unit digit in c is 0. Thus, what we need to do is only focus on the unit digits of a^2 and b^2 and find all the favorable combinations when the sum of these two unit digits leads to a 0 for the unit digit in c. Let u_a, u_b denote the unit digits of a and b. The total number of pairs of u_a and u_b is simply $10 \times 10 = 100$, since both u_a and u_b can be selected in 10 ways each. Now, let us find the number of those pairs for which $u_a^2 + u_b^2$ leads to a 0, for the unit digit of c. This can be done by explicit listing:

u_a	u_b	**Number of Pairs Possible**
0	0	1
1 or 9	3 or 7	$2 \times 2 = 4$
2 or 8	4 or 6	$2 \times 2 = 4$
3 or 7	1 or 9	$2 \times 2 = 4$
4 or 6	2 or 8	$2 \times 2 = 4$
5	5	1
		Total = 18

The total number of favorable pairs thus possible is 18. Therefore, there is a 18/100 or 18% chance that the sum of the squares of two non-negative integers picked at random, will be divisible by 10. The correct option is (B).

To check whether you have *really* understood the solution, try to answer this question: will the answer change if instead of selecting any two arbitrary a and b from the non-negative integers we impose a constraint, like, for example, both $a, b \geq 100$? What about $a, b \geq 25$? And $a \geq 0, b \geq 5$?

S7. Let us approach the problem by drawing the corresponding probability tree:



Observe carefully the probabilities of all the relevant steps. The required probability p is

$$p = \frac{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2} + \frac{3}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2}} = \frac{11}{50}.$$

Hence, the chances of the missing card being a Spade are 22%. The correct option is (B). Can you appreciate this answer? In particular, can you see why it is less than 25%?

S8. Let A be the event that at least 4 balls are white in the first 6 draws. Note that A can happen in three mutually exclusive ways:

A_1: exactly 4 white balls are drawn

A_2: exactly 5 white balls are drawn

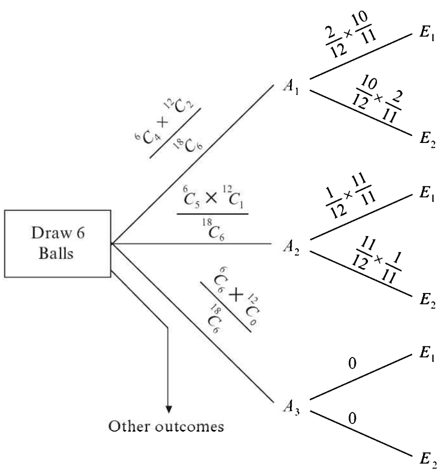
A_3: exactly 6 white balls are drawn

Now, in the next two draws, we want exactly one white ball which can happen in two ways:

E_1: First white, then black

E_2: First black, then white

Thus, we have the following probability tree



Probability tree for the given problem that leads us to the required situation.

Notice very carefully how the probabilities for the various segments have been calculated.

Note that A has already occurred, that is, we already know that at least 4 white balls have been drawn. We want to calculate $P(E_1 \mid A) + P(E_2 \mid A)$, that is, the probability that the next two draws give exactly one white ball, given that at least 4 white balls have already been drawn in the previous 6 draws. Thus, the required probability p is:

$$p = \frac{\begin{array}{l} P(A_1)\{P(E_1 \mid A_1) + P(E_2 \mid A_1)\} + P(A_2)\{P(E_1 \mid A_2) + P(E_2 \mid A_2)\} \\ \quad + P(A_3)\{P(E_1 \mid A_3) + P(E_2 \mid A_3)\} \\ \text{Numerator gives the probability that } E_1 \text{ or } E_2 \text{ occur given that } A \text{ has occurred} \end{array}}{\{P(A_1) + P(A_2) + P(A_3)\}}$$

Denominator gives the probability that A occurs

$$= \frac{\frac{{}^6C_4 \times {}^{12}C_2}{{}^{18}C_6}\left\{\frac{2}{12} \times \frac{10}{11} + \frac{10}{12} \times \frac{2}{11}\right\} + \frac{{}^6C_5 \times {}^{12}C_1}{{}^{18}C_6}\left\{\frac{1}{12} \times \frac{11}{11} + \frac{11}{12} \times \frac{1}{11}\right\} + \frac{{}^6C_6 \times {}^{12}C_0}{{}^{18}C_6}\{0+0\}}{\frac{{}^6C_4 \times {}^{12}C_2}{{}^{18}C_6} + \frac{{}^6C_5 \times {}^{12}C_1}{{}^{18}C_6} + \frac{{}^6C_6 \times {}^{12}C_0}{{}^{18}C_6}}$$

$$= \frac{15\left(\frac{40}{132}\right) + 72\left(\frac{1}{6}\right)}{15 + 72 + 1} = \frac{91}{484}.$$

The correct option is (B).

SUBJECTIVE TYPE EXAMPLES

S9. Consider a string with n symbols, x_i of which are X_i The permutations of such a string are:

$$N = \frac{n!}{x_1!\,x_2!\ldots x_r!}$$

in number. Each permutation has a probability

$$p = p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r}$$

of occurrence. The required probability is Np.

S10. **(a)** A total of 7 games are played if the 7th game is the last, that is, if A and B win 3 games each in the first 6 games. The probability P of this happening is:

$$P = {}^6C_3 p^3 (1-p)^3.$$

(b) By symmetry, P is maximum when $p = \frac{1}{2}$.

S11. Since $n \to \infty$, the given discrete probability distribution can be expressed as the continuous distribution:

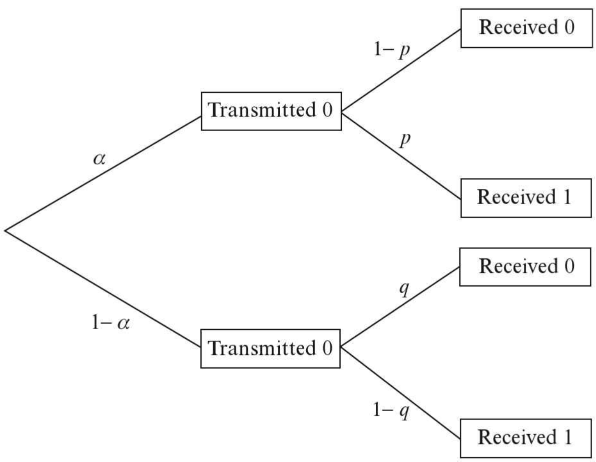
$$p(x) = kx(1-x).$$

(a) $\int p(x)dx = 1 \;\Rightarrow\; \int_0^1 kx(1-x)dx = 1 \;\Rightarrow\; k = 6$

(b) $P\left(x > \frac{3}{4}\right) = 6\int_{3/4}^{1} x(1-x)\,dx = \frac{5}{32}$

(c) $P\left(\frac{1}{4} < x < \frac{3}{4}\right) = 6\int_{1/4}^{3/4} x(1-x)dx = \frac{11}{16}$

S12. The probability tree corresponding to the given situation is:



A simple application of Baye's theorem yields the result in the first two cases.

(a) $\dfrac{\alpha(1-p)}{\alpha(1-p)+(1-\alpha)q}$

(b) $\dfrac{(1-\alpha)(1-q)}{\alpha p+(1-\alpha)(1-q)}$

(c) This is the probability that a transmitted 0 is received as 1, or vice-versa. Hence, the required probability is:

$$P=\alpha p+(1-\alpha)q.$$

S13. For a 4-engine plane to make a successful flight, at least 2 of its engines must stay operative, that is, 2, 3 or 4 engines may stay operative. For a 2-engine plane to do the same, 1 or 2 engines must stay operative. Let E_1 and E_2 be the events that a 4-engine and a 2-engine plane make a successful flight respectively. We have:

$$P(E_1)={}^4C_2p^2(1-p)^2+{}^4C_3p^3(1-p)+{}^4C_4p^4$$

$$P(E_2)={}^2C_1p(1-p)+{}^2C_2p^2.$$

It can now be shown that

$$P(E_1)>P(E_2) \text{ when } p>\frac{2}{3}.$$

S14. The number of ordered pairs (p,q) such that $LCM(p,q)=r^2s^2t^4$ is $(2\times3-1)\,(2\times3-1)\,(2\times5-1)=225$. The number of ordered pairs (p,q) such that GCD$(p,q)=rt$ is $2\times2\times2=8$. The reader is urged to verify these claims on her own. The required probability is $\frac{8}{225}$.

S15. The values of P_1 and P_2 should be obvious. To calculate P_n, we observe that there are two favorable cases:

Case 1: We obtain Tails at the nth toss.

Case 2: There are no two consecutive Heads in the first $(n-2)$ tosses. At the $(n-1)$th toss, we obtain Tails, and at the nth toss, we obtain Heads.

You must convince yourself that these are the only two favorable cases and that they are mutually exclusive. The required probability p_n is thus given by

$$P_n=(1-p)P_{n-1}+p(1-p)P_{n-2}.$$

S16. Denoting the number of elements in an event E by $n(E)$, we have

$$n(A_k)=1+4+8+12+\cdots+4k\;=1+\frac{4k(k+1)}{2}=2k(k+1)+1.$$

Therefore, $P(A_p/A_q)=\dfrac{n(A_p)}{n(A_q)}=\dfrac{2p(p+1)+1}{2q(q+1)+1}$.

S17. **(a)** Since all players are of equal strength, any player is equally likely to win or lose a particular game, this means that the probability of S_1 winning and being among the eight winners is simply $\frac{1}{2}$.

(b) The event that exactly one of S_1 and S_2 is among the eight winners can be divided into two cases.

Case 1: S_1 and S_2 are paired with each other. Since S_1 is equally likely to be paired with any of the remaining 15 players, Case 1 has a probability of $\frac{1}{15}$.

Case 2: S_1 and S_2 are not paired together, and exactly one of them wins his/her game. This will have a probability of $\frac{14}{15} \times \frac{1}{2} = \frac{7}{15}$.

The required probability is, thus $\frac{1}{15} + \frac{7}{15} = \frac{8}{15}$.

S18. **(a)** Matrices favorable to E_1 will be of the form $\begin{bmatrix} - & 0 & 0 \\ - & - & 0 \\ - & - & - \end{bmatrix}$. All we need is the three elements on the upper right side to be zero, which has a probability of $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$.

(b) For matrices of this form, the determinant is simply the product of the diagonal elements. Thus, E_2/E_1 occurs if each of the diagonal elements is non-zero which has a probability of $\left(\frac{9}{10}\right)^3$.

S19. Let us first consider the situation when both A and B have made n tosses. Three mutually exclusive cases are possible for this situation:

E_1: A gets more Heads than B

E_2: A and B get the same number of Heads

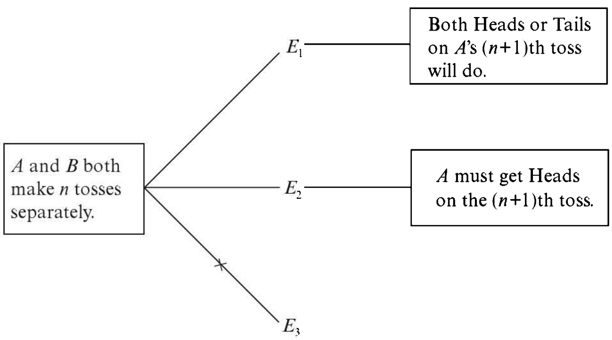
E_3: A gets lesser Heads than B

Now, for A to get a total of more heads than B, there are only two ways possible:

A_1: E_1 occurs; then the result of A's $(n + 1)$th toss doesn't matter

A_2: E_2 occurs; then A must get Heads on the $(n + 1)$th toss

Thus, the probability tree for the entire scenario that leads to A getting more Heads than B is:



What we need to do now is calculate the probabilities $P(E_1)$ and $P(E_2)$. To do so, we define two events:

$$H_{ij} = \{\text{A gets } i \text{ Heads, } B \text{ gets } j \text{ Heads}\}.$$

We thus have

$$P(E_1) = \sum_{i>j} P(H_{ij}) \text{ and } P(E_2) = \sum_{i=j} P(H_{ij}).$$

The probability $P(H_{ij})$ is straightforward to calculate, since how many Heads A gets and how many B gets are independent of each other. Thus,

$$P(H_{ij}) = \left(\frac{{}^nC_i}{2^n} \times \frac{{}^nC_j}{2^n} \right) = \frac{{}^nC_i \cdot {}^nC_j}{2^{2n}}.$$

Now,

$$P(E_1) = \sum_{i>j} P(H_{ij}) = \frac{1}{2^{2n}} \sum_{i>j} {}^nC_i \cdot {}^nC_j.$$

Also,

$$P(E_2) = \sum_{i=j} P(H_{ij}) = \frac{1}{2^{2n}} \sum_i ({}^nC_i)^2.$$

Thus, our task is to calculate $\sum_{i>j} {}^nC_i {}^nC_j$ and $\sum_i ({}^nC_i)^2$. To do so, recall the following manipulations from the chapter on Binomial Theorem:

$$\left(\sum_i {}^nC_i\right)^2 = \sum_i ({}^nC_i)^2 + 2\sum_{i>j} {}^nC_i \cdot {}^nC_j \tag{1}$$

and

$$\sum_{i=0}^{n} ({}^nC_i)^2 = \text{Coeff. of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n. \tag{2}$$

Thus,

$$2\sum_{i=j} {}^nC_i \cdot {}^nC_j = \left(\sum_i {}^nC_i\right)^2 - \sum_i ({}^nC_i)^2 = 2^{2n} - {}^{2n}C_n \text{ (why?)}$$

$$\Rightarrow \sum_{i>j} {}^nC_i \cdot {}^nC_j = \frac{2^{2n} - {}^{2n}C_n}{2}.$$

Thus, we have

$$P(E_1) = \frac{2^{2n} - {}^{2n}C_n}{2^{2n+1}} \quad \text{and} \quad P(E_2) = \frac{{}^{2n}C_n}{2^{2n}}.$$

Finally, from the probability tree, the required probability p is:

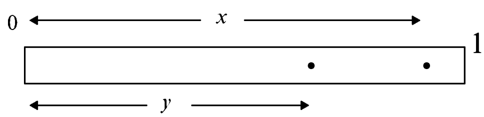
$$p = P(E_1) + P(E_2)\cdot\frac{1}{2} = \frac{2^{2n} - {}^{2n}C_n}{2^{2n+1}} + \frac{{}^{2n}C_n}{2^{2n+1}} = \frac{1}{2}.$$

The final result is extremely simple!

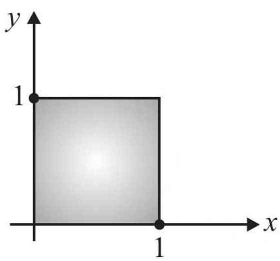
S20. **(a)** The probability of exactly one shot hitting the target is $p(1-q)+(1-p)q$. The probability that only B's shot hits the target is $(1-p)q$. The required probability is $\frac{(1-p)q}{p(1-q)+(1-p)q}$.

(b) The probability that the target is hit is $1-(1-p)(1-q)$. The probability that B's shot hits the target is q. The required probability is therefore $\frac{q}{1-(1-p)(1-q)}$.

S21. **(a)** Let the first and second points be at a distance of x and y units on the scale:



Note that $x \in [0,1]$ and $y \in [0,1]$, so if we wish to plot the sample space, we plot the region containing points with all possible coordinate pairings (x, y), where both x and y lie in [0, 1]. The sample space is thus a unit square:



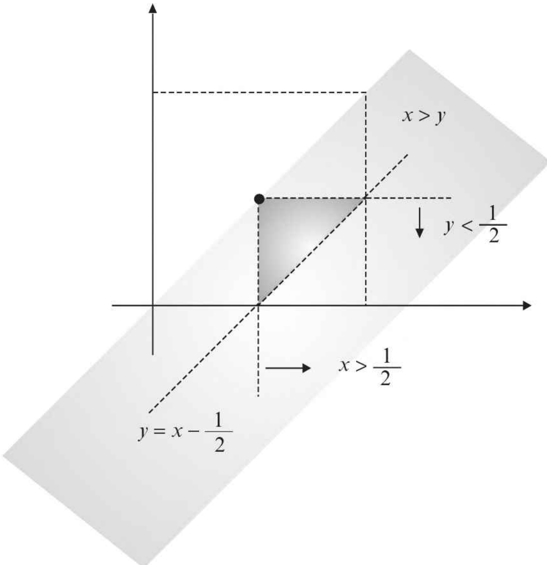
Sample space of this experiment. Total area = 1.

Now, we plot the favorable region. Looking at the first figure above, where $x > y$, we see that the three segments have the lengths $y, x-y, 1-x$. These three lengths must satisfy the triangle inequality if they are to form a triangle:

$$y + x - y > 1 - x, \quad y + 1 - x > x - y, \quad x - y + 1 - x > y$$

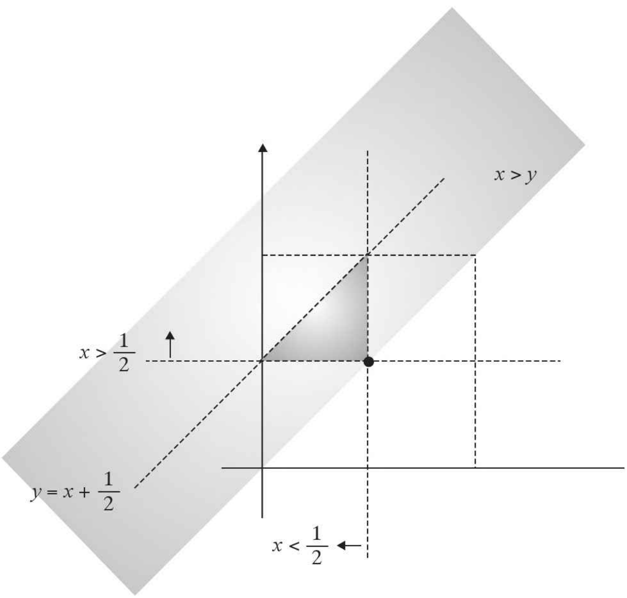
$$\Rightarrow \quad x > \frac{1}{2}, \quad x < y + \frac{1}{2}, \quad y < \frac{1}{2}$$

Along with $x > y$, these give rise to the following region:



Area of shaded region is $\frac{1}{8}$.

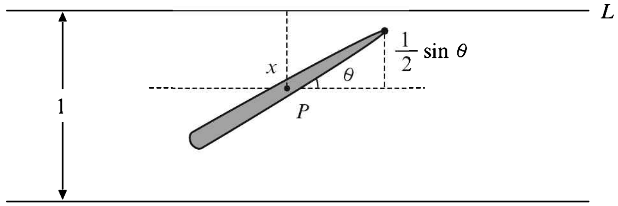
Similarly, for the case $x < y$, we'll have this:



Area of shaded region is $\frac{1}{8}$.

The total favorable region, thus has an area of $\frac{1}{4}$, while the area of the sample space is 1. The probability that the three segments can form a triangle is therefore $\frac{1}{4}$.

(b) Let us denote the center of the needle by P, the line nearest to the center by L, the (acute) angle the needle makes with the direction of the lines by θ, and the distance between P and L by x:



Note that:

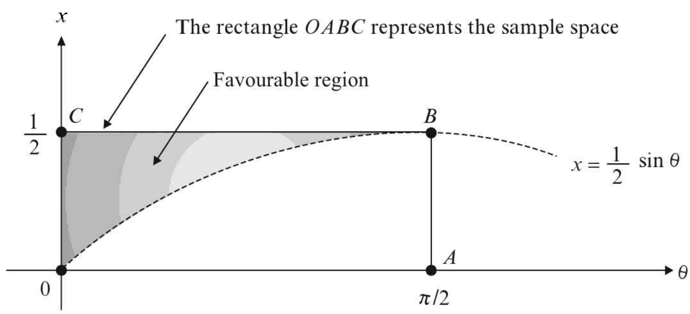
- $x \in \left[0, \frac{1}{2}\right]$
- $\theta \in \left[0, \frac{\pi}{2}\right]$

These constraints give the sample space.

For the needle to not cross L, we must have:

$$x > \frac{1}{2}\sin\theta$$

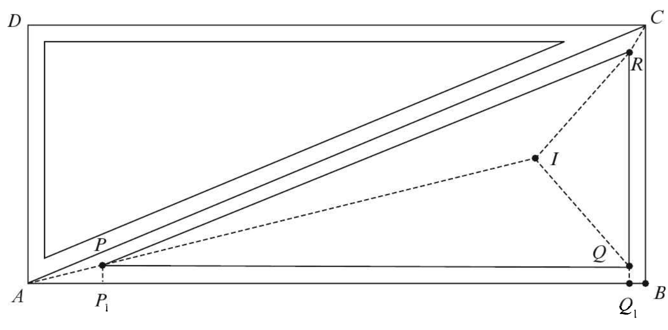
This constraint gives the favourable region:



The area of the favorable region is $(\frac{\pi}{2}\times\frac{1}{2}) - \int_0^{\pi/2}(\frac{1}{2}\sin\theta)\,d\theta$ or $\frac{\pi}{4}-\frac{1}{2}$. The required probability P is thus:

$$P = \frac{\frac{\pi}{4}-\frac{1}{2}}{\frac{\pi}{4}} = 1 - \frac{2}{\pi}.$$

(c) First of all, if the circle is placed completely inside the rectangle, the center of the circle must lie in a rectangle whose dimensions are $(15-2)\times(36-2) = 13\times 34$. Now, if the circle is to *not touch* the diagonal AC, the distance of the circle's center from AC must be greater than 1. If we consider the figure below carefully, we will now be able to effectively rephrase the problem.



We are required to calculate the probability that the distance of a randomly selected point in the 13×34 rectangle to each side of triangles ABC and CDA is greater than 1. Thus, the required probability p is

$$p = \frac{2area(\Delta PQR)}{13 \times 34} = \left(\frac{PQ}{AB}\right)^2 \times \frac{2(area\Delta ABC)}{13 \times 34} = \frac{270}{221} \cdot \left(\frac{PQ}{AB}\right)^2$$

To calculate the length of PQ, we start by observing that AP, BQ and CR (when produced) meet at the incenter I of ΔABC (how?). If $\angle IAB = \theta$ then $\tan 2\theta = \frac{BC}{AB} = \frac{5}{12}$, which will yield $\tan\theta = \frac{1}{5}$. Thus, in ΔAPP_1, $AP_1 = 5$. Also, $BQ_1 = QQ_1$ (why) and so $BQ_1 = 1$. Therefore,

$$PQ = P_1Q_1 = AB - (AP_1 + BQ_1) = 30.$$

Finally,

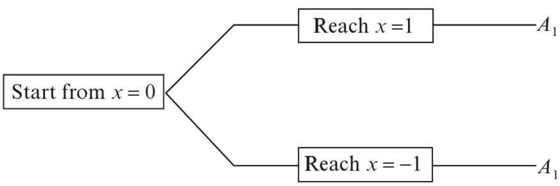
$$p = \frac{270}{221} \cdot \left(\frac{30}{36}\right)^2 = \frac{375}{442}$$

$$\Rightarrow \quad m + n = 375 + 442 = 817.$$

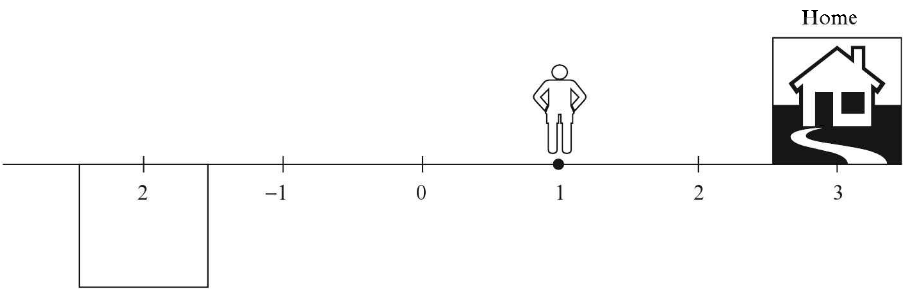
S22. Assume the required probability to be p. Thus, p denotes the probability $P(E)$ where

E : Pi reaches home before the pit

Now, let us draw the probability tree that will make E happen:



What should the events A_1 and A_2 be and what should be their probabilities? Once Pi as reached $x = 1$, the situation is as follows:

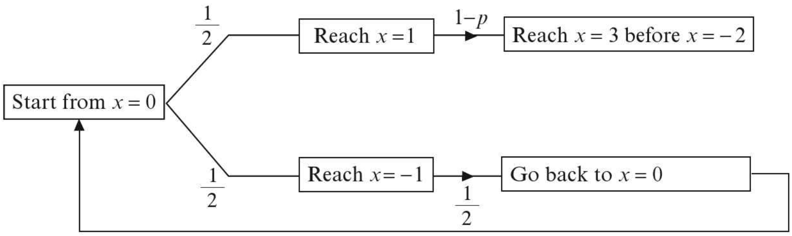


A_1 should therefore be the event that Pi reaches $x = 3$ before $x = -2$. Now comes the crucial insight. Compare the figure in the problem and the figure above. The situations have been symmetrically reversed. In the first situation, p was the probability that Pi reaches $x = 3$ before $x = -2$, that is, Pi travels 3 units in one direction before travelling 2 in the other. In the situation in the figure above, $P(A_1)$ is the probability that Pi travels 2 units in one direction before travelling 3 in the other, so we simply have

$$P(A_1) = 1 - p$$

Make sure you understand this argument carefully.

Coming to A_2: Pi has reached $x = -1$, so he *must* return to $x = 0$, because he *must not* reach $x = -2$. Once he is at $x = 0$, he comes back to the starting configuration, which means that now his probability of reaching $x = 3$ before $x = -2$ is again p. So let us complete the incomplete probability tree we drew earlier:



Thus (pay attention to how we write this!),

$$p = \frac{1}{2} \times \underbrace{(1-p)}_{\text{Reaching } x=3 \text{ from } x=1} + \frac{1}{2} \times \frac{1}{2} \times \underbrace{p}_{\text{Back to } x=0} \tag{1}$$

This relation is a form of recursion. We have already seen recursions in other contexts. Here, we see that in the definition (or expression) of p, we are using p again. Thus, we can say that recursion is a process a 'procedure' goes through when one of the steps of the 'procedure' involves rerunning the procedure. In this example, the 'procedure' of calculating p involved the step $(x = 0) \to (x = -1) \to (x = 0)$, at which point we have to rerun the 'procedure' of calculating p. From (1),

$$p = \frac{1}{2} - \frac{1}{2}p + \frac{1}{4}p \quad \Rightarrow \quad \frac{5}{4}p = \frac{1}{2} \quad \Rightarrow \quad p = \frac{2}{5}.$$

Thus, there is a 40% chance that Pi will return home safely!

S23. Before going through the solution, give the problem a great deal of thought. Will switching change the probability of wining? Or will it not matter because it is equally likely that the car may be behind Door 1 or Door 2?

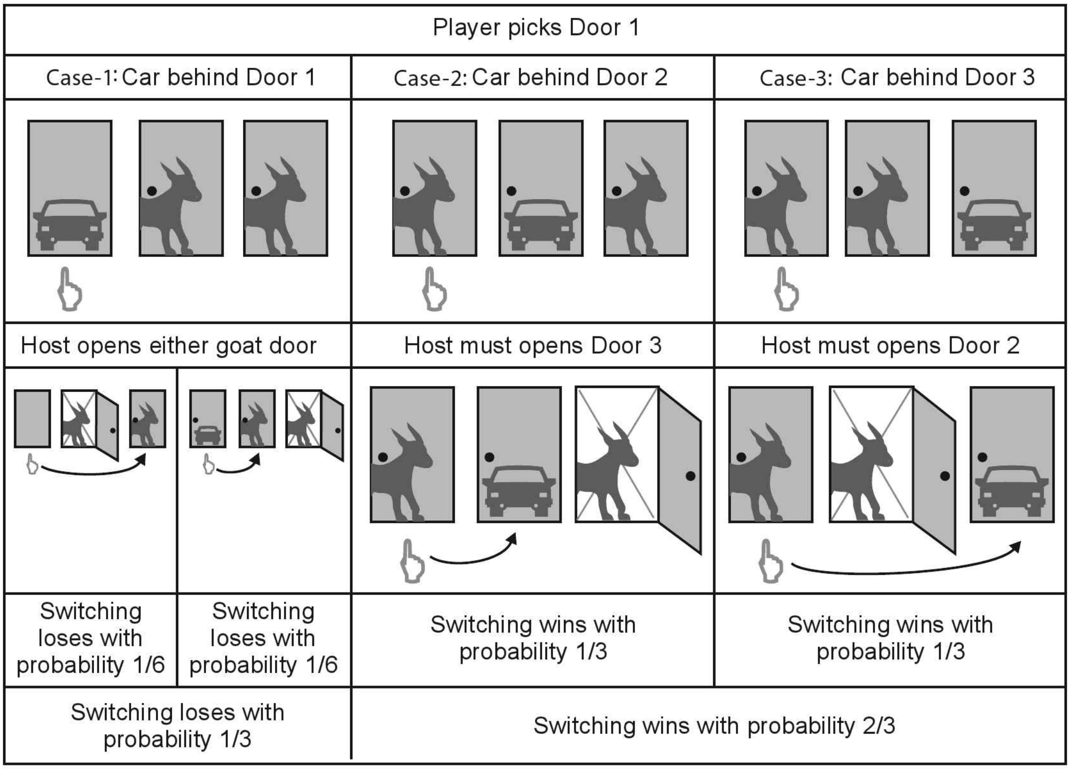
Let us assume that the player initially picked Door 1. (However, the player may initially choose any of the three doors). We'll now calculate the probability of winning by switching, by listing out all cases explicitly.

Case 1: The car is behind Door 1. In this case, the game host Monty Hall must open one of the two remaining doors randomly.

Case 2: The car is behind Door 2. Then the host must open Door 3.

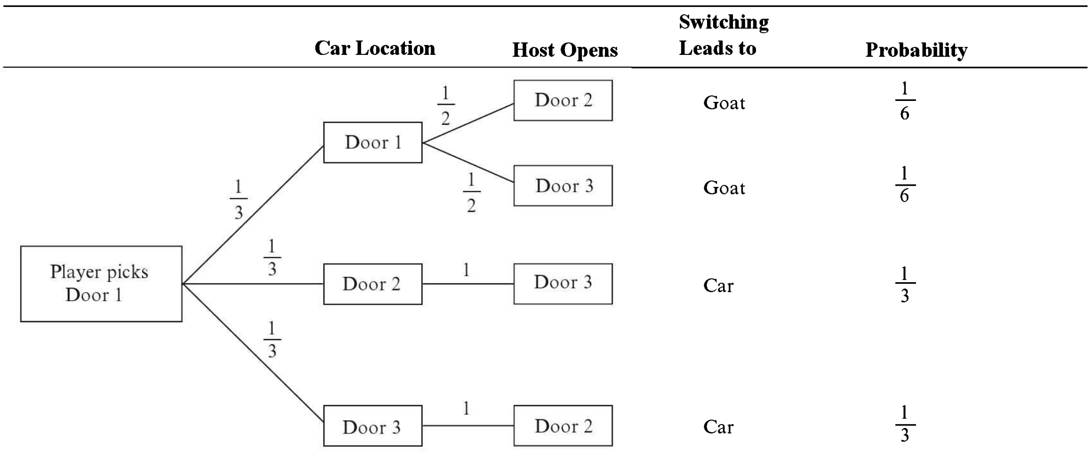
Case 3: The car is behind Door 3. Then the host must open Door 2.

If the player chooses to switch, he will win only if the car is behind either of the two unchosen doors rather than the one that was originally picked. The following diagram visually depicts the various cases and what happens with switching.



(*Image courtesy:* Wikipedia)

Let us explain how we arrived at the various probabilities, using a probability tree:



Thus, switching leads to winning with probability $\frac{2}{3}$, that is, the player should switch for a higher probability of winning! This result may seem very counter-intuitive to you. After all, you may say, the remaining two doors must each have a probability $\frac{1}{2}$ of containing the car. However, this intuition is wrong. In fact, you are not alone in this intuition. When first presented with this problem, an overwhelming majority of people assume that each door has an equal probability and conclude that switching does not matter. Even Nobel laureates have been known to have given the wrong answer and to have insisted upon it!

Determinants and Matrices

PART-A: Summary of Important Concepts

A matrix is a rectangular array of numbers, arranged in rows and columns. For example

$$M = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 3 & 0 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix}$$

Matrices with the same number of rows and columns are called square matrices. The determinant is a value associated with a square matrix, *i.e.*, every square matrix will have a corresponding determinant, which can be computed according to a specific arithmetic expression. For example, the determinant of

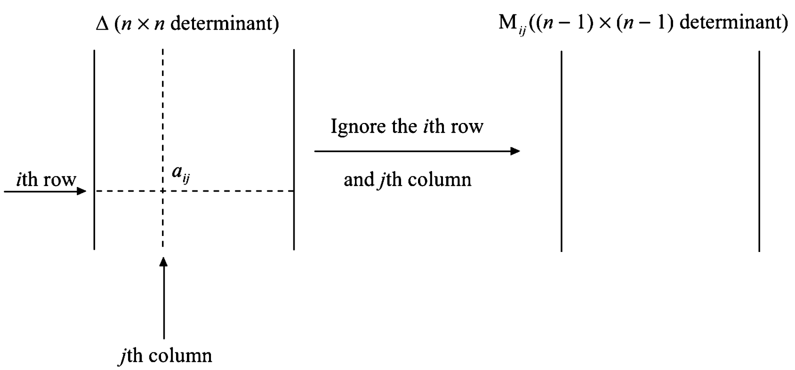
$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{can be written as } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and will be given by $\Delta = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$.

Matrices and determinants find application in an extremely wide variety of scientific fields, and this makes them indispensable tools for science students. Since determinants are quantities associated with a special type of matrices (square matrices), we summarize their properties first, and then turn to a discussion on matrices in general.

1. Expansion of Determinants

A 2×2 determinant $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ has the value $\Delta = ad - bc$. To expand a general determinant (of size $n \times n$) along any of its rows or columns, we need to understand the concept of *co-factor*. If a_{ij} is the element in the ith row and jth column, then ignoring the ith row and jth column in the $n \times n$ determinant leaves us with an $(n-1)\times(n-1)$ determinant:



If the reduced matrix is donated by M_{ij}, then C_{ij}, the co-factor of a_{ij}, is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Now, suppose that we denote the rows in an $n \times n$ determinant Δ by $R_1, R_2, \ldots, R_n$, and the columns by $C_1, C_2, \ldots, C_n$. To expand Δ along the pth row R_p, we find the co-factors of the terms in R_p. If the terms are $a_{p_1}, a_{p_2}, \ldots, a_{p_n}$, we denote the corresponding co-factors by $C_{p_1}, C_{p_2}, \ldots, C_{p_n}$. The value of Δ is now given by

$$\Delta = a_{p_1}C_{p_1} + a_{p_2}C_{p_2} + \cdots + a_{p_n}C_{p_n}$$

That is, take each term of the row, multiply it by its co-factor, and add all such terms. Similarly, Δ can be evaluated by expansion along any column. It is important to note that no matter what row or column you choose to expand Δ along, its value will always come out to be the same. As a concrete example, lets take

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \\ 7 & 5 & -2 \end{vmatrix}$$

and evaluate it by expansion along the 2nd column. We summarize the calculations in a table:

Position of Term	Term	Co-factor	Term × Co-factor
(1, 2)	3	$(-1)^{1+2}\begin{vmatrix} -1 & 4 \\ 7 & -1 \end{vmatrix} = 26$	78
(2, 2)	0	$(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 7 & -2 \end{vmatrix} = -16$	0
(3, 2)	5	$(-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = -6$	−30
			$\Delta = 48$

You can verify that the value of Δ will come out to be 48, no matter what other column (or row) we choose to expand Δ along. Let us find the co-factors in a 4×4 determinant.

Illustration 1: Let $D = \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 4 & 7 \\ 4 & 1 & 3 & 9 \\ 8 & 5 & -2 & 6 \end{vmatrix}$. Find the co-factors of 6, 7, and 8.

Working: 6 is at the position (4, 4) so $i + j = 8$. The co-factor of 6 becomes

$$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 4 & 1 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 1 \times 2 - 3 \times (-16) + 2 \times (-8) = 34$$

7 is at the position (2, 4), so $i + j = 6$. The co-factor of 7 is

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 8 & 5 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 8 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 8 & 5 \end{vmatrix} = 1 \times (-17) - 3 \times (-32) + 2 \times (12) = 103$$

For 8, $i+j=5$, so its co-factor is

$$(-1)\begin{vmatrix} 3 & 2 & -1 \\ 2 & 4 & 7 \\ 1 & 3 & 9 \end{vmatrix} = -65 \qquad \text{(verify)}$$

We note the following important facts:

(a) The sum of the products of the elements of any row or column with their corresponding co-factors is equal to the value of the determinant.

(b) The sum of the products of the elements of any row or column with the co-factors of the corresponding elements of any other row or column is zero.

2. Properties of Determinants

To evaluate a determinant, it is not always necessary to fully expand it. There are a number of properties of determinants, particularly row and column transformations, that can simplify the evaluation of any determinant considerably. These properties are summarized below:

Property 1

The value of determinant is not changed when rows are changed into columns and columns into rows.

For example,
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 2

If any two rows or columns of a determinant are interchanged, the sign of the determinant changes but its magnitude remains the same.

For example,
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Property 3

A determinant having two rows or two columns identical has the value zero. For example,

$$\Delta = \begin{vmatrix} p & q & r \\ p & q & r \\ x & y & z \end{vmatrix} = p\begin{vmatrix} q & r \\ y & z \end{vmatrix} - q\begin{vmatrix} p & r \\ x & z \end{vmatrix} + r\begin{vmatrix} p & q \\ x & y \end{vmatrix} = 0.$$

Property 4

Multiplying all the elements of a row (or column) by a scalar (a real number) is equivalent to multiplying the determinant by that scalar. For example,

$$\Delta = \begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note carefully that λ is multiplied with elements of just one row and not of the entire determinant. Another example:

$$\begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ \lambda b_1 & \lambda b_2 & \lambda b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix} = \lambda^3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 5

A determinant can be split into a sum of two determinants along any row or column. For example,

$$\begin{vmatrix} a_1 + d_1 & a_2 + d_2 & a_3 + d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The property is evident by expanding the determinant on the LHS along R_1.

Property 6

Row and column transformations. The transformations property is the most widely used property to simplify determinants. This property says that the value of a determinant does not change when any row (or column) is multiplied by a scalar (a real number) and is then added to or subtracted from any other row (or column). For example, lets consider

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Lets multiply the 3rd row by λ and add to R_1. This operation can be succinctly denoted as $R_1 \to R_1 + \lambda R_3$. Our property tells us that the determinant's value stays the same. Indeed:

$$\Delta = \begin{vmatrix} a_1 + \lambda c_1 & a_2 + \lambda c_2 & a_3 + \lambda c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad R_1 \to R_1 + \lambda C_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda c_1 & \lambda c_2 & \lambda c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{(Splitting along } R_1\text{)}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \lambda \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta$$

In the second-last step, we have used the fact that since two of the rows of the second determinant are the same, its value is 0.

Property 7

Multiplication of determinants. Suppose we have two 2×2 determinants

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

and we wish to find $\Delta_1\Delta_2$.

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \quad \begin{pmatrix}\text{We interchanged the rows and the} \\ \text{columns for the second determinant.}\end{pmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 & a_1\beta_1 + b_1\beta_2 \\ a_2\alpha_1 + b_2\alpha_2 & a_2\beta_1 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} (a_1, b_1)\times\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} & (a_1, b_1)\times\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix} \\ (a_2, b_2)\times\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} & (a_2, b_2)\times\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix} \end{vmatrix} = \begin{vmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{vmatrix}$$

In the last expression, R_1 and R_2 denote the rows of the first determinant, while C_1 and C_2 denote the columns of the second determinant. This is a row-by-column multiplication. Similarly, we'll have column-by-column multiplication.What if we have to multiply two 3 × 3 determinants row-by-row?

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

In terms of rows, we write

$$\Delta_1 = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} R_1' \\ R_2' \\ R_3' \end{vmatrix}$$

The multiplication process is analogous to the 2 × 2 case:

$$\Delta_1\Delta_2 = \begin{vmatrix} R_1R_1' & R_1R_2' & R_1R_3' \\ R_2R_1' & R_2R_2' & R_2R_3' \\ R_3R_1' & R_3R_2' & R_3R_3' \end{vmatrix}$$

Rows are multiplied similarly as before. For example,

$$R_1R_2' = (a_1, b_1, c_1)\times(\alpha_2, \beta_2, \gamma_2) = (a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2)$$

As in the 2 × 2 case, we can have row-by-column and column-by-column multiplication. Let us evaluate the numerical product of two 3 × 3 determinants:

$$\Delta_1 = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 2 & -1 \\ 5 & -1 & 3 \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 4 \\ 0 & 5 & 2 \end{vmatrix}$$

Note that $\Delta_1 = -74$ and $\Delta_2 = 15$, so $\Delta_1\Delta_2$ should be -1110. Now, let us evaluated $\Delta_1\Delta_2$ through row-by-row multiplication. We have,

$$R_1R_1' = (1\ 3\ 2)(2\ -1\ 3) = 5 \qquad R_1R_2' = (1\ 3\ 2)(3\ 1\ 4) = 14 \qquad R_1R_3' = (1\ 3\ 2)(0\ 5\ 2) = 19$$

$$R_2R_1' = (4\ 2\ -1)(2\ -1\ 3) = 3 \qquad R_2R_2' = (4\ 2\ -1)(3\ 1\ 4) = 10 \qquad R_2R_3' = (4\ 2\ -1)(0\ 5\ 2) = 8$$

$$R_3R_1' = (5\ -1\ 3)(2\ -1\ 3) = 20 \qquad R_3R_2' = (5\ -1\ 3)(3\ 1\ 4) = 26 \qquad R_3R_3' = (5\ -1\ 3)(0\ 5\ 2) = 1$$

So,

$$\Delta_1\Delta_2 = \begin{vmatrix} 5 & 14 & 19 \\ 3 & 10 & 8 \\ 20 & 26 & 1 \end{vmatrix} = -990 + 2198 - 2318 = -1110$$

3. Determinants and Linear Equations

Consider a system of linear equations in two variables:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Graphically, these represent two lines which may be parallel or which may intersect at a unique point. If we solve this system, we obtain

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\Rightarrow \quad x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad (1)$$

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then a unique solution will exist for x and y. Graphically speaking, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ means that $a_1b_2 - a_2b_1 \neq 0$, *or* $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$, which implies that the two lines are non-parallel, so a unique solution must exist.

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ the two lines are parallel, because $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. In this case, if $\frac{c_1}{c_2} \neq \frac{a_1}{a_2} = \frac{b_1}{b_2}$, the two lines are different and parallel, so no solution exists; this is confirmed in (1) because the denominator-determinant is 0, but the numerator-determinants are not.

The other case is when $\frac{c_1}{c_2} = \frac{a_1}{a_2} = \frac{b_1}{b_2}$; this means that in (1), all the determinants will be 0. Physically, this means that the two lines are the same, so there will be infinitely many solutions.

Let us now consider a system in three variables:

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

$$a_3x + b_3y + c_3z + d_3 = 0$$

Physically, this corresponds to three planes in 3-D space. If the three planes intersect in a point, a unique solution will exist for the system. But the three planes may not intersect in a point. For example, if two of the planes are parallel, there is no unique intersection point. In such cases, the system does not have a solution. Also, there will be cases where infinitely many solutions exist. For example, when the third plane passes through the line of intersection of the first two planes, the set of intersection points is the same line, which means there are infinitely many solutions.

Let us now determine the solutions of this system. Taking a cue from the result of the two-variables system, we define

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Now, on Δ_1, if we apply the transformation $C_3 \to yC_1 + zC_2 + C_3$, we obtain

$$\Delta_1 = \begin{vmatrix} b_1 & c_1 & b_1y + c_1z + d_1 \\ b_2 & c_2 & b_2y + c_2z + d_2 \\ b_3 & c_3 & b_3y + c_3z + d_3 \end{vmatrix}.$$

But using the original equations, the third column C_3 reduces to

$$\Delta_1 = \begin{vmatrix} b_1 & c_1 & -a_1x \\ b_2 & c_2 & -a_2x \\ b_3 & c_3 & -a_3x \end{vmatrix} = -x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -x\Delta \quad \Rightarrow \quad x = -\frac{\Delta_1}{\Delta}.$$

Using an exactly analogous process, we can arrive at expressions for y and z:

$$y = \frac{+\Delta_2}{\Delta} \quad \text{where} \quad \Delta_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_2 & c_3 & d_3 \end{vmatrix}$$

$$z = \frac{-\Delta_3}{\Delta} \quad \text{where} \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

Therefore, we arrive at what is known as the Cramer's rule:

$$\boxed{\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta}}$$

It is important to understand the justification we have presented for this rule (many students know the rule but not the reason why it is true). As is the case with a two variables system, if $\Delta \neq 0$, then obviously a unique solution will exist for (x, y, z), given by the Cramer's rule. That solution will physically correspond to the point of intersection of the three planes represented by the three equations of the system.

However, if $\Delta = 0$, then two cases may arise: in case all the other determinants $\Delta_1, \Delta_2, \Delta_3$ are also zero, then the system will have infinite solutions. However, if even one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then the system will have no solution, because Cramer's rule will give rise to undefined quantities.

Let us summarize:

$\Delta \neq 0$	$\Delta = 0$
$\to$ Unique solution exists	$\to$ If $\Delta_1, \Delta_2, \Delta_3$ are all zero, infinite solutions exist
	$\to$ If at least one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, no solution exists

A particular case of the 3-variables system is when all the constant terms are zero. Such a system is called a *homogeneous system*, since all the terms in each equation are of the same degree, *i.e.*, linear:

$$a_1x + b_1y + c_1z = 0 \qquad a_2x + b_2y + c_2z = 0 \qquad a_3x + b_3y + c_3z = 0$$

Note that $x = y = z = 0$ is always a solution of this system. But the system may also have infinite number of solutions, in which case there will exist solutions other than (0, 0, 0). The (0, 0, 0) solution is called

the *trivial solution* for this system, while other solutions (if they exist) are called *non-trivial*. For this system, $\Delta_1 = \Delta_2 = \Delta_3 = 0$, and so we have two possible cases:

$\Delta \neq 0$	$\Delta = 0$
The system has the trivial solution (0, 0, 0) and no other solution.	The system has non-trivial solutions, because infinite solutions exist.

4. Basic Properties of Matrices

(a) Sum/Difference of matrices: If we have two matrices A and B of the same order $m \times n$, their sum/difference is given by adding/subtracting the corresponding elements:

$$A \pm B = [a_{ij}] \pm [b_{ij}] = [a_{ij} \pm b_{ij}]$$

For example,

$$\begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 5 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 7 \\ 9 & -1 \end{bmatrix}$$

(b) Scalar multiplication: If we multiply a matrix $A = [a_{ij}]$ by λ, we get the matrix $\lambda A = [\lambda a_{ij}]$. For example:

$$4\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 8 \\ 8 & 4 & -4 \end{bmatrix}$$

(c) Multiplication of matrices: To understand how matrices are multiplied, let us first consider

a *row vector* $\quad R = [r_1 \;\; \vec{r}_2 \ldots \vec{r}_n] \quad \left\{\begin{matrix}\text{A row vector is a matrix} \\ \text{with just one row}\end{matrix}\right\}$

and a *column vector* $\quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \left\{\begin{matrix}\text{A column vector is a matrix} \\ \text{with just one column}\end{matrix}\right\}$

which are both of order n. The product of R and C can be defined as

$$RC = [r_1 \;\; r_2 \ldots r_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

Therefore, RC is a scalar quantity. For example,

$$[1 \;\; 3 \;\; 2]\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 7$$

Now, we will discuss matrix multiplication. To multiply two matrices A and B to find AB, the number of columns in A should equal the number of rows in B. Let A be of order $m \times n$ and B be of order $n \times p$. The matrix AB will be of order $m \times p$ and will be obtained by multiplying each row vector of A successively with column vectors in B. Let us understand this using a concrete example:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

To obtain the element a_{11} of AB, we multiply R_1 of A with C_1 of B:

$$\underset{3\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}} \times \underset{3\times 2}{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \alpha_1\alpha_1+\alpha_2\alpha_2+\alpha_3\alpha_3 & \square \\ \square & \square \\ \square & \square \end{bmatrix}}$$

To obtain the element a_{12} of AB, we multiply R_1 of A with C_2 of B:

$$\underset{3\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}} \times \underset{3\times 2}{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \square & \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \\ \square & \square \\ \square & \square \end{bmatrix}}$$

To obtain the element a_{21} of AB, we multiply R_2 of A with C_1 of B:

$$\underset{3\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}} \times \underset{3\times 2}{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \square & \square \\ b_1\alpha_1+b_2\alpha_2+b_3\alpha_3 & \square \\ \square & \square \end{bmatrix}}$$

Proceeding this way, we obtain all the elements of AB. In general, if A is of order $m \times n$, and B of order $n \times p$, then to obtain the element a_{ij} in AB, we multiply R_i in A with C_j in B:

$$\mathrm{R}_i \longrightarrow \underset{m\times n}{\begin{bmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ & \vdots & \end{bmatrix}} \times \underset{n\times p}{\begin{bmatrix} & b_{1j} & \\ \cdots & \vdots & \cdots \\ & b_{nj} & \end{bmatrix}} = \underset{m\times p}{\begin{bmatrix} & \vdots & \\ \cdots & \boxed{\mathrm{R}_i\mathrm{C}_j} & \cdots \\ & \vdots & \end{bmatrix}}$$

(with $\mathrm{C}_j \downarrow$ above the column of B)

(d) Identity matrix: For any matrix A of order $m \times n$, the identity matrix I is a square matrix of order $n \times n$ such that $AI = A$. If A is itself a square matrix of order $n \times n$, then $AI = IA = A$. In the identity matrix, all diagonal elements are 1 and all non-diagonal elements are 0. For example, the 3×3 identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Illustration 2: If

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 4 & 3 \end{bmatrix}, \text{ find } AB \text{ and } BA.$$

Working: Since A is of order 2×3, and B is of order 3×2, AB is defined and will be of order 2×2. BA is also defined, and will be of order 3×3.

$$AB = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 19 & 13 \\ 8 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 8 \\ 1 & 1 & -3 \\ 17 & 10 & 19 \end{bmatrix}$$

This calculation should have made it clear to you that AB is in general not equal to BA, that is, multiplication of matrices is a non-commutative operation.

We note the following important facts about matrix multiplication:

(a) For three matrices A, B, C, the equality $AB = AC$ does not imply $B = C$.
(b) Matrix multiplication is associative, that is $A(BC) = (AB)C$.

5. Applications of Matrices to Linear Equations

The application of matrices to linear equations is extremely important, and is therefore discussed in some detail here. Consider the system of equations

$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x + 2y + z &= 4 \end{aligned}$$

We can write this system in matrix form as

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

While studying determinants, we learnt how to solve linear systems using Cramer's rule. We'll now see how to use matrices to do the same. Note that the matrix equation can be written as $AX = B$ where

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

A simple linear equation $ax = b$, where a, x, b are reals, has the solution

$$x = \frac{b}{a} = a^{-1}b$$

Can we do something similar for matrices? Can we define a matrix inverse so that

$$X = A^{-1}B$$

In case of square matrices, it turns out that we can! The inverse of a square matrix A should be another square matrix A^{-1} of the same order such that

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Let us understand how to arrive at A^{-1} using the example of a 3×3 matrix:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Note that the determinant of A, denoted as det(A) or $|A|$, by expansion along R_1 is

$$|A| = a_1\left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right) + b_1\left(-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}\right) + c_1\left(\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right)$$

This can be written as a row-column product:

$$|A| = [a_1 \quad b_1 \quad c_1]\begin{bmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \end{bmatrix}$$

We had also seen that the sum of co-factors of any row (or column) with the corresponding elements in a different row (or column) is zero. For example,

$$[a_2 \quad b_2 \quad c_2]\begin{bmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \end{bmatrix} = 0 \text{ whereas } [a_2 \quad b_2 \quad c_2]\begin{bmatrix} -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \\ +\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \\ -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \end{bmatrix} = |A| \text{again}$$

This suggests an interesting step forward. For each element a_{ij} in A, define its co-factor as C_{ij}. Consider the matrix composed of the co-factors, but with co-factors of rows arranged as columns. For example, consider the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ with corresponding co-factors } C_1, C_2, ..., C_9$$

Now, consider the matrix

$$\tilde{A} = \begin{bmatrix} C_1 & C_4 & C_7 \\ C_2 & C_5 & C_8 \\ C_3 & C_6 & C_9 \end{bmatrix}$$

Note that

$$A\tilde{A} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

It is important that you are absolutely sure of this step. The matrix $\tilde{A}$ is called the adjoint of A and is sometimes denoted as adj(A). Note that $\tilde{A}A$ is the same as $A\tilde{A}$. Thus,

$$A\tilde{A} = \tilde{A}A = |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I$$

$$\boxed{\begin{aligned} &\Rightarrow \quad A\left(\frac{\tilde{A}}{|A|}\right) = \left(\frac{\tilde{A}}{|A|}\right)A = 1 \\ &\Rightarrow \quad AA^{-1} = A^{-1}A = 1 \quad \text{where } A^{-1} = \frac{\tilde{A}}{|A|} \end{aligned}}$$

We have succeeded in evaluating the inverse of a matrix. Let us apply this to our original problem.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad |A| = 8$$

Carefully observe each of the elements of A^{-1}:

$$A^{-1} = \frac{\tilde{A}}{|A|} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} = \begin{bmatrix} \frac{-1}{8} & \frac{3}{8} & \frac{-5}{8} \\ \frac{-3}{8} & \frac{1}{8} & \frac{7}{8} \\ \frac{7}{8} & \frac{-5}{8} & \frac{-11}{8} \end{bmatrix}$$

Since, we had $AX = B$,

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B = \begin{bmatrix} \frac{-1}{8} & \frac{3}{8} & \frac{5}{8} \\ \frac{-3}{8} & \frac{1}{8} & \frac{7}{8} \\ \frac{7}{8} & \frac{-5}{8} & \frac{-11}{8} \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \Rightarrow \quad x = 1, y = 2, z = -1$$

We make the following observations:

- If $|A| \neq 0$, A^{-1} always exists which means that the system has a unique solution.
- If $|A| = 0$, two cases arise as in the case of determinants. Since, $X = A^{-1}B = \frac{\tilde{A}}{|A|}B$,
  (a) If $\tilde{A}B \neq 0$, no solution exists since X becomes undefined.
  (b) If $\tilde{A}B = 0$, the solution has an infinite number of solutions.
- For a homogenous system, *i.e.*, $B = 0$,
  (a) If $|A| \neq 0$, the system has only one solution, namely the trivial solution $X = 0$.
  (b) If $|A| = 0$, then since $\tilde{A}B = 0$ too, the system has an infinite number of solutions.

6. General Properties of Matrices

6.1 Standard Terminology

(1) **Zero matrix:** All elements of the matrix are zero.

(2) **Row matrix:** Also called a row vector, such a matrix has only row. Example:

$[1 \;\; -1 \;\; 2 \;\; 4]$

(3) **Column matrix:** Also called a column vector, such a matrix has only one column. Example:

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

(4) **Diagonal matrix:** All non-diagonal elements are zero. Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(5) **Upper/lower triangular matrix:** All elements below/above the diagonal are zero.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ -1 & 7 & 1 \end{bmatrix}$$

Upper Triangular — Lower Triangular

(6) **Singular matrix:** A matrix whose determinant is zero.

(7) **Transpose:** For a matrix A, the transpose of A, which we represent as A^T, is obtained by interchanging the rows and columns of A. Example,

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \end{bmatrix}$$

Note that $(A^T)^T = A$.

6.2 Important Properties and Results

(a) $(AB)^T = B^T A^T$. In general, we have $(A_1 A_2 \ldots A_n)^T = A_n^T A_{n-1}^T \ldots A_2^T A_1^T$.

(b) A square matrix A is symmetric if $A = A^T$, and skew-symmetric if $A = -A^T$. *Every* square matrix M can be expressed as the sum of symmetric and a skew-symmetric matrix as follows:

$$M = \underbrace{\frac{1}{2}(M + M^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(M - M^T)}_{\text{Skew-Symmetric}}$$

(c) $(A^T)^{-1} = (A^{-1})^T$

(d) A square matrix A is said to be orthogonal if $A^T A = I$.

(i) The transpose of an orthogonal matrix is orthogonal.

(ii) If A is orthogonal, then $|A| = \pm 1$.

(iii) The product of two orthogonal matrices is orthogonal.

(iv) The inverse of an orthogonal matrix is orthogonal.

It will be a very useful exercise to prove these results about orthogonal matrices.

IMPORTANT IDEAS AND TIPS

1. *The Larger Picture.* Why do we need matrices and determinants? Many of the readers might be surprised to know that matrices and determinants are an exceedingly important component of higher mathematics. For almost all engineering disciplines, a thorough knowledge of this subject is required. The utility of matrices and determinants stems from the fact that they allows us to handle and operate on a large array of numbers simultaneously. This is particularly useful in solving practical problems whose mathematical models contain a larger number of unknown variables. Suppose that you are trying to solve a system of linear equations with 200 variables (you will encounter systems with a large number of variables in higher mathematics). Matrices and determinants give us a way to solve this system using a defined set of rules. We have solved three-variable systems, but the techniques can be applied to systems with any

number of varialbles. All we have to do is to program a computer to implement the various operations of matrices or determinants, and give the 200-variable system to this program as the input. Many computer scientists and mathematicians work on such problems.

2. *Mistakes in Evaluating Determinants.* When evaluating determinants, students frequently calculate the co-factors incorrectly. The alternating + and – signs can lead to calculation mistakes. A particularly useful exercise to ensure that you never calculate determinants incorrectly is as follows: construct a few 4×4 determinants (select terms with small magnitudes) and evaluate them using a calculating utility (a calculator, or an online determinant evaluator). Every now and then, select one of these 4×4 determinants and evaluate it as fast as you can (by expanding along a randomly selected row or column), and then compare your answer with the correct answer. Doing this exercise frequently will minimize the calculation mistakes you might otherwise make.

3. *Mistakes in Matrix Multiplication.* Mistakes are commonly made while multiplying two matrices. That is why we have explained matrix multiplication in detail in the preceeding pages, and you are urged to reread that discussion multiple times. If you have to calculate AB, the product of two matrices A and B, remember that:

   (a) The number of columns in A and the number of rows in B must be the same.
   (b) Rows in A multiply with columns in B to produce elements in AB. Each element in AB is formed by multiplying one row in A with one column in B.

4. *Left and Right Cancellation in Matrices.* A very important concept to understand is that matrix multiplication is not like normal (arithmetic) multiplication. For example, if I have the *algebraic* equation $ab = ac$, I can say that $b = c$. What I've done is the left cancellation of a, and doing that is possible in such equations. However, if I have the *matrix* equation $AB = AC$, where A, B, C are matrices, then I *cannot* say that $B = C$. That is, (left or right) cancellation does not hold in general for matrix multiplication.

5. *Scalar Multiplication.* When a matrix is multiplied by a scalar, each element of the matrix gets multiplied individually by the scalar. However, when a determinant is multiplied by a scalar, only the elements of any one row (or column) may be multiplied by that scalar:

$$\lambda\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ \lambda p & \lambda q & \lambda r \\ \lambda x & \lambda y & \lambda z \end{bmatrix} \quad \text{whereas} \quad \lambda\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \begin{vmatrix} \lambda a & \lambda b & \lambda c \\ p & q & r \\ x & y & z \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a & \lambda b & c \\ p & \lambda q & r \\ x & \lambda y & z \end{vmatrix} \text{ etc}$$

6. *Determinant Transformations.* One numerical fact which you need to work out on your own is: can row/column transformations be clubbed together into a single step? For example, consider the determinant on the left, on which two row transformations are applied in the same step to obtain the determinant on the right:

$$\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 + \lambda_1 R_2 + \mu_1 R_3 \\ R_3 \to R_3 + \lambda_2 R_1 + \mu_2 R_2 \end{matrix} \quad \begin{vmatrix} a+\lambda_1 p+\mu_1 x & b+\lambda_1 q+\mu_1 y & c+\lambda_1 r+\mu_1 z \\ p & q & r \\ x+\lambda_2 a+\mu_2 p & y+\lambda_2 b+\mu_2 q & z+\lambda_2 c+\mu_2 r \end{vmatrix}$$

Is it correct to do so? Secondly, can row *and* column tranformations be clubbed in the same step? Try working with some examples.

7. *Linear Equations.* The most important application of matrices and determinants we have studied is related to finding the solutions of systems of linear equations. You must understand the geometrical significance of systems of linear equations, and how matrices or determinants are used to solve them. Also, you must appreciate how the two approaches are deeply connected. For this purpose, we summarize the results from the two approaches side by side. Compare the determinant approach with the matrix approach:

(a) **A two variables system:** This system corresponds to two lines in a 2-D plane.

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

	Determinants	Matrices
(i) Problem Formulation	$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta}$	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$ or $AX=B$
(ii) Case 1: The lines are non-parallel, which means that there is a unique point of intersection.	$\Delta \neq 0$	The inverse of A exists.
(iii) Case 2: The lines are different but parallel.	$\Delta = 0$ At least one of Δ_1, Δ_2 is not zero	The inverse of A does not exist and $\tilde{A}B \neq 0$
(iv) Case 3: The lines are the same	$\Delta = \Delta_1 = \Delta_2 = 0$	The inverse of A does not exist and $\tilde{A}B = 0$.

(b) **A three variables system:** This system corresponds to three planes in 3-D space.

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

	Determinants	Matrices
(i) Problem Formulation	$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta}$	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -d_1 \\ -d_2 \\ -d_3 \end{bmatrix}$ or $AX=B$
(ii) Case 1: The planes are non-parallel, which means that there is a unique point of intersection.	$\Delta \neq 0$	The inverse of A exists.
(iii) Case 2: At least two of the three planes are parallel, so there is no unique solution.	$\Delta = 0$ At least one of $\Delta_1\Delta_2\Delta_3$ is not zero.	The inverse of A does not exist and $\tilde{A}B \neq 0$.
(iv) Case 3: The planes intersect in a line. This could happen if one plane passes through the intersecting line of the other two planes; or it could happen if two planes are the same and the third is non-parallel to them. Alternatively, all the three planes could be coincident.	$\Delta = \Delta_1 = \Delta_2 = 0$	The inverse of A does not exist and $\tilde{A}B = 0$.

8. *Cramer's Rule*. We have seen how the Cramer's Rule applied to a system of linear equations helps us calculate the solution to that system by calculating the appropriate determinants. We would like to emphasize that you must be familiar with the justification behind this rule (for a system of 3 variables, we have presented the justification in the preceeding pages). In particular, you must understand that this rule is based on the transformation properties of determinants. Also, knowing the justification will help avoid calculation mistakes like taking the incorrect signs or incorrectly constructing the determinants.

9. *Inverse of a Matrix*. You must remember that inverses exist only for square matrices. Secondly, you must be familiar with the justification behind the sequence of steps used to calculate matrix inverses. We have presented this justification for 3×3 matrices in the preceding section. There is a very specific reason why the adjoint matrix is constructed, and you must understand that reason properly.

Determinants and Matrices

PART-B: Illustrative Examples

Example 1

The value of $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ is

(A) $(a-b)(b-c)(c-a)(a+b+c)$ (C) $2(a-b)(b-c)(c-a)(a+b+c)$

(B) $-(a-b)(b-c)(c-a)(a+b+c)$ (D) $-2(a-b)(b-c)(c-a)(a+b+c)$

Solution: We have

$$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

Expanding along C_1,

$$\begin{aligned} \Delta &= (b-a)(c^3-a^3) - (b^3-a^3)(c-a) \\ &= (b-a)(c-a)\{(c^2+a^2+ac)-(b^2+a^2+ab)\} \\ &= (a-b)(b-c)(c-a)(a+b+c) \end{aligned}$$

The correct option is (A). ■

Example 2

Let $\theta \in (0, \frac{\pi}{2})$ satisfy $\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin^{4\theta} \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$. The value of $24\frac{\theta}{\pi}$ is

(A) 3 (B) 5 (C) 7 (D) 11

Solution: Using the transformation $C_1 \to C_1 + C_2 + C_3$ and factoring out the common factor $(2+4\sin 4\theta)$ from C_1, we obtain,

$$(2+4\sin 4\theta)\begin{vmatrix} 1 & \sin^2\theta & 4\sin 4\theta \\ 1 & 1+\sin^2\theta & 4\sin 4\theta \\ 1 & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0.$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have

$$2(1+2\sin 4\theta)\begin{vmatrix} 1 & \sin^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

Explanding along C_1, we have

$$2(1+2\sin 4\theta) = 0 \quad \text{or} \quad \sin 4\theta = -\frac{1}{2}$$

$$\Rightarrow \quad 4\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \quad \Rightarrow \quad \theta = \frac{7\pi}{24}$$

The value of $24\frac{\theta}{\pi}$ is therefore 7. The correct option is (C). ■

Example 3

Consider the homogenous system

$$x - cy - bz = 0$$
$$cx - y + az = 0$$
$$bx + ay - z = 0$$

If this system has a non-trivial solution, then the value of $a^2 + b^2 + c^2 + 2abc$ is

(A) 0 (B) 1 (C) 1 (D) 2 (E) None of these

Solution: If a homogenous system has non-trivial solutions, this implies that $\Delta = 0$:

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

Using $C_2 \to C_2 + cC_1$ and $C_3 \to C_3 + bC_1$, we have

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ c & -1+c^2 & a+bc \\ b & a+bc & -1+b^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1+c^2)(-1+b^2) - (a+bc)^2 = 0$$

Upon simplifying, we have

$$a^2 + b^2 + c^2 + 2abc = 1$$

Thus, the correct option is (B). ■

Example 4

Let A, B, C be the angles of a ΔABC. What is the value of the following determinant?

$$\Delta = \begin{vmatrix} e^{i2A} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{i2B} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{i2C} \end{vmatrix}$$

(A) 0 (B) 2 (C) 4 (D) 8 (E) None of these

Solution: We'll make use of the fact that $A+B+C=\pi$. Taking out factors of $e^{i2A}, e^{-iC}, e^{-iB}$ from C_1, C_2, C_3 respectively, we have:

$$\Delta = e^{i(2A-B-C)}\begin{vmatrix} 1 & 1 & 1 \\ e^{-i(2A+C)} & e^{i(2B+C)} & e^{i(B-A)} \\ e^{-i(2A+B)} & e^{i(C-A)} & e^{i(B+2C)} \end{vmatrix}$$

Now, we note that $2A+C=\pi-(B-A), 2B+C=\pi-(A-B), 2C+B=\pi-(A-C)$, and $2A+B=\pi+(A-C)$. Therefore,

$$\Delta = e^{i(2A-B-C)}\begin{vmatrix} 1 & 1 & 1 \\ -e^{i(B-A)} & -e^{i(B-A)} & e^{i(B-A)} \\ -e^{i(C-A)} & e^{i(C-A)} & -e^{i(C-A)} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = 4$$

The correct option is (C). ■

Example 5

Let f, g, h be differentiable functions of x and Δ and be such that

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$$

We also define two more determinants as follows:

$$\Delta_1 = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$$

Which of the following expressions for Δ' (the derivative of Δ') is correct in terms of Δ_1 and Δ_2?

(A) $\Delta' = 2x\Delta_1 + 3x^2\Delta_2$ (C) $\Delta' = x^2\Delta_1 + 4x^3\Delta_2$

(B) $\Delta' = x^2\Delta_1 + 2x\Delta_2$ (D) $\Delta' = 3x^2\Delta_1 + x^3\Delta_2$

Solution: We have

$$\Delta = \begin{vmatrix} f & g & h \\ f+xf' & g+xg' & h+xh' \\ x^2f''+4xf'+2f & x^2g''+4xg'+2g & x^2h''+4xh'+2h \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 4R_2 + 2R_1$, we have

$$\Delta = x^3\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = x^3\Delta_1$$

$$\Rightarrow \quad \Delta' = 3x^2\Delta_1 + x^3\Delta_1'$$

where

$$\Delta_1' = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$$

$$= 0 + 0 + \Delta_2$$

$$\Rightarrow \quad \Delta' = 3x^2\Delta_1 + x^3\Delta_2$$

The correct option is (D). ■

SUBJECTIVE TYPE EXAMPLES

Example 6

Let a, b, c be the pth, qth, rth terms respectively of a GP. Evaluate $\Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$.

Solution: If we let the first term of the GP be f and the common ratio be t, then we have

$$a = f\,t^{p-1} \quad \Rightarrow \quad \log a = \log f + (p-1)\log t.$$

Similar relations can be written for log b and log c:

$$\Delta = \begin{vmatrix} \log f + (p-1)\log t & p & 1 \\ \log f + (q-1)\log t & q & 1 \\ \log f + (r-1)\log t & r & 1 \end{vmatrix} = \begin{vmatrix} (p-1)\log t & p & 1 \\ (q-1)\log t & q & 1 \\ (r-1)\log t & r & 1 \end{vmatrix} \quad C_1 \to C_1 - (\log t)C_3$$

$$= \log t \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix} = \log t \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} \quad C_1 \to C_1 + C_3 = 0$$

$$= 0$$

■

Example 7

Evaluate

$$\Delta = \begin{vmatrix} ax - by - cz & ay + bx & cx + az \\ ay + bx & by - cz - ax & bz + cy \\ cx + az & bz + cy & cz - ax - by \end{vmatrix}$$

Solution: Let us try to reduce C_1 to a simpler form. By the transformation $C_1 \to aC_1 + bC_2 + cC_3$, we generate the factor $(a^2 + b^2 + c^2)$ in all the terms of C_1:

$$\Delta = \frac{1}{a}\begin{vmatrix} x(a^2+b^2+c^2) & ay + bx & cx + az \\ y(a^2+b^2+c^2) & by - cz - ax & bz + cy \\ z(a^2+b^2+c^2) & bz + cy & cz - ax - by \end{vmatrix} \quad \text{(Note that there is a factor of } \tfrac{1}{a}\text{)}$$

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} x & ay + bx & cx + az \\ y & by - cz - ax & bz + cy \\ z & bz + cy & cz - ax - by \end{vmatrix}$$

Using a similar artifice, we can simplify R_1 now. By $R_1 \to xR_1 + yR_2 + zR_3$, we have

$$\Delta = \frac{1}{ax}(a^2+b^2+c^2)\begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by - cz - ax & bz + cy \\ z & bz + cy & cz - ax - by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ y & by - cz - ax & bz + cy \\ z & bz + cy & cz - ax - by \end{vmatrix}$$

Now, we simplify C_1 further by the row transformations $R_2 \to R_2 - yR_1$ and $R_3 \to R_3 - zR_1$:

$$\Delta = \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

Finally, we expand along C_1 and simplify

$$\Delta = (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz).$$

■

Example 8

Evaluate

$$\Delta = \begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}.$$

Solution: The trick is in recognizing the fact that Δ can be expressed as a product of two 3×3 determinants Δ_1 and Δ_2. To arrive at this product, we note how elements of row 1 can be expressed as products of rows.

$$(a_1-b_1)^2 = a_1^2 - 2a_1b_1 + b_1^2 = (a_1^2 \quad 2a_1 \quad 1) \times (1 \quad -b_1 \quad b_1^2)$$
$$(a_1-b_2)^2 = a_1^2 - 2a_1b_2 + b_2^2 = (a_1^2 \quad 2a_1 \quad 1) \times (1 \quad -b_2 \quad b_2^2)$$
$$(a_1-b_3)^2 = a_1^2 - 2a_1b_3 + b_3^2 = (a_1^2 \quad 2a_1 \quad 1) \times (1 \quad -b_3 \quad b_3^2)$$

This means that we now know how to fill R_1 of Δ_1 and the three rows of Δ_2

$$\Delta = \begin{vmatrix} a_1^2 & 2a_1 & 1 \\ \leftarrow & ? & \rightarrow \\ \leftarrow & ? & \rightarrow \end{vmatrix} \times \begin{vmatrix} 1 & -b_1 & b_1^2 \\ 1 & -b_2 & b_2^2 \\ 1 & -b_3 & b_3^2 \end{vmatrix}$$

But now, the other rows of Δ_1 should be evident immediately by symmetry:

$$\Delta = \begin{vmatrix} a_1^2 & 2a_1 & 1 \\ a_2^2 & 2a_2 & 1 \\ a_3^2 & 2a_3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -b_1 & b_1^2 \\ 1 & -b_2 & b_2^2 \\ 1 & -b_3 & b_3^2 \end{vmatrix}$$

These two determinants are separately very easy to evaluate; the product comes out to

$$\Delta = 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1).$$

■

Example 9

(a) Prove that the system of equations

$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$

has at least one solution for any real λ.

(b) Find the solution(s) for $\lambda = -5$.

Solution: We have

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 7(\lambda+5)$$

If $\lambda \neq -5, \Delta \neq 0,$ so a unique solution exists. For $\lambda = -5$, we have

$$\Delta_1 = \begin{vmatrix} -1 & 4 & -3 \\ 2 & -3 & 2 \\ 5 & -5 & 3 \end{vmatrix} = 0$$

Similarly, $\Delta_2 = \Delta_3 = 0$. That is, for $\lambda = -5, \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, so in this case infinitely many solutions exist. Therefore, in all cases, the system has at least one solution.

Now, how do we express the set of infinite solutions in the case of $\lambda = -5$? What we have to realize is that if we express two of the variables in terms of the third, and vary the third as a parameter, we get all the possible solutions. For $\lambda = -5,$ observe that y and z can be written in terms of x as

$$y = \frac{1-13x}{5}, \quad z = \frac{4-7x}{5} \quad \text{(Verify!)}$$

As x is varied, y and z vary in a manner such that (x, y, z) is a valid solution to the given system. Therefore, the set of all possible solutions can be expressed as

$$\left(x, \frac{1-13x}{5}, \frac{4-7x}{5}\right), x \in \mathbb{R}$$

■

Example 10

Find the value of

$$\Delta = \begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix}$$

Solution: The given determinant can be expressed as product of two simpler determinants:

$$\Delta = \begin{vmatrix} a_1^2 & -2a_1 & 1 & 0 \\ a_2^2 & -2a_2 & 1 & 0 \\ a_3^2 & -2a_3 & 1 & 0 \\ a_4^2 & -2a_4 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 & 0 \\ 1 & b_2 & b_2^2 & 0 \\ 1 & b_3 & b_3^2 & 0 \\ 1 & b_4 & b_4^2 & 0 \end{vmatrix} = 0$$

■

Example 11

Prove that

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} p & q & q \\ q & p & q \\ q & q & p \end{vmatrix},$$

where $p = a^2+b^2+c^2$ and $q = ab+bc+ca$.

Solution: If you consider the determinant on the LHS, say Δ_L, carefully, you'll see it is the determinant of the co-factors of

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Also, $\Delta\Delta_L = \Delta^3(\text{why?}),\ \text{so } \Delta_L = \Delta^2.$ Thus,

$$\Delta_L = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta_R \quad \text{(the determinant on the RHS)}$$ ■

Example 12

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ 2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 4 & 3 \\ -1 & 0 & 3 & 1 \end{bmatrix}$. Evaluate $(AB)\,C$ and $A\,(BC)$ and verify that they are the same.

Solution: The orders of A, B, C are 2×3, 3×2 and 2×4.

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 13 & -7 \\ 6 & 11 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 13 & -7 \\ 6 & 11 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 3 \\ -1 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 33 & 13 & 31 & 32 \\ 1 & 6 & 57 & 29 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 3 \\ -1 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 16 & 7 \\ 7 & 3 & 9 & 8 \\ 7 & 2 & -1 & 3 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 16 & 7 \\ 7 & 3 & 9 & 8 \\ 7 & 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 33 & 13 & 31 & 32 \\ 1 & 6 & 57 & 29 \end{bmatrix}$$

As expected, $(AB)\,C$ and $A\,(BC)$ are the same. ■

Example 13

Consider the system of equations

$$x + y + z = 5$$
$$x + 2y + 3z = 9$$
$$x + 3y + \lambda z = \mu$$

Find λ and μ for which this system has
(a) a unique solution (b) no solution (c) infinite solutions

Solution: We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = \lambda - 5$$

(a) For a unique solution, $\Delta \neq 0 \quad \Rightarrow \quad \lambda \neq 5.$

(b) For no solution, $\Delta = 0$ and $\tilde{A}B \neq 0.$ Now, $\Delta = 0$ means that $\lambda = 5.$ For $\lambda = 5,$ the coefficients matrix A becomes

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} \quad \Rightarrow \quad \tilde{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \tilde{A}B = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ \mu \end{bmatrix} = \begin{bmatrix} \mu - 13 \\ -2\mu + 26 \\ \mu - 13 \end{bmatrix}$$

Since $\tilde{A}B \neq 0,\ \mu \neq 13.$ Therefore, the system will have no solution if $\lambda = 5,\ \mu \neq 13.$

(c) For infinitely many solutions,

$$\Delta = 0,\ \tilde{A}B = 0 \quad \Rightarrow \quad \lambda = 5,\ \mu = 13.$$

■

Example 14

For two matrices A and B whose product is defined, prove that $(AB)^T = B^T A^T.$

Solution: This general result is extremely important, and so you must understand its proof carefully. Let A be of order $m \times n$ and B of order $n \times p$. Then AB is of order $m \times p$. Now, A^T is of order $n \times m$ and B^T is of order $p \times n$. So $B^T A^T$ is of order $p \times m$ which is the same as the order of $(AB)^T$. Let us understand this using a concrete example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix}$$

$$\Rightarrow \quad (AB) = \begin{bmatrix} ap + bx & aq + by & ar + bz \\ cp + dx & cq + dy & cr + dz \end{bmatrix}$$

$$\Rightarrow \quad (AB)^T = \begin{bmatrix} ap + bx & cp + dx \\ aq + by & cq + dy \\ ar + bz & cr + dz \end{bmatrix}$$

Also,

$$B^T A^T = \begin{bmatrix} p & x \\ q & y \\ r & z \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ap + bx & cp + dx \\ aq + by & cq + dy \\ ar + bz & cr + dz \end{bmatrix}$$

Focus on a particular term, say $ap + bx$, and note how it is generated at the position in both $(AB)^T$ and $B^T A^T$. Now let us generalize this. Consider the term in $(AB)^T$ at the position (i, j), say t. In AB, this same term is at the position (j, i). So, t is generated from R_j in A and C_i in B:

$$t = \sum_{k=1}^{n} a_{jk} b_{ki}.$$

Let us now consider the term at the position (i, j) in $B^T A^T$. This will be generated from R_i in B^T and C_j in A^T, or C_i in B and R_j in A, which will generate t again. Thus,

$$(AB)^T = B^T A^T$$

■

Example 15

A square matrix A is said to be symmetric if $A = A^T$, and skew-symmetric if $A = -A^T$. Show that every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Solution: If A be the given matrix, then A can be written as

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew-symmetric}}$$

(i) $A + A^T$ is symmetric because $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.

(ii) $A - A^T$ skew symmetric because $(A - A^T)^T = A^T - A = -(A - A^T)$.

For example, let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 6 \\ 2 & 4 & -3 \end{bmatrix}.$$

Then,

$$X = \frac{1}{2}(A + A^T) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & -3 \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Note that $X + Y = A$. ■

Example 16

Let A be a skew-symmetric matrix. Let B be a matrix of the same order such that $B(I - A) = I + A$. Show that B is orthogonal.

Solution: We have to show that $B^T B = BB^T = I$. Note that since A is skew-symmetic, $A^T = -A$. We are given that

$$B(I - A) = I + A \tag{1}$$

Taking the transpose on both sides, we have

$$\begin{aligned} &(I - A)^T B^T = I + A^T \\ &\Rightarrow \quad (I + A)B^T = I - A \end{aligned} \tag{2}$$

From (1) × (2), we have

$$(I + A)B^T B(I - A) = (I - A)(I + A) = (I + A)(I - A) \quad \text{(How?)}$$

$$\Rightarrow \quad B^T B = I$$

By (1) × (2) again in another order, we have

$$B(I - A)(I + A)B^T = I^2 - A^2$$

$$\Rightarrow \quad B(I^2 - A^2)B^T B = (I^2 - A^2)B \tag{3}$$

Since $B^T B = I$, we have

$$B(I^2 - A^2) = (I^2 - A^2)B \tag{4}$$

Using (4) in (3), we have

$$(I^2 - A^2)BB^T = I^2 - A^2$$

$$\Rightarrow \quad BB^T = I$$

Thus, $BB^T = B^T B = I$. ■

Example 17

Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of the these entries are 1 and four of them are 0.

(a) Find the number of matrices in A.

(b) Find the number of matrices in A for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution.

(c) Find the number of matrices in A for which this system of linear equations is inconsistent.

Solution: (a) Since all the matrices are symmetric, this means that on the diagonal, there can be either one '1' or three '1's. There cannot be two '1's.

If all entries on the diagonal are 1: In this case, 3 matrices are possible, since the other two '1's can be placed in 3 possible ways.

If one entry on the diagonal is 1: In this case, $3 \times 3 = 9$ matrices are possible, since the '1' on the diagonal can be placed in 3 ways, and the other two pairs of '1's can be placed in three ways.

$\Rightarrow$ The number of matrices in A is 12.

(b) For a unique solution, $|A|$ should be non-zero. We count such cases explicitly:

(i) All diagonal elements are 1

$$\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix}$$ One of x, y, z is 1, the other two are 0.

If $x = y = 0$, then $|A| = 0$. Verify this explicitly. Similarly, if $y = z = 0$ or if $x = z = 0$, then also $|A| = 0$. So in this case, there are no matrices for which a unique solution exists

(ii) One diagonal element is 1, the other two are 0

$$\begin{bmatrix} 1 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & x & y \\ x & 1 & z \\ y & z & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 1 \end{bmatrix}$$

In each case, one of x, y, z is 0, the other two are 1. In the first case, if $x = 0$, $y = z = 1$, then $|A| \neq 0$. If $y = 0$, $x = z = 1$, then $|A| \neq 0$. If $z = 0$, $x = y = 1$, then $|A| = 0$. Thus, there are 2 matrices from the first case for which $|A| \neq 0$. Similarly, each of the other two cases give rise to two matrices with non-zero determinants.

$\Rightarrow$ There are 6 matrices A for which $|A| \neq 0$ (for which the system has a unique solution).

(c) There are 6 cases where $|A| = 0$. Out of these, in 2 cases, namely,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix},$$

the number of solutions is infinite. The other 4 cases are inconsistent systems. ■

Determinants and Matrices

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. What does the following equation represent?

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0.$$

(A) A point (B) A straight line (C) A circle (D) A parabola (E) None of these

P2. If $\Delta = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$, the value of $\lim_{h\to 0}(\frac{\Delta}{h^2})$ is

(A) $6\sin x\cos^2 x$ (B) $6\cos x\sin^2 x$ (C) $9\sin x\cos^2 x$ (D) $9\cos x\sin^2 x$

P3. Suppose that $f(x)$ is a function satisfying the following conditions:

(a) $f(0) = 2, f(1) = 1$ (b) f has a minimum value at $x = \frac{5}{2}$, and (c) For all x,

$$f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$$

where a, b are some constants.

The value of $f(3)$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

P4. A 2×2 real matrix is given such that $AA^T = I$. What is the maximum value of the sum of the absolute values of entries of matrix A?

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) $2\sqrt{2}$ (D) None of these

P5. Let A, B be skew symmetric matrices. Which of the following are skew-symmetric?

(A) $A + B$ (B) AB (C) $AB + BA$ (D) $AB - BA$ (E) A^n

P6. Let $A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ such that $|A| = 0$. If a, b, c are distinct, then the sum of the coordinates of the fixed point through which the line $ax + by + c = 0$ passes is

(A) 2 (B) 3 (C) 4 (D) 5

P7. Let us define the power of a matrix A as the minimum $m \in \mathbb{Z}^+$ such that $A^m = I$. For two matrices A and B, if $A^5 = I$ and $ABA^{-1} = B^2$, the power of matrix B is between

(A) 20 and 24 (B) 28 and 32 (C) 36 and 40 (D) 44 and 48

P8. Let A be a $n \times n$ matrix with $|A| = 4$. B is the adjoint of the matrix $2A$ such that $|B| = 1024$. What is the value of n?

(A) 3 (B) 4 (C) 5 (D) More than 5

P9. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and define matrix C equal to $(AB\,A^T)^n$ where $n \in \mathbb{Z}^+$. The minimum value of the product of entries of C is

(A) $\frac{1}{n}$ (C) $\frac{2}{n}$ (E) None of these

(B) $\frac{\sqrt{2}}{n}$ (D) $\frac{1}{n\sqrt{2}}$

P10. Let A and B be two non-singular matrices such that $(AB)^k = A^k B^k$ for three consecutive positive integer values of k. Match the matrices in the first column to the corresponding matrices in the second column.

(A) ABA^{-1} (P) A^2
(B) BAB^{-1} (Q) B
(C) AB^2A^{-1} (R) A
(D) BA^2B^{-1} (S) B^2

P11. If M is a 3×3 matrix, where $M^T M = I$ and det $(M) = 1$, then the value of det $(M - I)$ is

(A) 1 (B) 2 (C) 4 (D) None of these

SUBJECTIVE TYPE EXAMPLES

P12. If $S_k = a^k + b^k + c^k$, evaluate $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix}$.

P13. Consider the following facts:

a cows graze b fields in c days

a_1 cows graze b_1 fields in c_1 days

a_1 cows graze b_2 fields in c_2 days

Assume that all the fields provide the same amount of grass, that the daily growth of the fields remains constant, and all the cows eat the same amount each day. Find the value of

$$\begin{vmatrix} b & cb & ca \\ b_1 & c_1 b_1 & c_1 a_1 \\ b_2 & c_2 b_2 & c_2 a_2 \end{vmatrix}$$

P14. Let $(l_i, m_i, n_i), i = 1, 2, 3$ be the direction cosines of 3 mutually perpendicular lines. Find the values of

(a) $\sum l_i^2, \sum m_i^2, \sum n_i^2$ **(b)** $\sum l_i m_i, \sum m_i n_i, \sum l_i n_i$

P15. For any value of θ, find the value of the following determinant:

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

P16. If $a, b, c, x, y, z \in R$, then prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}.$$

P17. If $f(x)$ is a polynomial of degree < 3, prove that

$$\begin{vmatrix} 1 & a & \frac{f(a)}{x-a} \\ 1 & b & \frac{f(b)}{x-b} \\ 1 & c & \frac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}.$$

P18. For what values of p and q does the system of equations

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

(a) have no solution?
(b) have a unique solution?
(c) have infinitely many solutions?

P19. For a matrix A, define the *eigen-vector* as a non-zero column matrix X and *eigen-value* a scalar λ such that $AX = \lambda X$.

(a) Prove that the eigen-values of a symmetric matrix are real.
(b) Prove that the eigen-values of a skew-symmetric matrix are 0 or purely imaginary.
(c) Find the absolute value(s) of the eigen-values of an orthogonal matrix.

P20. A matrix M is said to be orthogonal if $MM^T = M^T M = I$. Let S be a skew-symmetric matrix and I the identity matrix. If $I - S$ is invertible, can we say that the matrix $A = (I + S)\,(I - S)^{-1}$ is orthogonal?

P21. Let U_1, U_2, U_3 be column matrices satisfying

$$AU_1 = \lambda_1 U_1,\ AU_2 = \lambda_2 U_2,\ AU_3 = \lambda_3 U_3$$

where A is another matrix. Find A.

P22. Let A, B be square matrices of order n. Can $AB - BA$ be equal to the identity matrix?

P23. Let S be the set of invertible matrices satisfying the following properties.

(i) $A \cdot I = A$ for every $A \in S$

(ii) For every $A \in S,$ there exists $B \in S$ such that $A \cdot B = I$

Show that

(a) $B \cdot A = I$ (b) $I \cdot A = A$

P24. Let A be the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.
(a) Show that $A^2 - 2A + 2I = 0$.
(b) Hence evaluate the value of B given by

$$B = A^{10} - A^9 + 2A^8 - A^7 + 4A^6 - 2A^5 + 4A^4 + A^3 - A^2 + A + I.$$

P25. Let A and B be distinct 3 × 3 matrices such that $A^3 = B^3$ and $A^2 B = B^2 A$. Can we say that the matrix $A^2 + B^2$ is invertible?

P26. Let $A = [a_{ij}]$ be a square matrix. Define by the *conjugate* of A the matrix $\bar{A} = [\bar{a}_{ij}]$. A square matrix H is said to be *Hermitian* if $M = (\bar{M})^T$. Let A be a Hermitian matrix written in the form $A = P + iQ$ where P, Q are real matrices. Which of the following statements is true?

(a) P is symmetric.

(b) Q is skew-symmetric.

P27. Consider a 2 × 2 matrix $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Consider a polynomial $P(\lambda)$ generated by the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0.$$

(a) Show that $P(A) = 0$

(b) Define e^A through the following equality:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Show that e^A can be expressed as $aA + bI$, where a, b are constants.

(c) Determine e^A for

$$\text{(i) } A = \begin{bmatrix} x & 0 \\ x & -x \end{bmatrix}$$

$$\text{(ii) } \begin{bmatrix} 0 & x \\ -x & -2x \end{bmatrix}$$

P28. Let C be the matrix defined as

$$C = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & & & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{vmatrix}$$

Let $f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$. Let α_i, $i = 1$ to n be the n^{th} (complex) roots of unity. Find the value of C in terms of $f(\alpha_i), i = 1, 2, \ldots, n$.

Determinants and Matrices

PART-D: Solutions to Advanced Problems

S1. In this problem, we will only be providing the sequence of steps, and you can work out the details yourself.

Step 1: $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

Step 2: $R_1 \to R_1 + R_2 + R_3$, and use $a^2 + b^2 + c^2 = 1$

Step 3: $R_2 \to \frac{R_2}{b}, R_3 \to \frac{R_3}{c}$

Step 4: $R_2 \to R_2 - bR_1, R_3 \to R_3 - cR_1$

Step 5: $C_3 \to C_3 - xC_1 + yC_2$

Step 6: Expand along C_3 to obtain $(x^2 + y^2)(x + by + c) = 0$.

This is (apart from the point (0, 0)), the line $x + by + c = 0$. The correct options are (A) and (B).

S2. We apply the following transformations to Δ in sequence: $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, $C_2 \to \frac{C_2}{h}$ and $C_3 \to \frac{C_3}{h}$. Now using $\lim_{h\to 0}$ on the resulting determinant, we have

$$\lim_{h\to 0} \frac{\Delta}{h^2} = \begin{vmatrix} \sin x & \cos x & \cos x \\ \sin x & -2\cos x & \cos x \\ \sin x & \cos x & -2\cos x \end{vmatrix}$$

$$= \begin{vmatrix} 3\sin x & 0 & 0 \\ 3\sin x & 0 & -3\cos x \\ \sin x & \cos x & -2\cos x \end{vmatrix} \qquad \begin{matrix} R_1 \to R_1 + R_2 + R_3, \\ R_2 \to R_2 + 2R_3 \end{matrix}$$

$$= 9\sin x \cos^2 x$$

The correct option is (C).

S3. Applying the transformation $R_3 \to R_3 - R_1 - 2R_2$ on the given determinant reduces it to

$$f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 2ax + b$$

$$\Rightarrow \quad f(x) = ax^2 + bx + c$$

From the first two conditions, a, b, c, are easily determined, and we have $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$. Thus, $f(3) = \frac{1}{2}$. The correct option is (B).

S4. Let $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = I$$

$$\Rightarrow \; a^2 + b^2 = c^2 + d^2 = 1 \text{ and } ac + bd = 0.$$

Solving these tells us that A will be of a form similar to $\left[\begin{smallmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{smallmatrix}\right]$ (verify). The sum of absolute values, $2(|\sin\theta| + |\cos\theta|)$, has a maximum value of $2\sqrt{2}$. Proving this is straightforward, and left to the reader as an exercise. The correct option is (C).

S5. (A) $$(A+B)^T = A^T + B^T = -(A+B).$$

This matrix is skew-symmetric.

(B) $$(AB)^T = B^T A^T = (-B)\,(-A) = BA.$$

This matrix is symmetric.

(C) $$\begin{aligned}(AB+BA)^T &= (AB)^T + (BA)^T = B^T A^T + A^T B^T \\ &= (-B)(-A) + (-A)(-B) \\ &= BA + AB.\end{aligned}$$

This matrix is symmetric

(D) $$\begin{aligned}(AB-BA)^T &= (AB)^T - (BA)^T = B^T A^T - A^T B^T \\ &= BA - AB.\end{aligned}$$

This matrix is skew-symmetric

(E) $$(A^n)^T = (A^T)^n = (-A)^n = \begin{Bmatrix} A^n, \text{ if } n \text{ even} \\ -A^n, \text{ if } n \text{ odd} \end{Bmatrix}$$

Thus, A^n is skew-symmetric only if n is odd, and not in general.
We see that the correct options are (A) and (D).

S6. A upon simplification is $(a+b+c)\,(a-b)\,(b-c)\,(c-a)$. The only way this can equal *0* is if

$$a + b + c = 0 \Rightarrow a(1) + b(1) + c = 0$$

The fixed point is therefore (1, 1). The required sum is $1+1 = 2$. This means that the correct option is (A).

S7. We have $B^4 = (ABA^{-1})\,(ABA^{-1}) = AB^2A^{-1} = A(ABA^{-1})A^{-1} = A^2BA^{-2}$

$$\Rightarrow \; B^8 = A^3BA^{-3} \;\Rightarrow\; B^{16} = A^4BA^{-4} \;\Rightarrow\; B^{32} = A^5BA^{-5} = B \;\Rightarrow\; B^{31} = I$$

Thus, the power is 31. The correct option is (B).

S8. $B = adj(2A) \;\Rightarrow\; |B| = |2A|^{n-1}$ (why?)

$$\Rightarrow\; 1024 = 4^{n-1}\, 2^{n(n-1)} \quad \text{(again, why?)}$$

$$\Rightarrow\; 2^{10} = 2^{n^2+n-2}$$

$$\Rightarrow\; n = 3$$

The correct option is (A).

S9. We have $AA^T = I$, *i.e.*, $A^T = A^{-1}$. Thus,

$$C = (ABA^T)^n = \underbrace{(ABA^T)(ABA^T)\cdots(ABA^T)}_{n \text{ times}} = AB^n A^T$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 + n\sin\theta\cos\theta & n\cos^2\theta \\ -n\sin^2\theta & 1 - n\sin\theta\cos\theta \end{bmatrix}$$

$$\Rightarrow\; P(\text{product of entries}) = -n^2 \sin^2\theta\cos^2\theta(1 - n^2\sin^2\theta\cos^2\theta)$$

$$\Rightarrow\; P_{\min} = -\frac{1}{4} \text{ at } \sin 2\theta = \frac{\sqrt{2}}{n}$$

The correct option is (B).

S10. This is a very interesting problem on the commutative/non-commutative behaviour of matrices. Specifically, we always need to keep in mind that matrix multiplication is in general non-commutative.

Let $(AB)^k = A^k B^k$ hold true for $k = n, n+1, n+2$, where $n \in \mathbb{Z}$. Thus,

$$\left.\begin{array}{ll} \text{(a)} & (AB)^n = A^n B^n \\ \text{(b)} & (AB)^{n+1} = A^{n+1}B^{n+1} \end{array}\right\} \begin{array}{lll} \Rightarrow & (AB)(AB)^n = A^{n+1}B^{n+1} & \text{(using(a))} \\ \Rightarrow & (AB)A^nB^n = A^{n+1}B^{n+1} & \text{(left-cancellation of } A\text{;} \end{array}$$

$$\text{(c)} \quad (AB)^{n+2} = A^{n+2}B^{n+2} \quad \Rightarrow \quad BA^n = A^nB \qquad \text{right cancellation of } B^n\text{)}$$

We therefore have

$$BA^n = A^nB \tag{1}$$

Similarly, using (b) and (c), we will have

$$BA^{n+1} = A^{n+1}B \tag{2}$$

Thus,

$$\begin{aligned} BA^{n+1} = A^{n+1}B \;&\Rightarrow\; (BA^n)A = A^{n+1}B \\ &\Rightarrow\; A^nBA = A^{n+1}B \quad \text{(using(1))} \\ &\Rightarrow\; BA = AB \qquad \text{(left-cancellation of } A^n\text{)} \end{aligned}$$

We have managed to show that A and B are *commutative*. Now we proceed to the actual question(s):

(A) $ABA^{-1} = (AB)A^{-1} = (BA)A^{-1} = B(AA^{-1}) = B$

(B) $BAB^{-1} = (BA)B^{-1} = (AB)B^{-1} = A(BB^{-1}) = A$

(C) $AB^2A^{-1} = (AB)BA^{-1} = (BA)BA^{-1} = B(AB)A^{-1} = B(BA)A^{-1} = BB(AA^{-1}) = B^2$

(D) $BA^2B^{-1} = (BA)AB^{-1} = (AB)AB^{-1} = A(BA)B^{-1} = A(AB)B^{-1} = AA(BB^{-1}) = A^2$

The correct matching is (A) to (Q), (B) to (R), (C) to (S) and (D) to (P).

S11. We have

$$(M-I)^T = M^T - I$$

Multiplying by M, we obtain

$$(M-I)^T M = M^T M - M = I - M = -(M-I)$$

Taking the determinant on both sides, we have

$$|(M-I)^T M| = |-(M-I)|$$

$$\Rightarrow \quad |(M-I)^T|\,|M| = -|M-I|$$

$$\Rightarrow \quad |M-I| = -|M-I| \quad \Rightarrow \quad |M-I| = 0$$

None of the first three options is therefore correct, so (D) would be the correct response.

SUBJECTIVE TYPE EXAMPLES

S12. Split the determinant into a product:

$$\Rightarrow \quad \Delta = \begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \text{(How?)}$$

$$= \{(a-b)(b-c)(c-a)\}^2$$

S13. Let the initial quantity of grass per field be P, the daily growth per field be Q, and the grazing capacity of each cow be R per day. We thus have:

$$\begin{aligned} b(P+cQ) &= c\cdot aR \\ b_1(P+c_1Q) &= c_1\cdot a_1R \\ b_2(P+c_2Q) &= c_2\cdot a_2R \end{aligned} \quad \Rightarrow \quad \begin{vmatrix} b & bc & ca \\ b_1 & b_1c_1 & c_1a_1 \\ b_2 & b_2c_2 & c_2a_2 \end{vmatrix} = 0$$

S14. We let

$$A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

Since $l_i^2 + m_i^2 + n_i^2 = 1$ for $i = 1, 2, 3$, while

$$l_il_j + m_im_j + n_in_j = 0, \quad \text{where} \quad i, j \in \{1, 2, 3\}, i \neq j,$$

we conclude that

$$AA^T = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AA^T = I$, we have

$$(AA^T)^T = I^T = I \Rightarrow A^TA = I$$

Thus,

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sum l_i^2 & \sum l_im_i & \sum l_in_i \\ \sum l_im_i & \sum m_i^2 & \sum m_in_i \\ \sum l_in_i & \sum m_in_i & \sum n_i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} \sum l_i^2 = \sum m_i^2 = \sum n_i^2 = 1 \\ \sum l_im_i = \sum l_in_i = \sum m_in_i = 0 \end{array} \right\}$$

S15. We simply apply $R_1 \to R_1 + R_2 + R_3$ to obtain all elements in the top-row as 0, and thus $\Delta = 0$.

S16. This is an interesting problem on determinant row/column operations. The trick is to write the left determinant Δ_1 as a product:

$$\Delta_1 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix}$$

and now multiplying 'row-by-row', from which the right determinant Δ_2 is immediately obtained. We leave it up to the reader to figure out the details.

S17. We construct a new function $h(x)$ as follows:

$$h(x) = \begin{vmatrix} x-a & a(x-a) & f(a) \\ x-b & b(x-b) & f(b) \\ x-c & c(x-c) & f(c) \end{vmatrix} - f(x)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad (1)$$

Why did we construct this particular function? Well, the final result we need to prove has guided us, and the reader is urged to make the connection, especially after going through the entire solution.

Now, we note that $h(x)$ is again a polynomial of degree <3. The essence of the solution is in this very fact. To proceed, we evaluate $h(a)$:

$$\begin{aligned} h(a) &= \begin{vmatrix} 0 & 0 & f(a) \\ a-b & b(a-b) & f(b) \\ a-c & c(a-c) & f(c) \end{vmatrix} - f(a)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \\ &= \{c(a-b)(a-c) - b(a-b)(a-c)\}f(a) - f(a)(a-b)(b-c)(c-a) \\ &= (a-b)(b-c)(c-a)f(a) - (a-b)(b-c)(c-a)f(a) \\ &= 0 \end{aligned}$$

Similarly, $h(b) = h(c) = 0$. What can we infer from this, given that $h(x)$ is at the most a second degree polynomial? Well, since a second degree polynomial cannot have three zeroes, we must have the coefficients in $h(x)$ to be all zero, so that

$$h(x) = 0 \quad \forall \ x \in \mathbb{R}$$

Using (1) and slightly rearranging, we obtain the desired expression we were required to prove.

S18. We evaluate the appropriate determinants:

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (3-q)(2-p)$$

If $p \neq 2$ and $q \neq 3$, then the system definitely has a unique solution. If $p = 2$ or $q = 3$ or both, the system will have no solution or infinitely many solutions, depending on the values of the other determinants:

$$\Delta_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (15-4q)(2-p)$$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = 0 \quad \text{(regardless of the values of } p \text{ and } q\text{)}$$

$$\Delta_3 = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = p-2$$

The answers are:

(a) $p \neq 2, q = 3$

(b) $p \neq 2, q \neq 3$

(c) $p = 2, q$ can have any value.

S19. (a) Since A is symmetric, we have $A = A^T$, while $A = \bar{A}$ because A is real. Now,

$$AX = \lambda X \quad \Rightarrow \quad X^T A = \lambda X^T \quad \Rightarrow \quad \bar{X}^T A = \bar{\lambda}\bar{X}^T$$

$$\Rightarrow \quad \bar{X}^T(AX) = \bar{\lambda}\bar{X}^T X \quad \Rightarrow \quad \lambda(\bar{X}^T X) = \bar{\lambda}(\bar{X}^T X)$$

$\bar{X}^T X$ represents the *inner product* of X and $\bar{X}^T$, and must be non-zero for non-zero X (why?). Thus,

$$\lambda = \bar{\lambda} \quad \Rightarrow \quad \text{The eigen-values are real}$$

(b) In this case, $A^T = -A$:

$$AX = \lambda X \quad \Rightarrow \quad -X^T A = \lambda X^T \quad \Rightarrow \quad -\bar{X}^T A = \bar{\lambda}\bar{X}^T$$

$$\Rightarrow \quad -\bar{X}^T(AX) = \bar{\lambda}\bar{X}^T X \quad \Rightarrow \quad -\lambda(\bar{X}^T X) = \bar{\lambda}(\bar{X}^T X)$$

This means that $\lambda + \bar{\lambda} = 0$, *i.e.*,

$$\lambda = 0 \text{ or } ki \text{ for } k \in \mathbb{R} \quad \Rightarrow \quad \text{The eigen-values are 0 or purely imaginary.}$$

(c) If A is orthogonal, then $A^T A = AA^T = I$:

$$AX = \lambda X \quad \Rightarrow \quad X^T A^T = \lambda X^T \quad \Rightarrow \quad X^T A^T(AX) = \lambda X^T(\lambda X)$$

$$\Rightarrow \quad X^T(A^T A)X = \lambda^2(X^T X) \quad \Rightarrow \quad X^T X = \lambda^2(X^T X)$$

This gives $\lambda^2 = 1 \quad \Rightarrow \quad$ The eigen-values have absolute value 1.

S20. We have

$$A^T A = ((I+S)(I-S)^{-1})^T (I+S)(I-S)^{-1}$$

Since the transpose of a product is the product of the transposes in the opposite order, we have

$$A^T A = ((I-S)^{-1})^T (I+S)^T (I+S)(I-S)^{-1}$$
$$= ((I-S)^T)^{-1}(I+S^T)(I+S)(I-S)^{-1}$$
$$= (I+S)^{-1}(I-S)(I+S)(I-S)^{-1} \quad (1)$$

We observe that $I-S$ and $I+S$ must commute (why?). Thus, we switch the orders of the second and third factors in (1):

$$A^T A = (I+S)^{-1}(I+S)(I-S)(I-S)^{-1} \quad (2)$$

Finally, it is given that $I-S$ is invertible, and as a consequence, its transpose $(I-S)^T = I+S$ must also be invertible:

$$\left.\begin{aligned}(I+S)^{-1}(I+S) = I\\ (I-S)(I-S)^{-1} = I\end{aligned}\right\} \text{ Defined}$$

From (2), we have

$$A^T A = I$$

Similarly, $AA^T = I$ can be proved, *i.e.*, A is orthogonal.

S21. Let U be the matrix with columns as U_1, U_2, U_3. Then,

$$AU = A[U_1\ U_2\ U_3] = [\lambda_1 U_1 \quad \lambda_2 U_2 \quad \lambda_3 U_3]$$

$$= [U_1\ U_2\ U_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Denote the second matrix on the right hand side by D. Then, we have

$$AU = UD$$
$$\Rightarrow \quad A = UDU^{-1}$$

S22. The diagonal elements of AB and BA are the same, so $AB-BA$ will have its diagonal elements 0. Hence, it cannot be equal to the identity matrix.

S23. Let $A \in S$. There exists a $B \in S$ such that $A \cdot B = I$. Further, for B, there exists a $C \in S$ such that $B \cdot C = I$. Now,

$$B \cdot A = B \cdot A \cdot I \text{ (Property (i))}$$
$$= (B \cdot A)\cdot(B \cdot C) = B\cdot(A \cdot B)\cdot C = B \cdot I \cdot C = B \cdot C = I$$

Further,

$$A \cdot B = I \Rightarrow \quad A \cdot B \cdot A = I \cdot A$$

$$\Rightarrow \quad A \cdot (B \cdot A) = I \cdot A \quad \Rightarrow \quad A \cdot I = I \cdot A = A.$$

S24. Part (a) is trivial. To obtain B, consider

$$f(x) = x^{10} - x^9 + 2x^8 - x^7 + 4x^6 - 2x^5 + 4x^4 + x^3 - x^2 + x + 1$$

and divide $f(x)$ by $x^2 - 2x + 2$ to obtain a quotient and a remainder:

$$f(x) = (x^2 - 2x + 2)\,(x^8 + x^7 + 2x^6 + x^5 + 2x^4 + x + 1) + x - 1$$

Now, substitute $x = A$ to obtain

$$f(A) = B = A - I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

S25. No. We have $(A^2 + B^2)\,(A - B) = A^3 - A^2B + B^2A - B^3 = 0$. Since A and B are distinct, this implies that $A^2 + B^2$ is non-invertible, for if it were invertible, then assuming the inverse is C, we would have

$$C \cdot (A^2 + B^2)\,(A - B) = C \cdot 0 = 0$$

$$\Rightarrow \quad I \cdot (A - B) = 0 \quad \Rightarrow \quad A = B, \qquad \text{a contradiction}$$

S26. Since A is Hermitian, $A = (\overline{A})^T$:

$$\Rightarrow \quad P + iQ = (\overline{P + iQ})^T = (P - iQ)^T = P^T - iQ^T$$

$$\Rightarrow \quad P = P^T, \quad Q = -\,Q^T$$

Thus, both the statements are true.

S27. (a) We have $P(\lambda) = 0$:

$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\Rightarrow \quad P(A) = A^2 - (a + d)A + (ad - bc)I$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= 0$$

(b) The crucial step here is to note that since e^A is a polynomial, and A satisfies a polynomial equation of degree 2, we can write

$$e^A = P(A)Q(A) + R(A),$$

where the degree of $R(A)$ is less than 2, that is, $R(A)$ is of the form $aA + bI$. Since $P(A) = 0$, the given assertion is true.

(c) (i) $P(\lambda) = \lambda^2 - x^2$. Thus,

$$e^\lambda = (\lambda^2 - x^2)Q(\lambda) + a\lambda + b, \text{ i.e.,}$$

$$e^x = ax + b \text{ and } e^{-x} = -ax + b$$

$$\Rightarrow \quad a = \frac{e^x - e^{-x}}{2x} \quad \text{and} \quad b = \frac{e^x + e^{-x}}{2}$$

Hence,

$$e^A = aA + bI = \begin{bmatrix} e^x & 0 \\ \dfrac{e^x - e^{-x}}{2} & e^{-x} \end{bmatrix}$$

(ii) $P(\lambda) = \lambda^2 + 2\lambda x + x^2 = (\lambda + x)^2$. Thus,

$$e^\lambda = (\lambda + x)^2 Q(\lambda) + a\lambda + b \tag{1}$$

Letting $\lambda = -x$, we have $e^{-x} = -ax + b$. Differentiating (1), and letting $\lambda = -x$ again, we have

$$a = e^{-x}$$

$$\Rightarrow \quad b = xe^{-x} + e^{-x} = (x+1)e^{-x}$$

$$\Rightarrow \quad e^A = aA + bI = e^{-x}\begin{bmatrix} x+1 & x \\ -x & 1-x \end{bmatrix}$$

S28. The trick in this problem is very simple yet non-trivial. We construct a new determinant V given by

$$V = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{vmatrix}$$

Now, we evaluate the product $C \times V$:

$$CV = \begin{vmatrix} f(\alpha_1) & f(\alpha_2) & \dots & f(\alpha_n) \\ \alpha_1 f(\alpha_1) & \alpha_2 f(\alpha_2) & \dots & \alpha_n f(\alpha_n) \\ \vdots & & & \vdots \\ \alpha_1^{n-1} f(\alpha_1) & \alpha_2^{n-1} f(\alpha_2) & \dots & \alpha_n^{n-1} f(\alpha_n) \end{vmatrix}$$

$$= f(\alpha_1) f(\alpha_2) \dots f(\alpha_n) V$$

$$\Rightarrow \quad C = f(\alpha_1) f(\alpha_2) \dots f(\alpha_n)$$

Note how V is generated again in the product, so it cancels out. This artifice is what makes the problem very interesting!

S29. Let $|a_1| > |a_2| + |a_3|$, $|b_2| > |b_1| + |b_3|$, and $|c_3| > |c_1| + |c_2|$ and let

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

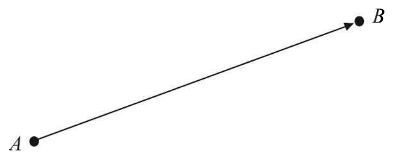
Can Δ be equal to 0?

Vectors

PART-A: Summary of Important Concepts

1. Introduction

A vector can be thought of as an arrow, in a 2-D plane or in 3-D space, with a starting (initial) point A and an ending (final) point B:



This vector will be represented by $\overrightarrow{AB}$, and we see that it has two associated attributes:

(a) a magnitude, represented by $\overrightarrow{AB}$
(b) a direction

In our study of vectors, we encounter two kinds of vectors: either fixed or free. As the name suggests, a fixed vector has its absolute position fixed with respect to any chosen coordinate system; a free vector is one which can be translated to any position in space, keeping its magnitude and direction fixed. For example, suppose that O is the origin and A is a fixed point in the coordinate system. Then the vector $\overrightarrow{OA}$ is fixed because its starting point, O is fixed. On the other hand, suppose a vector $\vec{a}$ corresponds to going 1 unit right and 2 units up in the coordinate system. Then $\vec{a}$ is free since it can be translated to anywhere in the coordinate system; it will still represent going 1 unit right and 2 units up.

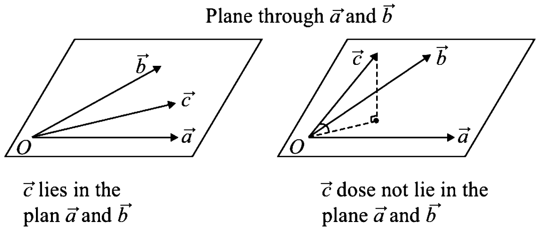
Below, we summarize some of the important terminologies associated with vectors:

(a) Unit vectors: Vectors with unity magnitude. If $|\vec{a}|=1$, it is generally written as $\hat{a}$.

(b) Collinear vectors: Parallel vectors (have the same direction)

(c) Equal vectors: Vectors with the same magnitude and direction

(d) Coplanar vectors: The concept of coplanarity is important, so this will be elaborated slightly. A system of *free* vectors is coplanar if they are parallel to the same plane. Note that defined this way, two free vectors will always be coplanar. This is because you can always bring these two vectors together to have the same initial point, and then a plane can always be drawn through the two vectors. On the other hand, three free vectors might or might not be coplanar; let us think of this more elaborately. Assume three free vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. Suppose you bring together $\vec{a}$ and $\vec{b}$ to have the same initial point O; you then draw the plane passing through $\vec{a}$ and $\vec{b}$. Now, when $\vec{c}$ is translated so that its initial point is O, it is not necessary for $\vec{c}$ also to lie in the plane that you drew through $\vec{a}$ and $\vec{b}$. Thus, $\vec{a}$, $\vec{b}$, and $\vec{c}$ might or might not be coplanar.

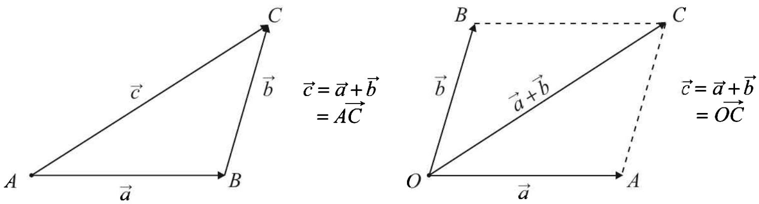


(e) Position vector of a point P: A *fixed* vector joining the origin of the reference frame to the point P.

2. Basic Vector Operations and Results

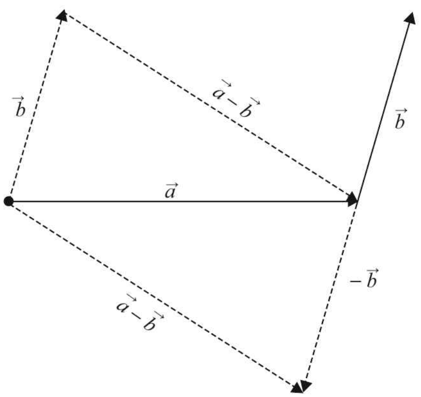
(a) Addition

Addition of vectors can be accomplished through the triangle law or the parallelogram law:



(b) Subtraction

The subtraction of vectors can be seen as an extension of the addition operation:



(i) Reveres $\vec{b}$ to obtain $-\vec{b}$

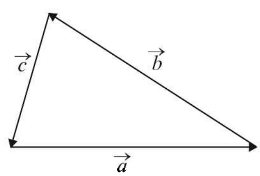
(ii) Add $\vec{a}$ and $(-\vec{b})$ to obtain $\vec{a}-\vec{b}$

OR

(i) Make $\vec{a}$ and $\vec{b}$ co-initial.

(ii) Join the tip of $\vec{b}$ to the tip of $\vec{a}$ to obtain $\vec{a}-\vec{b}$

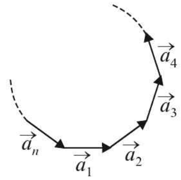
Note that from the triangle law, it follows that for three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ representing the sides of a triangle as shown,



we must have

$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$

In fact, for the vectors $\vec{a}_i$, $i=1,2,\ldots,n$, representing the sides of an n-sided polygon as shown,



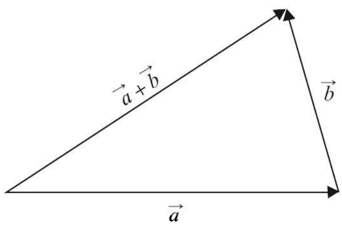
we must have

$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n = \vec{0}$$

since the *net effect* of all these vectors is to bring us back from where we started, and thus our net displacement is the zero vector.

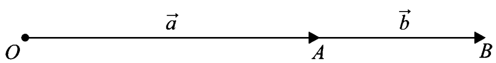
(c) Triangle Inequality

The triangle inequality satisfied by vectors is an algebraic expression of the geometric fact that in any triangle, the sum of two sides is greater than the third, while the difference of two sides is less than the third:



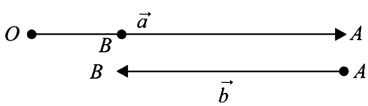
$$\underbrace{\left||\vec{a}|-|\vec{b}|\right|}_{\text{Difference of two sides}} \le \underbrace{\left|\vec{a}+\vec{b}\right|}_{\text{Third side}} \le \underbrace{\left||\vec{a}|+|\vec{b}|\right|}_{\text{Sum of two sides}}$$

The equality $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ holds when the two vectors $\vec{a}$ and $\vec{b}$ are parallel:



$$\begin{aligned} |\vec{a}+\vec{b}| &= \text{OB} \\ &= OA + AB \\ &= |\vec{a}|+|\vec{b}| \end{aligned}$$

On the other hand, the equality $|\vec{a}+\vec{b}| = ||\vec{a}|-|\vec{b}||$ holds when the two vectors $\vec{a}$ and $\vec{b}$ are anti-parallel:

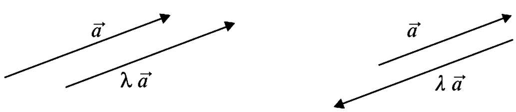


$$\begin{aligned} |\vec{a}+\vec{b}| &= |\overrightarrow{OA}+\overrightarrow{AB}| \\ &= |\overrightarrow{OB}| \\ &= |OA - AB| \\ &= ||\vec{a}|-|\vec{b}|| \end{aligned}$$

(d) Multiplication of a Vector with a Scalar

Suppose that a vector $\vec{a}$ is multiplied by a real number λ. The effect on $\vec{a}$ will be as follows:

- Its magnitude will become scaled by a factor equal to the magnitude of λ.
- If $\lambda > 0$, its direction will be the same; if $\lambda < 0$, its direction will get reversed.



$\lambda > 0$ $\qquad\qquad$ $\lambda < 0$

Based on these observations, we note the following:

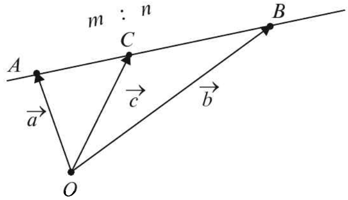
(a) Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\,\hat{a}$. Equivalently, the unit vector along any vector $\hat{a}$ can be written as $\hat{a} = \frac{|\vec{a}|}{|\vec{a}|}$.

(b) Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists some $\lambda \in \mathbb{R}$ such that $\vec{a} = \lambda\vec{b}$. Equivalently, we can say that if $\vec{a}$ and $\vec{b}$ are non-collinear vectors, and $\lambda\vec{a} + \mu\vec{b} = 0$ for some scalars λ and μ, then $\lambda = \mu = 0$.

(e) Section Formula

Let $A(\vec{a})$ and $B(\vec{b})$ be two fixed points. The position vector of the point C lying on the line AB which divides it internally in the ratio $m:n$ is

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$$



Similarly, the point $D(\vec{d})$ which divides AB externally in the ratio $m : n$ is given by $\vec{d} = \frac{m\vec{b}-n\vec{a}}{m-n}$.
The section formula has some important corollaries:

(i) The mid-point of $A(\vec{a})$ and $B(\vec{b})$ is $\frac{\vec{a}+\vec{b}}{2}$.

(ii) The centroid of a triangle with the vertices $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

(iii) In a tetrahedron with the vertices $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$, the lines joining the vertices to the centroids of the opposite faces are concurrent, with the point of concurrency being $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$.

3. Linear Combinations of Vectors

3.1 What are linear combinations of vectors?

The concept of linear combinations is one of the most important concepts in the study of vector algebra. In itself, the idea is extremely simple: if we have n arbitrary vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$, a linear combination of these n vectors is a vector $\vec{r}$ such that

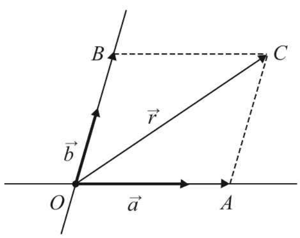
$$\vec{r} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \cdots + \lambda_n\vec{a}_n$$

where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ are arbitrary scalars (real numbers).

3.2 The basis of a vector space

Using the concept of linear combinations, we can talk about the **basis** of a vector space:

(a) In a 2-D plane, any vector $\vec{r}$ can be expressed as a linear combination of *any two* non-collinear vectors $\vec{a}$ and $\vec{b}$. For example:



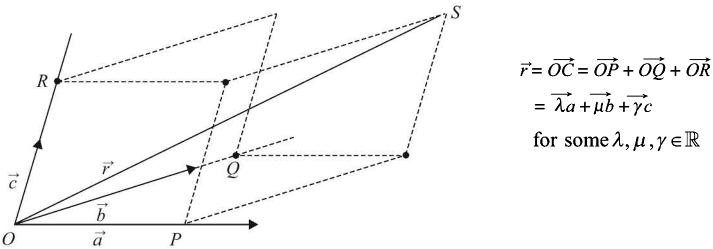
$$\begin{aligned} r^2 &= \overrightarrow{OC} \\ &= \overrightarrow{OA} - \overrightarrow{OB} \\ &= \overrightarrow{\lambda a} + \overrightarrow{\mu b} \end{aligned}$$

for some $\lambda, \mu \in \mathbb{R}$

We see that the *components* of $\vec{r}$ along $\vec{a}$ and $\vec{b}$ must be some scalar multiples of $\vec{a}$ and $\vec{b}$. In mathematical terms, we can say that the vectors $\vec{a}$ and $\vec{b}$ form a *basis* of our 2-D vector space.

The term 'basis' implies that using only the vectors $\vec{a}$ and $\vec{b}$, we can construct any vector lying in the plane of $\vec{a}$ and $\vec{b}$. We also note that if $\vec{a}$ and $\vec{b}$ were collinear, they could not have formed a basis for the 2-D vector space.

(b) In a 3-D space, any vector $\vec{r}$ can be expressed as a linear combination of *any three* non-coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. For example:



In this case, the triplet of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ forms a basis for our 3-D vector space. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ were coplanar (lying in some plane P), then any linear combination of these three vectors would lie in the same plane P, and we would be unable to express an arbitrary vector as a linear combination of these three vectors. This means that three coplanar vectors cannot form a basis for a 3-D space.

3.3 Linearly Independent/Dependent Vectors

We now discuss the extremely important concept of linearly independent and linearly dependent vectors.

Linearly Independent Vectors: A set of non-zero vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, ..., \vec{a}_n$ is said to be linearly independent if

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \cdots + \lambda_n\vec{a}_n = \vec{0} \quad \text{implies} \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Thus, a linear combination of linearly independent vectors cannot be zero unless all the scalars used to form the linear combination are zero.

Linearly Dependent Vectors: A set of non-zero vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, ..., \vec{a}_n$ is said to be linearly dependent if there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n$, not all zero, such that

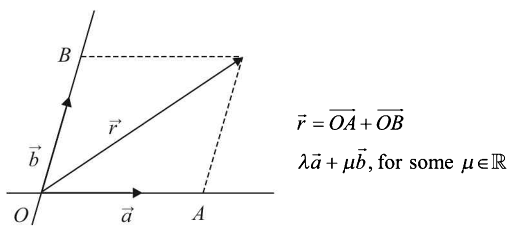
$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \cdots + \lambda_n\vec{a}_n = \vec{0}$$

We note that

(i) Two non-zero, non-collinear vectors are linearly independent.
(ii) Two collinear vectors are linearly dependent.
(iii) Three non-zero, non-coplanar vectors are linearly independent.
(iv) Three coplanar vectors are linearly dependent.
(v) Any four vectors in 3-D space are linearly dependent.

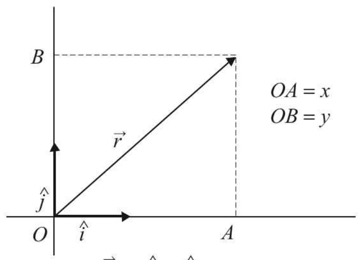
3.4 Resolution of Vectors

An arbitrary vector $\vec{r}$ (in a 2-D plane or a 3-D space, or even n-D space for that matter) can be resolved in terms of the vectors of any valid basis of that vector space, by finding the components of $\vec{r}$ along the vectors of the basis. For example, in the figure below, the vector $\vec{r}$ has been resolved along the basis formed by $\vec{a}$ and $\vec{b}$:



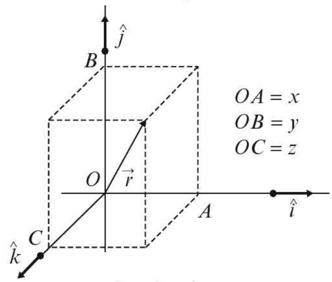
Frequently, we use rectangular resolution, where we take as our basis for the vector space a set of perpendicular unit vectors:

2-D plane



$\vec{r} = x\hat{i} + y\hat{j}$

3-D space

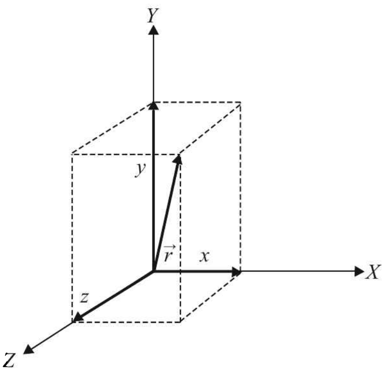


$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Rectangular resolution is very convenient to work with; the basis $(\hat{i}, \hat{j})$ for a 2-D vector space and $(\hat{i}, \hat{j}, \hat{k})$ for a 3-D vector space are by far the most widely used basis.

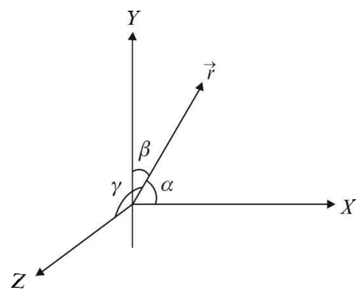
Using rectangular resolution, we can easily specify the magnitude and direction of any vector:

(a) Magnitude: Consider a vector $\vec{r}$ in 3-D space:



We have $\vec{r} = \sqrt{x^2 + y^2 + z^2}$.

(b) Direction: In a rectangular basis, the direction of a vector can be specified using direction cosines (or direction ratios) of a vector. Suppose $\vec{r}$ makes angles α, β, γ with the axes, as shown:



The quantities $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are called the **direction cosines** of $\vec{r}$.

We have:

- $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{|\vec{r}|} = l\hat{i} + m\hat{j} + n\hat{k}$
- $l^2 + m^2 + n^2 = 1$

Direction ratios of a vector $\vec{r}$ are simply any set of three numbers, say a, b and c, which are proportional to the direction cosines:

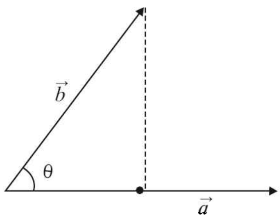
$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

We see that direction ratios of a vector are not unique, while the direction cosines are. It is easy to obtain the direction cosines from a set of direction ratios a, b, c:

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \qquad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \qquad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

4. Dot Product

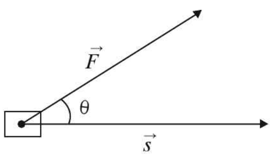
The dot product of two vectors $\vec{a}$ and $\vec{b}$, represented by $\vec{a}\cdot\vec{b}$, is a scalar quantity which measures the *effect* of one vector along the other:



$$P = \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

We see that P is the product of the modulus of either vector and the projection of the other in its direction, and is maximum when $\theta = 0$, and minimum when $\theta = \pi$.

The dot product arises in situations where the effect of one vector along another is to be measured. For example, for a force $\vec{F}$ moving an object through a displacement vector $\vec{s}$ at at angle θ to $\vec{F}$, we can define the scalar quantity *work* (W) using the dot product, as a measure of *how much of an effect* the force has had, or how much the force has worked in causing that movement:



$$W = \vec{F}\cdot\vec{s} = |\vec{F}||\vec{s}|\cos\theta$$

Listed below are some important properties of the dot product.

(i) The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

(ii) $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$, the equality holding only if $\theta = 0$ or π.

(iii) The projection of $\vec{a}$ on $\vec{b}$ is $p_{ab} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right) = \vec{a}\cdot\hat{b}$

(iv) The projection of $\vec{b}$ on $\vec{a}$ is $p_{ba} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \left(\frac{\vec{a}}{|\vec{a}|}\right)\cdot\vec{b} = \hat{a}\cdot\vec{b}$

(v) Scalar product is commutative, *i.e.*,

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

(vi) Scalar product is distributive, *i.e.*,

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} \quad \text{and} \quad (\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$$

(vi) The scalar product of two vectors is zero if and only if the two vectors are perpendicular. This also gives

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{i}=\hat{i}\cdot\hat{k}=\hat{k}\cdot\hat{i}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{j}=0$$

(vii) For any vector $\vec{a}$,

$$\boxed{\vec{a}\cdot\vec{a}=|\vec{a}|^2}$$

Thus,

$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

(viii) $|\vec{a}\pm\vec{b}|^2=(\vec{a}\pm\vec{b})\cdot(\vec{a}\pm\vec{b})=|\vec{a}|^2+|\vec{b}|^2\pm 2(\vec{a}\cdot\vec{b})$

$$(\vec{a}+\vec{b})(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2$$

(ix) This property is very important. If two vectors $\vec{a}$ and $\vec{b}$ have been specified in rectangular form, *i.e.*,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\ \text{ and }\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},$$

then we have,

$$\begin{aligned}\vec{a}\cdot\vec{b}&=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\&=a_1b_1+a_2b_2+a_3b_3\end{aligned}$$

The angle θ between the two vectors will be given by $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$:

$$\Rightarrow\quad\cos\theta=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

(x) The direction cosines l, m, n of a vector $\vec{a}$ will be given by

$$l=\hat{a}\cdot\hat{i},\quad m=\hat{a}\cdot\hat{j},\quad n=\hat{a}\cdot\hat{k}$$

(xi) Let $\vec{r}$ be a vector coplanar with the vectors $\vec{a}$ and $\vec{b}$. If $\vec{r}\cdot\vec{a}=0$ and $\vec{r}\cdot\vec{b}=0$, this would imply that $\vec{r}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. This can only happen if $\vec{a}$ and $\vec{b}$ are collinear.

Analogously, let $\vec{r}$ be an arbitrary vector and $\vec{a}, \vec{b}, \vec{c}$ be three vectors, such that:

$$\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$$

This means that $\vec{r}$ is perpendicular to each of $\vec{a}$, $\vec{b}$ and $\vec{c}$ which can only happen if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

(xii) Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors. We have already discussed that $\vec{a}, \vec{b}, \vec{c}$ can form a basis for 3-D space. Any vector $\vec{r}$ can be written in this basis as

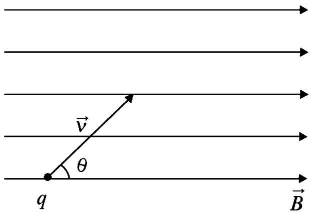
$$\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}+(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot\hat{c})\hat{c}=\left(\frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}+\left(\frac{\vec{r}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}+\left(\frac{\vec{r}\cdot\vec{c}}{|\vec{c}|^2}\right)\vec{c}$$

This representation is of significant importance and you must understand why it is correct.

5. Cross Product

To understand the significance of the cross product, it is best to keep the following real-world example in mind:

Consider a horizontal magnetic field, which we can represent by $\vec{B}$, and a charge q projected into this field with a velocity $\vec{v}$ (at an angle θ with the horizontal).



Experiments show that the force $\vec{F}$ acting on this particle.

(a) is perpendicular to the plane of $\vec{v}$ and $\vec{B}$ and goes into the plane for the figure above.
(b) increases with increase in $|\vec{v}|$ and $|\vec{B}|$.
(c) is such that its magnitude increases as θ goes from 0 to $\frac{\pi}{2}$. In fact, when $\vec{a}$ and $\vec{c}$ are parallel, the force on the particle is zero. For fixed magnitudes of $\vec{v}$ and $\vec{B}$, the force is the maximum when $\theta = \frac{\pi}{2}$.
(d) increases with increase in charge.

This suggests the dependence

$$|\vec{F}| \propto q\,|\vec{v}|\,|\vec{B}|\sin\theta,$$

which has been confirmed experimentally. In fact, the relation is (exactly)

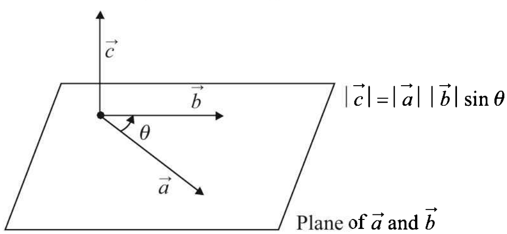
$$|\vec{F}| = q\,|\vec{v}|\,|\vec{B}|\sin\theta$$

The direction of $\vec{F}$ is found out to satisfy the right hand thumb rule. Holding out your thumb, use your right hand fingers to map out the rotation from $\vec{v}$ to $\vec{B}$. The direction of $\vec{F}$ is given by the direction in which the thumb points. Now, since $\vec{F}$ is a vector with direction perpendicular to both $\vec{v}$ and $\vec{B}$, we write the expression for $\vec{F}$ as

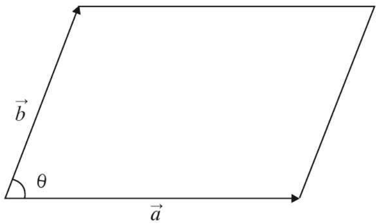
$$\boxed{\vec{F} = q(\vec{v}\times\vec{B})}$$

where the *vector* $\vec{v}\times\vec{B}$, the cross product of $\vec{v}$ and $\vec{B}$, is understood to be a vector, such that its magnitude is $|\vec{v}||\vec{B}|\sin\theta$ and its direction is given by the right hand thumb rule.

In general, the cross product of $\vec{a}$ and $\vec{b}$, *i.e.*, $\vec{c} = \vec{a}\times\vec{b}$ is a vector with magnitude $|\vec{a}|\,|\vec{b}|\sin\theta$, ($\theta$ being the angle between $\vec{a}$ and $\vec{b}$) and direction perpendicular to the plane of $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and this direction form a right handed system.



It is important to keep in mind that the cross product is a vector; the dot product was a scalar. The cross product is also referred to as the vector product. The cross product of $\vec{a}$ and $\vec{b}$, say $\vec{c}$, has an interesting geometrical interpretation. Since $|\vec{c}| = |\vec{a}|\,|\vec{b}|\sin\theta$, $|\vec{c}|$ represents the area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$:



Area of parallelogram $= |\vec{a}||\vec{b}| \sin\theta$

$= \vec{c}$

Where $\vec{c} = \vec{a} = \vec{b}$

In fact, the area of the parallelogram can itself be treated as a vector (as it is in physical phenomena):

$$\vec{A} = \vec{a} \times \vec{b}$$

The area of the triangle formed with $\vec{a}$ and $\vec{b}$ as two sides is simply $\frac{1}{2}|\vec{A}| = \frac{1}{2}|\vec{a} \times \vec{b}|$.

We now note some important properties of the cross product:

(i) If $\vec{a}$ and $\vec{b}$ are parallel, their cross product is zero, *i.e.*,

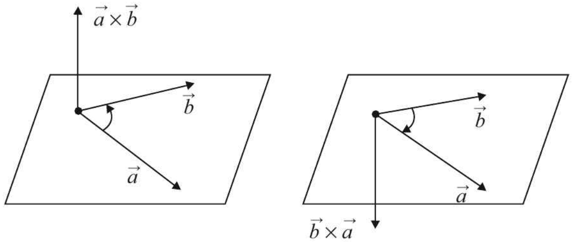
$$\vec{a} \times \vec{b} = \vec{0}$$

since $\sin\theta = 0$. Conversely, if $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a}$ and $\vec{b}$ must be parallel.

(ii) The cross product is *not commutative*. In fact,

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

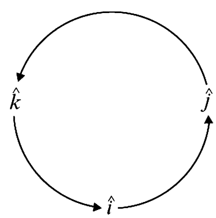
This is because the direction of $\vec{a} \times \vec{b}$ was defined so that $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right handed system:



(iii) The cross product is distributive over vector addition:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

and $$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$



(iv) $$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

These relations can be remembered using the adjacent figure. Going in the reverse direction, we have

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{i} \times \hat{k} = -\hat{j},\ \hat{k} \times \hat{j} = -\hat{i}$$

Thus, for two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, we have

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\\ &= a_1\, b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i}\\ &= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)\end{aligned}$$

This can be written concisely in determinant notation as

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix}$$

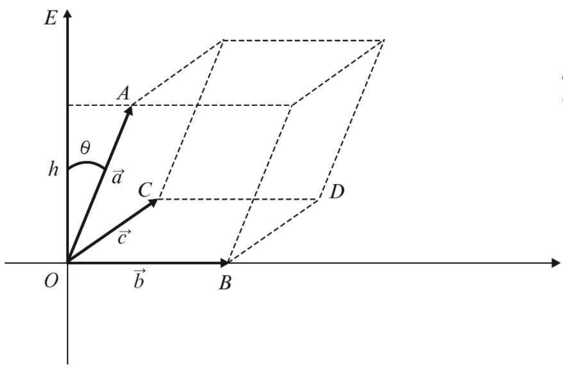
(v) The unit vector(s) $\hat{r}$ normal to the plane of $\vec{a}$ and $\vec{b}$ can be written as

$$\hat{r} = \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

6. Scalar Triple Product

This is one of the most important concepts of ths chapter, and will be discussed in some detail. As the name suggests, a scalar triple product involves the (scalar) product of three vectors. How may such a product be defined? Consider three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. Consider the quantity $\vec{a}\cdot(\vec{b}\cdot\vec{c})$. Since $\vec{b}\cdot\vec{c}$ is a scalar, you cannot define its dot product with another vector. Thus, $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is a meaningless quantity. However, consider the expression $\vec{a}\cdot(\vec{b}\times\vec{c})$. Since, $\vec{b}\times\vec{c}$ is a vector, its dot product with $\vec{a}$ is defined. Thus, $\vec{a}\cdot(\vec{b}\times\vec{c})$ is defined and is termed the scalar triple product of $\vec{a}, \vec{b}$,and $\vec{c}$. This product is represented concisely as $[\vec{a}\ \vec{b}\ \vec{c}]$. Let us try to assign a geometrical interpretation to the scalar triple product (STP) $[\vec{a}\ \vec{b}\ \vec{c}]$.

First of all, make $\vec{a}, \vec{b}, \vec{c}$ co-initial. Assume for a moment that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Complete the parallelopiped with $\vec{a}, \vec{b}, \vec{c}$ as adjacent edges:



h represents the height of the parallelopiped

Consider $\vec{b}\times\vec{c}$. This is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. We have represented it by $\overrightarrow{OE}$. Let the angle between $\vec{a}$ and $\overrightarrow{OE}$ be θ. What can $\vec{a}\cdot(\vec{b}\times\vec{c})$, *i.e.*, $\vec{a}\cdot\overrightarrow{OE}$ represent? $|\overrightarrow{OE}|$ represents the area of the parallelogram *OBDC*. Thus,

$$\vec{a}\cdot\overrightarrow{OE} = |\vec{a}|\,|\overrightarrow{OE}|\cos\theta = (|\vec{a}|\cos\theta)\,\overrightarrow{OE}$$

= (Height of the parallelopiped h) × (Area of the base parallelopiped)
= Volume of the parallelopiped.

The STP $[\vec{a}\ \vec{b}\ \vec{c}]$ therefore represents the volume of the parallelopiped with $\vec{a}, \vec{b}, \vec{c}$ as adjacent edges.

Note that the volume V of the parallelopiped could equally well have been specified as

$$V = \vec{b}\cdot(\vec{c}\times\vec{a}) = [\vec{b}\ \vec{c}\ \vec{a}] = \vec{c}\cdot(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{a}\ \vec{b}]$$

Thus, we come to an important property of the STP:

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

That is, if the vectors are *cyclically* permuted, the value of the STP remains the same. However, note that

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -[\vec{a}\ \vec{c}\ \vec{b}]$$

Let us see some more significant properties of the STP:

(i) The STP of three vectors is zero if any two of them are parallel. This implies as a corollary that $[\vec{a}\ \vec{a}\ \vec{b}] = 0$ (always).

(ii) For any $\lambda \in \mathbb{R}$,

$$[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$$

(iii) $\boxed{[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]}$

This property is very important and is used extremely frequently. The justification is straight forward:

$$\begin{aligned}[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] &= (\vec{a}+\vec{b})\cdot(\vec{c}\times\vec{d})\\ &= \vec{a}\cdot(\vec{c}\times\vec{d}) + \vec{b}\cdot(\vec{c}\times\vec{d})\ \{\text{Dot product is distributive over vector addition}\}\\ &= [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]\end{aligned}$$

(iv) Three vector are coplanar if and only if their STP is zero. This is because the volume of the parallelopiped formed by the three vectors becomes zero if they are coplanar. You are urged to rigorously prove the other way implication, *i.e.*, prove that if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ where $\vec{a}\ \vec{b}\ \vec{c}$ are non-zero non-collinear vectors, then $\vec{a}, \vec{b}, \vec{c}$ must be coplanar.

(v) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

This is widely used to evaluate the STP of vectors.

7. Vector Triple Product

The vector product $\vec{v}$ of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as

$$\vec{v} = \vec{a} \times (\vec{b} \times \vec{c})$$

The most important property that the vector triple product satisfies is this:

$$\boxed{\underset{\text{I}}{\vec{v} = \vec{a}} \times (\underset{\text{II}}{\vec{b}} \times \underset{\text{III}}{\vec{c}}) = \underset{(\text{I}\cdot\text{III})}{(\vec{a}\cdot\vec{c})}\, \underset{\text{II}}{\vec{b}} - \underset{(\text{I}\cdot\text{II})}{(\vec{a}\cdot\vec{b})}\underset{\text{III}}{\vec{c}}} \tag{1}$$

Let us see how this property is justified. Since $\vec{v}$ is perpendicular to $(\vec{b} \times \vec{c})$ (and $\vec{a}$ too), $\vec{v}$ must lie in the plane containing $\vec{b}$ and $\vec{c}$ (convince yourself about this). We may assume $\vec{b}$ and $\vec{c}$ to be non-collinear (since, if they are collinear, (1) automatically holds because both sides will be zero). Thus, $\vec{b}$ and $\vec{c}$ form a basis of the plane in which they lie, which means that

$$\vec{v} = \lambda\vec{b} + \mu\vec{c} \quad \text{for some } \lambda, \mu \in \mathbb{R} \tag{2}$$

Now, since $\vec{v}$ is perpendicular to $\vec{a}$ too, we have

$$\vec{v}\cdot\vec{a} = 0$$

$$\Rightarrow \quad (\lambda\vec{b} + \mu\vec{c})\cdot\vec{a} = 0 \quad \Rightarrow \quad \lambda(\vec{a}\cdot\vec{b}) + \mu(\vec{a}\cdot\vec{c}) = 0$$

$$\Rightarrow \quad \frac{\lambda}{(\vec{a}\cdot\vec{c})} = \frac{\mu}{-(\vec{a}\cdot\vec{b})} = l(\text{say}) \quad \Rightarrow \quad \lambda = l(\vec{a}\cdot\vec{c}),\ \mu = -l(\vec{a}\cdot\vec{b})$$

Using the values of λ and μ in (2), we have

$$\vec{v} = l\{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\} \tag{3}$$

The only task that now remains is to find out the value of l. This can be done by taking particular values of $\vec{a}, \vec{b}$ and $\vec{c}$ ((3) must hold for all arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$). Let $\vec{a} = \hat{i}, \vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{k}$. Thus,

$$\vec{v} = \vec{a} \times (\vec{b} \times \vec{c}) = \hat{i} \times \{(\hat{i} + \hat{j}) \times \hat{k}\} = \hat{i} \times \{-\hat{j} + \hat{i}\} = -\hat{k}$$

$$\text{and} \quad l\{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\} = l\{0 - \hat{k}\} = -l\hat{k}$$

This gives $l = 1$. Thus, we see that the relation in (1) holds.

It should be obvious that the vector triple product is a non-associative operation. For example, $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane of $\vec{b}$ and $\vec{c}$ while $(\vec{a} \times \vec{b}) \times \vec{c} = (-\vec{c}) \times (\vec{a} \times \vec{b})$ lies in the plane of $\vec{a}$ and $\vec{b}$. In general, therefore,

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Some important properties of the vector triple product are summarized below:

(a) $[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}] = [\vec{a} \; \vec{b} \; \vec{c}]^2$

(b) $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

(c) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a} \; \vec{b} \; \vec{d}]\vec{c} - [\vec{a} \; \vec{b} \; \vec{c}]\vec{d}$

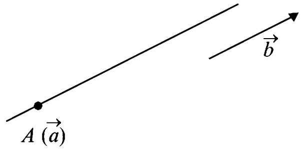
(d) $\vec{a}\times(\vec{b}\times(\vec{c}\times\vec{d})) = (\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d})$

(e) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

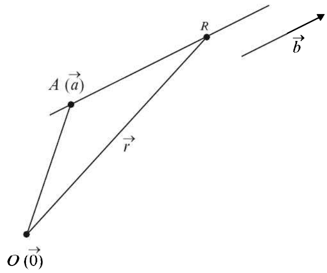
8. Geometry with Vectors

8.1 Lines in Vector Form

Consider a straight line passing through the point $A(\vec{a})$ and parallel to the vector $\vec{b}$, as shown:



Any point $\vec{r}$ on this line can be written in terms of a real parameter λ.



$$\vec{r} = \overrightarrow{OA} + \overrightarrow{AR}$$
$$= \vec{a} + \lambda\vec{b} \text{ where } \lambda \in \mathbb{R}$$

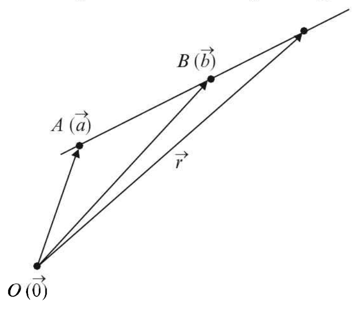
The equation

$$\boxed{\vec{r} = \vec{a} + \lambda\vec{b}}$$

can be viewed as the (vector) equation of this line. As we vary λ, we get varying position vectors $\vec{r}$ and hence varying points on this line. This form of the equation of a line is called the *parametric form* since it involves the use of a parameter λ. We could also have specified the equation in *non-parametric form*. Observe that since $\overrightarrow{AR}$ is parallel to $\vec{b}$, we have

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$
$$\Rightarrow \quad \boxed{\vec{r} \times \vec{b} = \vec{a} \times \vec{b}}$$

This is the required equation of the line. You must convince yourself that this equation is valid; in particular, understand that only points lying on the line and none other will satisfy this equation. We can use these equations to obtain the equation of a line passing through the points $A(\vec{a})$ and $B(\vec{b})$:



$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})} \qquad \text{Parametric form}$$

OR

$$(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = \vec{0}$$

$$\boxed{\vec{r} \times (\vec{b} - \vec{a}) = \vec{a} \times \vec{b}} \qquad \text{Non-parametric form}$$

Note the following important results:

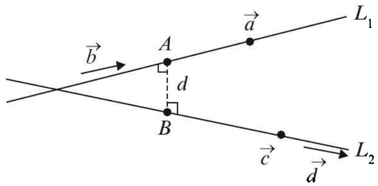
(a) The vector equations of the bisectors of the angles between the straight lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are

$$\vec{r} = \vec{a} + \lambda\left\{\frac{\vec{b}}{|\vec{b}|} \pm \frac{\vec{c}}{|\vec{c}|}\right\}, \lambda \in \mathbb{R}$$

(b) The perpendicular distance of the point $A(\vec{a})$ from the line $\vec{r} = \vec{b} + \lambda\vec{c}$ is given by

$$d = |(\vec{a} - \vec{b}) \times \frac{\vec{c}}{|\vec{c}|}| = |(\vec{a} - \vec{b}) \times \hat{c}|$$

(c) Two straight lines in space are called *skew-lines*, if they are neither parallel nor intersecting:



$L_1 : \vec{r} = \vec{a} + \lambda\vec{b}$

$L_2 : \vec{r} = \vec{c} + \mu\vec{d}$

The shortest distance between L_1 and L_2 is given by

$$d = \left|\frac{(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

8.2 Planes in vector form

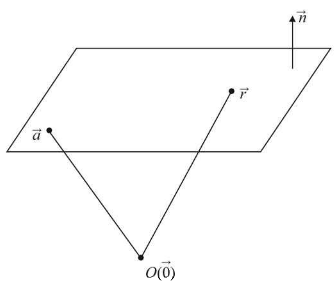
Consider an arbitrary plane. How do you think that the equation of this plane can be specified? We need

(a) either a point on the plane and the orientation of the plane (the orientation of the plane can be specified by the orientation of the normal of the plane).

(b) or a point on the plane and *two* vectors coplanar with the plane.

Depending on whether we have the information as in (a) or as in (b), we have two different forms for the equation of the plane.

(a) Let the plane be such that it passes through the point $\vec{a}$ and $\vec{n}$ is a vector perpendicular to the plane:



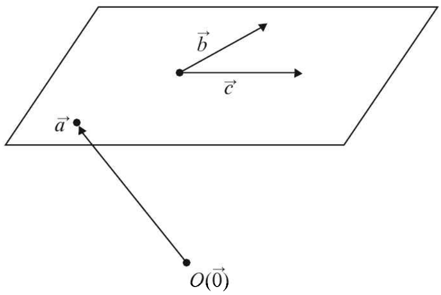
It is evident that for any point $\vec{r}$ lying on the plane, the vectors $(\vec{r} - \vec{a})$ and $\vec{n}$ are perpendicular. Thus,

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \boxed{\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}}$$

This is the required equation of the plane. Convince yourself that all (and only) points lying on the plane will satisfy this equation.

(b) Let the plane be such that it passes through the point $\vec{a}$ and is parallel to the vectors $\vec{b}$ and $\vec{c}$ (in other words, is coplanar with vectors $\vec{b}$ and $\vec{c}$). It is assumed that $\vec{b}$ and $\vec{c}$ are non-collinear:



$\vec{b}$ and $\vec{c}$ lie in the plane whose equation we wish to determine

Since $\vec{b}$ and $\vec{c}$ are non-collinear, any vector in the plane of $\vec{b}$ and $\vec{c}$ can be written as

$$\lambda\vec{b} + \mu\vec{c}, \quad \text{where } \lambda, \mu \in \mathbb{R}$$

Thus, any point lying in the plane can be written in the form

$$\boxed{\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}} \text{ for some } \lambda, \mu \in \mathbb{R}$$

This is the equation of the plane in parametric form. As we vary λ and μ we get different points lying in the plane.

To specify the equation of the plane in non-parametric form, note that for any point $\vec{r}$ in the plane, $(\vec{r} - \vec{a})$ lies in the plane of $\vec{b}$ and $\vec{c}$. Thus, $(\vec{r} - \vec{a})$ is perpendicular to $\vec{b} \times \vec{c}$:

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

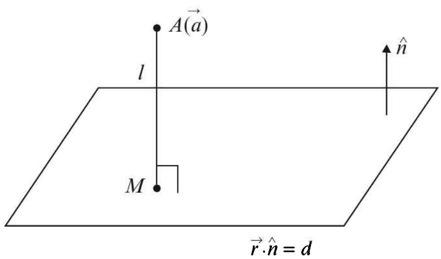
$$\Rightarrow \quad \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow \quad \boxed{[\vec{r} \;\; \vec{b} \;\; \vec{c}] = [\vec{a} \;\; \vec{b} \;\; \vec{c}]}$$

Convince yourself that all (and only) points $\vec{r}$ lying on the plane will satisfy this relation.

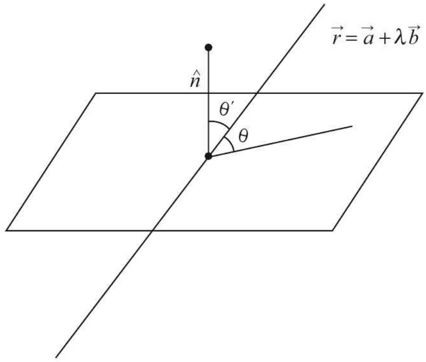
Note the following important results:

(a) The length of the perpendicular dropped from a point $A(\vec{a})$ onto the plane $\vec{r} \cdot \hat{n} = d$ is given by



$$l = |\vec{a} \cdot \hat{n} - d|$$

(b) The angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \hat{n}$ is the angle θ (the complementary angle of θ') as shown in the figure below, and is given by $\theta = \sin^{-1}(\hat{b} \cdot \hat{n})$.



(c) The angle between two planes $\vec{r}\cdot\hat{n}_1 = d_1$ and $\vec{r}\cdot\hat{n}_2 = d_2$ (which is the same as the angle between their normals) is $\theta = \cos^{-1}(\hat{n}_1\cdot\hat{n}_2)$.

(d) The equation of the plane passing through the line of intersection of the planes

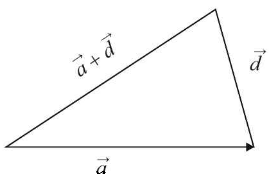
$$P_1 \equiv \vec{r}_1\cdot\vec{n}_1 - d_1 = 0$$

$$P_2 \equiv \vec{r}\cdot\vec{n}_2 - d_2 = 0.$$

can be written as $P_1 + \lambda P_2 = 0$.

IMPORTANT IDEAS AND TIPS

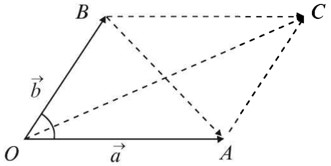
1. *Vector Addition/Subtraction*: The essence of vectors lies in the way they add (or subtract) through the triangle or parallelogram law. You must always think of vector addition and subtraction as geometrical rather than algebraic operations. When you see an expression of the form $\vec{a}+\vec{b}$, rather than thinking of an algebraic sum, what you should actually be thinking about is the following picture:

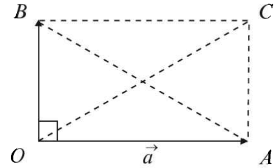


This will help a lot; for example, when you think of the addition and subtraction of vectors geometrically, the triangle equality is straightforward to interpret:

$$\underbrace{\left||\vec{a}|-|\vec{b}|\right|}_{\text{Difference of two sides}} \le \underbrace{|\vec{a}+\vec{b}|}_{\text{Third side}} \le \underbrace{|\vec{a}|+|\vec{b}|}_{\text{Sum of two sides}}$$

In addition, this way of thinking will enable you to quickly interpret many vector expressions. Consider the vector equation $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$. What does this tell us about $\vec{a}$ and $\vec{b}$? If we picture $\vec{a}$ and $\vec{b}$ forming parallelogram as shown below, we see that $|\vec{a}+\vec{b}|$ is OC, while $|\vec{a}-\vec{b}|$ is BA. OC can equal BA only if $\vec{a}$ and $\vec{b}$ are perpendicular. Thus, one look at the expression and we can say that the two vectors are perpendicular. This is possible only through a geometric way of thinking.





2. *Vectors as Free Entities:* It is important to remember that vectors generally behave as *free entities* (though in the case of position vectors, they are fixed). You can keep moving around a vector in the plane—as long as its magnitude and direction is preserved, it will remain the same vector. This fact is especially useful in situations like the one below, where we have a hexagon, and in terms of just two of the sides, we can express all the other sides:

   (i) $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$ (given)

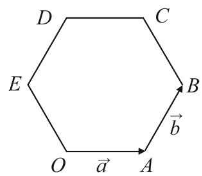
   (ii) $\overrightarrow{OC} = 2\vec{b} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

   $\Rightarrow \overrightarrow{BC} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$

   (iii) $\overrightarrow{CD} = -\vec{a}$

   (iv) $\overrightarrow{DE} = -\vec{b}$

   (v) $\overrightarrow{EO} = -\overrightarrow{BC} = \vec{a} - \vec{b}$



3. *Linear Combinations of Vectors:* Without a doubt, the most important concept in vectors is linear combinations of vectors and the basis of a vector space. You must be absolutely comfortable with the terminology used in discussing these concepts. In particular, internalizing the following facts will help you avoid a lot of pitfalls:
   (a) Two collinear vectors cannot form the basis of a 2-D vector space; two non-collinear vectors can.
   (b) Three coplanar vectors cannot form the basis of a 3-D vector space; three non-collinear vectors can.
   (c) Collinear/coplanar vectors are linearly dependent. Any three vectors in 2-D space are linearly dependent; any four vectors in 3-D space are linearly dependent.
4. *Direction Cosines and Ratios:* Keep in mind the difference between the direction cosines (DCs) and direction ratios (DRs) of a vector:
   (a) DCs are unique and represented conventionally by *l*, *m*, *n*, such that

   $$l^2 + m^2 + n^2 = 1$$

   (b) DRs are any three numbers proportional to the DCs. They are not unique.

   Many students ask the question: If DCs are sufficient to specify the direction of a vector, then why do we even use DRs? The answer is straightforward. Suppose that the DCs of a vector $\vec{r}$ are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$. If I were to tell you the direction of $\vec{r}$, I could say that the direction is given by (1, 2, 3) instead of saying that the direction is given by $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$. The former is easier to say, and using the former (the non-unique DRs), I can anyway deduce the latter (the unique DCs). I could even have said that the direction is given by (2, 4, 6) or (200, 400, 600) etc. That is all there is to the concept of DRs, a set of numbers which help us easily specify the direction of a vector.
5. *Mistakes in Dot and Cross Products:* Frequently, students overlook the fact that for two vectors $\vec{a}$ and $\vec{b}$ the dot product $\vec{a}\cdot\vec{b}$ is a scalar quantity, a real number, whereas the cross product $\vec{a}\times\vec{b}$ is a vector quantity. Therefore, expressions like $(\vec{a}\cdot\vec{b})\times(\vec{a}\cdot\vec{b})$ and $(\vec{a}\times\vec{b})^2$ do not make sense, while expressions like $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$ and $(\vec{a}\cdot\vec{b})^2$ are valid. Being able to identify such incorrect expressions is crucial when handling large vector relations.

6. *Non-Commutativity of Cross Product:* A common mistake is to forget the fact that the cross product is non-commutative. In fact, we have $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$; the two vectors are negatives of each other.

7. *Interpreting Zero and Non-Zero STP*. If the scalar triple product (STP) of three non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 0, *i.e.*, $[\vec{a}, \vec{b}, \vec{c}] = 0$, you should always associate this with the fact that $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. In fact, the STP gives the volume of the parallelopiped formed by the three vectors-if that volume is 0, it means that no parallelopiped is formed, which happens if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. On the other hand, if the STP is non-zero, it means that a non-zero volume is formed by the three vectors-they must be non-coplanar in that case.

8. *Evaluating Vector Triple Product*. The order in which a vector triple product (VTP) is carried out is sometimes mixed up. The correct relation is:

$$\boxed{\vec{v} = \underset{\mathrm{I}}{\vec{a}}\times(\underset{\mathrm{II}}{\vec{b}}\times\underset{\mathrm{III}}{\vec{c}}) = \underset{(\mathrm{I}\cdot\mathrm{III})\mathrm{II}}{(\vec{a}\cdot\vec{c})\vec{b}} - \underset{(\mathrm{I}\cdot\mathrm{II})\mathrm{III}}{(\vec{a}\cdot\vec{b})\vec{c}}}$$

One very important fact about the VTP $\vec{a}\times(\vec{b}\times\vec{c})$ is that the resulting vector will lie in the plane of $\vec{b}$ and $\vec{c}$. This is evident from the relation above (the resulting vector is a linear combination of $\vec{b}$ and $\vec{c}$). It should also be obvious geometrically: $\vec{b}\times\vec{c}$ will be perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. Therefore, $\vec{a}\times(\vec{b}\times\vec{c})$ being perpendicular to $\vec{b}\times\vec{c}$ will lie in the plane of $\vec{b}$ and $\vec{c}$. Note that the VTP will not (in general) be 0 if the three vectors are coplanar.

9. *Importance of Visualization:* Visualizing in three dimensions is important when working with vector algebra of 3-D lines, planes, tetrahedrons etc. Many students find it difficult initially to understand the concept of skew lines, which are neither parallel nor perpendicular. Similarly, the concept of specifying a line through the intersection of two planes may seem confusing at first. But once you start taking cues from your surroundings to encourage your visualization process, vector algebra will become much more simple. As a simple example, if you are sitting in a room right now, look at the various edges of this room, and try to identify some pairs of skew lines.

Vectors

PART-B: Illustrative Examples

Example 1

Suppose that for three non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, any two of them are non-collinear. If the vectors $(\vec{a}+2\vec{b})$ and $\vec{c}$ are collinear and the vectors $(\vec{b}+3\vec{c})$ and $\vec{a}$ are collinear, then which of the following relations is true?

(A) $\vec{a}+\vec{b}+2\vec{c}=\vec{0}$ (B) $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$ (C) $\vec{a}+3\vec{b}+4\vec{c}=\vec{0}$ (D) $\vec{a}+2\vec{b}+6\vec{c}=\vec{0}$

(E) None of these

Solution: We must have some $\lambda, \mu \in \mathbb{R}$ such that

$$\vec{a}+2\vec{b}=\lambda\vec{c} \qquad (1)$$

$$\vec{b}+3\vec{c}=\mu\vec{a} \qquad (2)$$

From (1), we have

$$\vec{c}=\frac{1}{\lambda}(\vec{a}+2\vec{b}) \qquad (3)$$

We use this in (2):

$$\vec{b}+\frac{3}{\lambda}(\vec{a}+2\vec{b})=\mu\vec{a} \quad \Rightarrow \quad \left(\frac{3}{\lambda}-\mu\right)\vec{a}+\left(1+\frac{6}{\lambda}\right)\vec{b}=\vec{0}.$$

Since $\vec{a}$ and $\vec{b}$ are non-collinear, their linear combination can be zero if and only if the two scalars are zero. This gives:

$$\frac{3}{\lambda}-\mu=0 \quad \text{and} \quad 1+\frac{6}{\lambda}=0$$

$$\Rightarrow \quad \lambda=-6,\ \mu=-\frac{1}{2}.$$

Using the value of λ in (3), we have

$$\vec{a}+2\vec{b}+6\vec{c}=0.$$

The correct option is (D). ■

Example 2

It is known that in a ΔABC with centroid G, circumcentre O and orthocentre H,

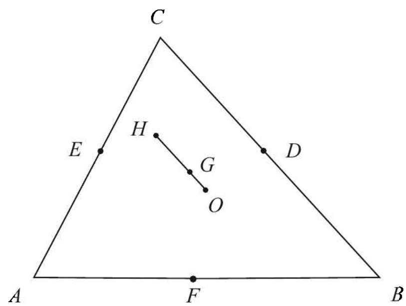
$$OG : GH = 1 : 2$$

Let P be any point in the plane of ΔABC. Which of the following assertions is true?

(A) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$ (C) $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = 3\overrightarrow{PG}$

(B) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HO}$ (D) None of these

Solution: All three assertions in (A), (B), and (C) are true. Consider the following diagram:



Let D, E and F be the mid-point of BC, CA and AB respectively

We will prove all three assertions one-by-one:

(A) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + (\overrightarrow{OB} + \overrightarrow{OC})$

$= \overrightarrow{OA} + 2\overrightarrow{OD}$ (Since D is BC's mid-point)

$= 3\overrightarrow{OG}$ (Since G lies on AD and divides it in the ratio 2:1)

$= \overrightarrow{OH}$ (Since O, G and H are collinear and $OH = 3OG$)

(B) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA} + (\overrightarrow{HB} + \overrightarrow{HC})$

$= \overrightarrow{HA} + 2\overrightarrow{HD}$

$= 3\overrightarrow{HG}$ (Same logic as above)

$= 3 \times \dfrac{2}{3}\overrightarrow{HO}$ (again, same as above)

$= 2\overrightarrow{HO}$

(C) For any arbitrary point P in the plane of ΔABC, we have

$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{PA} + (\overrightarrow{PB} + \overrightarrow{PC})$

$= \overrightarrow{PA} + 2\overrightarrow{PD}$

$= 3\overrightarrow{PG}$ ■

Example 3

If any point O inside or outside a tetrahedron $ABCD$ is joined to the vertices and AO, BO, CO, DO are produced so as to cut the planes of the opposite faces in P, Q, R, S respectively, the value of $\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}$ is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution: Assume O to be the origin, and the position vectors of A, B, C, D to be $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively. Since $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are non-coplanar vectors, we must have scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, such that

$$\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+\lambda_4\vec{d}=\vec{0}. \tag{1}$$

Since AO is produced to meet the plane of the opposite face in P, $\overrightarrow{AO}$ and $\overrightarrow{OP}$ must be collinear vectors. Thus,

$$\overrightarrow{AO}=\mu\overrightarrow{OP} \quad \text{for some } \mu\in\mathbb{R}$$

$$\Rightarrow \quad -\vec{a}=\mu\overrightarrow{OP}=\mu\vec{p} \quad (\vec{p} \text{ is the position vector of } P)$$

This when used in (1) gives

$$(-\mu\lambda_1)\vec{p}+\lambda_2\vec{b}+\lambda_3\vec{c}+\lambda_4\vec{d}=\vec{0}.$$

However, since B, C, D and P will be coplanar, we have

$$-\mu\lambda_1+\lambda_2+\lambda_3+\lambda_4=0$$

$$\Rightarrow \quad \mu=\frac{\lambda_1}{\lambda_2+\lambda_3+\lambda_4}$$

$$\Rightarrow \quad \frac{OP}{AP}=\frac{|\overrightarrow{OP}|}{|\overrightarrow{AP}|}=\frac{\mu}{1+\mu}=\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}=\frac{\lambda_1}{\Sigma\lambda_i}.$$

We similarly have,

$$\frac{OQ}{BQ}=\frac{\lambda_2}{\Sigma\lambda_i},\frac{OR}{CR}=\frac{\lambda_3}{\Sigma\lambda_i},\frac{OS}{DS}=\frac{\lambda_4}{\Sigma\lambda_i}$$

$$\Rightarrow \quad \frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1.$$

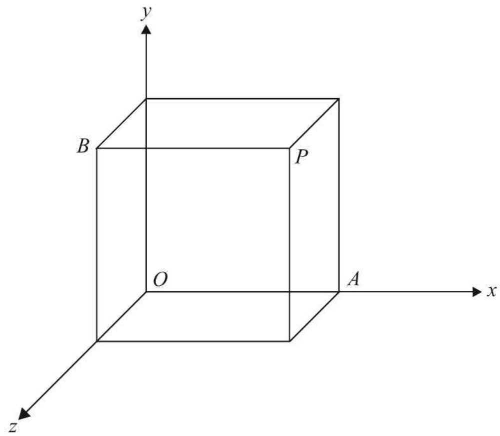
The correct option is (B). ■

Example 4

The angle between the two diagonals of a cube is

(A) $\cos^{-1}\frac{1}{3}$ (B) 60° (C) 90° (D) $\cos^{-1}\frac{2}{3}$ (E) None of these

Solutions:



Let us find the angle between the diagonals OP and AB (not shown). Note that the position vectors of A, B and P are respectively

$$A \equiv a\hat{i}$$
$$B \equiv a\hat{j} + a\hat{k}$$
$$P \equiv a\hat{i} + a\hat{j} + a\hat{k}$$

where a is the side of the square.

We now have

$$\overrightarrow{OP} \equiv a\hat{i} + a\hat{j} + a\hat{k} \Rightarrow \left|\overrightarrow{OP}\right| = \sqrt{3}a$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -a\hat{i} + a\hat{j} + a\hat{k} \ \Rightarrow \ \left|\overrightarrow{AB}\right| = \sqrt{3}a$$

Let θ denote the angle between OP and AB. Thus,

$$\cos\theta = \frac{\overrightarrow{OP}\cdot\overrightarrow{AB}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{AB}\right|} = \frac{(a\hat{i} + a\hat{j} + a\hat{k})\cdot(-a\hat{i} + a\hat{j} + a\hat{k})}{(\sqrt{3}a) \qquad\qquad (\sqrt{3}\,a)}$$

$$= \frac{-a^2 + a^2 + a^2}{3a^2} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\frac{1}{3}$$

This is the angle between any two diagonals of (any) cube. The correct option is (A). ■

Example 5

Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector such that $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$ and $\vec{r}\cdot\vec{a} = 0$. The magnitude of $x + y + z$ is:

(A) 3 (B) 5 (C) 7 (D) 9

Solution: We have

$$\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$$

$$\Rightarrow \quad (\vec{r} - \vec{c})\times\vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{r} - \vec{c} \text{ is parallel to } \vec{b}$$

Thus,

$$\vec{r}-\vec{c}=\lambda\vec{b} \text{ for some } \lambda\in\mathbb{R}$$

$$\Rightarrow \quad \vec{r}=\vec{c}+\lambda\vec{b} \qquad (1)$$

Now, we are given that $\vec{r}\cdot\vec{a}=0$:

$$\Rightarrow \quad (\vec{c}+\lambda\vec{b})\cdot\vec{a}=0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{c}+\lambda\vec{a}\cdot\vec{b}=0$$

$$\Rightarrow \quad \lambda=-\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}=-\frac{8+7}{2+0+1}=-5$$

Using (1), we can now determine $\vec{r}$:

$$\vec{r}=\vec{c}+\lambda\vec{b}=\vec{c}-5\vec{b}=(4\hat{i}-3\hat{j}+7\hat{k})-5(\hat{i}+\hat{j}+\hat{k})=-\hat{i}-8\hat{j}+2\hat{k}.$$

We note that $x+y+z=-7$, and so the required magnitude is 7. The correct option is (C). ■

Example 6

If the vectors

$$\vec{\alpha}=a\hat{i}+\hat{j}+\hat{k}, \quad a\neq 1$$

$$\vec{\beta}=\hat{i}+b\hat{j}+\hat{k}, \quad b\neq 1$$

$$\vec{\gamma}=\hat{i}+\hat{j}+c\hat{k}, \quad c\neq 1$$

are coplanar, the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

Solution: The coplanarity of the three vectors means that their STP must be zero:

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0$$

$$\Rightarrow \quad a(bc-1)+(1-c)+(1-b)=0$$

$$\Rightarrow \quad a+b+c=abc+2. \qquad (1)$$

Now we have

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$$

$$=\frac{(1-b)(1-c)+(1-a)(1-c)+(1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$=\frac{3-2(a+b+c)+(ab+bc+ac)}{1-(a+b+c)+(ab+bc+ac)-abc}$$

$$= \frac{3-2(a+b+c)+(ab+bc+ac)}{1-(a+b+c)+(ab+bc+ac)+2-(a+b+c)} \quad \{\text{Using (1) for } abc\}$$

$$= 1$$

The correct option is (D). ■

Example 7

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, the value of $\frac{|\vec{a}|^2|\vec{b}|^2}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$ is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: Since $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, $\vec{c}$ must be parallel to $\vec{a}\times\vec{b}$, *i.e.*, the angle between $\vec{c}$ and $(\vec{a}\times\vec{b})$ must be 0. Thus,

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{c}\ \vec{a}\ \vec{b}] = \vec{c}\cdot(\vec{a}\times\vec{b}) = |\vec{c}|\,|\vec{a}\times\vec{b}|\cos 0$$

$$= 1\cdot|\vec{a}|\,|\vec{b}|\sin\frac{\pi}{6}\cdot 1 = \frac{1}{2}|\vec{a}|\,|\vec{b}|$$

$$\Rightarrow\ [\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{2}|\vec{a}|\,|\vec{b}| \qquad (1)$$

Squaring both sides of (1) and slightly rearranging the resulting expression gives us the required value as 4. ■

Example 8

For arbitrary vectors $\vec{a},\vec{b},\vec{c},\vec{d}$, which of the following relations is true?

(A) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

(B) $\vec{a}\times(\vec{b}\times(\vec{c}\times\vec{d})) = (\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d})$

(C) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

(D) None of these

Solution: All three relations in (A), (B) and (C) are true. We will prove them one by one:

(A) This relation is simply obtained by expanding the left hand side:

$$(\underset{\mathrm{I}}{\vec{a}\times\vec{b}})\times(\underset{\mathrm{II}}{\vec{c}}\times\underset{\mathrm{III}}{\vec{d}}) = ((\vec{a}\times\vec{b})\cdot\vec{d})\vec{c} - ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{d} = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

(B) $\vec{a}\times(\underset{\mathrm{I}}{\vec{b}}\times(\underset{\mathrm{II}}{\vec{c}}\times\underset{\mathrm{III}}{\vec{d}})) = \vec{a}\times((\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}) = (\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d})$

(C) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot(\vec{b}\times(\vec{c}\times\vec{d})) = \vec{a}\cdot((\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

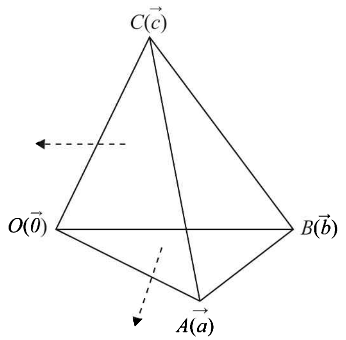
■

Example 9

The angle between any two faces of a regular tetrahedron is

(A) 45° (B) $\cos^{-1}\frac{1}{3}$ (C) 60° (D) $\cos^{-1}\frac{2}{3}$ (E) None of these

Solution: The angle between any two faces will obviously equal the angle between the normals to the two faces. Let us find the (acute) angle θ between the normals to the faces *OAC* and *OAB*.



$$\hat{n}_{OAC} = \frac{\vec{c}\times\vec{a}}{|\vec{c}\times\vec{a}|} = \frac{\vec{c}\times\vec{a}}{|\vec{c}||\vec{a}|\sin\pi/3}$$

$$\hat{n}_{OAB} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|\sin\pi/3}$$

$$\Rightarrow \hat{n}_{OAC}\cdot\hat{n}_{OAB} = \frac{(\vec{c}\times\vec{a})\cdot(\vec{a}\times\vec{b})}{|\vec{a}|^2|\vec{b}|^2\sin^2\pi/3} \qquad (\because\ |\vec{a}|=|\vec{b}|=|\vec{c}|)$$

$$= \frac{(\vec{c}\cdot\vec{a})(\vec{a}\cdot\vec{b})-(\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{a})}{|\vec{a}|^4\sin^2\pi/3} = \frac{\dfrac{|\vec{a}|^2|\vec{b}||\vec{c}|}{4}-\dfrac{|\vec{a}|^2|\vec{b}||\vec{c}|}{2}}{|\vec{a}|^4\sin^2\pi/3} = \frac{-\frac{1}{4}|\vec{a}|^4}{|\vec{a}|^4\cdot\frac{3}{4}} = \frac{-1}{3}$$

$$\Rightarrow \cos\theta = \frac{1}{3} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

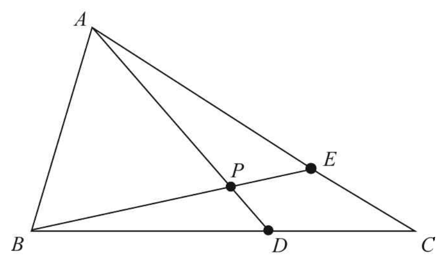
Thus, the angle between any two faces equals $\cos^{-1}(\frac{1}{3})$. The correct option is (B). ■

Example 10

In ΔABC, D and E are points on BC and AC respectively such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. The ratio $BP : PE$ has the value

(A) 5 : 2 (B) 7 : 3 (C) 8 : 3 (D) 9 : 4 (E) None of these

Solution: Consider the following diagram:



There's no loss of generality in assuming A to be the origin $\vec{0}$, and B and C to be the points $\vec{b}$ and $\vec{c}$ respectively. By the section formula,

$$D \equiv \frac{b+2\vec{c}}{3}, \qquad E \equiv \frac{3\vec{c}}{4}$$

The equation of AD can be written (in parametric form) as:

$$\vec{r} = \vec{0} + \lambda\left(\frac{\vec{b}+2\vec{c}}{3}\right),\ \lambda \in \mathbb{R} \quad \Rightarrow \quad \vec{r} = \left(\frac{\lambda}{3}\right)\vec{b} + \left(\frac{2\lambda}{3}\right)\vec{c}.$$

Similarly, the equation of BE can be written as

$$\vec{r} = \vec{b} + \lambda'\left(\frac{3\vec{c}}{4} - \vec{b}\right),\ \lambda' \in \mathbb{R} \quad \Rightarrow \quad \vec{r} = (1-\lambda')\vec{b} + \left(\frac{3\lambda'}{4}\right)\vec{c}.$$

AD and BE intersect at P. Thus, the position vector of P must satisfy the equations of both AD and BE. This means that we must have

$$\left(\frac{\lambda}{3}\right)\vec{b} + \left(\frac{2\lambda}{3}\right)\vec{c} = (1-\lambda')\vec{b} + \left(\frac{3\lambda'}{4}\right)\vec{c} \text{ for some } \lambda, \lambda' \in \mathbb{R}$$

$$\Rightarrow \quad \left(\frac{\lambda}{3} + \lambda' - 1\right)\vec{b} + \left(\frac{2\lambda}{3} - \frac{3\lambda'}{4}\right)\vec{c} = \vec{0}$$

Since $\vec{b}$ and $\vec{c}$ are non-collinear, we must have

$$\frac{\lambda}{3} + \lambda' - 1 = 0 \quad \text{and} \quad \frac{2\lambda}{3} - \frac{3\lambda'}{4} = 0.$$

This system upon solving yields

$$\lambda = \frac{9}{11}, \quad \lambda' = \frac{8}{11}$$

Thus, the position vector of P can be obtained by substituting the value of $\lambda(\text{or } \lambda')$ in the equations for AD (or BE):

$$P \equiv \frac{3}{11}(\vec{b} + 2\vec{c})$$

We now know the position vectors of B, P and E. We simply need to find $BP : PE$. Suppose this is $m : 1$. Then,

$$\frac{m\left(\frac{3\vec{c}}{4}\right) + 1(\vec{b})}{m+1} = \frac{3}{11}(\vec{b} + 2\vec{c})$$

$$\Rightarrow \quad 33m\vec{c} + 44\vec{b} = 12(m+1)\vec{b} + 24(m+1)\vec{c}$$

$$\Rightarrow \quad (32-12m)\vec{b} + (9m-24)\vec{c} = \vec{0} \quad \Rightarrow \quad m = \frac{8}{3}$$

Thus,

$$BP : PE = 8 : 3$$

The correct option is (C). ■

Example 11

Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is

(A) $3\hat{i} + \hat{j} - \hat{k}$ (B) $2\hat{i} + \hat{j} - 2\hat{k}$ (C) $\hat{i} + 2\hat{j} - 3\hat{k}$ (D) $2\hat{i} + 3\hat{j} - \hat{k}$

Solution: Both the lines have been specified in non-parametric form, which we can easily convert to parametric form:

$$\vec{r} \times \vec{a} = \vec{b} \times \vec{a} \quad \Rightarrow \quad \vec{r} = \vec{b} + \lambda \vec{a} \qquad \text{where } \lambda \in \mathbb{R}$$

$$\vec{r} \times \vec{b} = \vec{a} \times \vec{b} \quad \Rightarrow \quad \vec{r} = \vec{a} + \lambda' \vec{b} \qquad \text{where } \lambda' \in \mathbb{R}$$

If these two lines intersect, then we must have some values of λ, λ', say λ_0 and λ_0', such that

$$\vec{b} + \lambda_0 \vec{a} = \vec{a} + \lambda_0' \vec{b}$$

$$\Rightarrow \quad (1 - \lambda_0')\vec{b} + (\lambda_0 - 1)\vec{a} = \vec{0}$$

Since $\vec{a}$ and $\vec{b}$ are non-collinear, we must have $\lambda_0 = \lambda_0' = 1$. The position vector of the point of intersection P can now be evaluated by substituting λ_0 or λ_0' in the corresponding equation:

$$P \equiv \vec{b} + \vec{a} = \vec{a} + \vec{b}$$

$$\Rightarrow \quad P \equiv 3\hat{i} + \hat{j} - \hat{k}$$

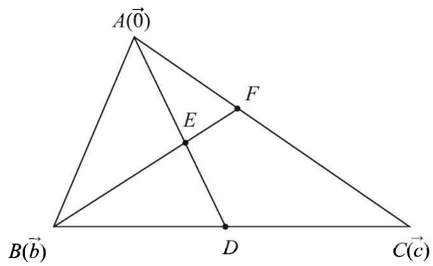
The correct option is (A). ■

Example 12

The median AD of a triangle ABC is bisected at E and BE is produced to meet AC in F. The value of $\frac{BF}{EF}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Consider the following diagram:



Assume A to be the origin and B, C to be the points $\vec{b}$, $\vec{c}$ respectively. We have,

$$D \equiv \frac{\vec{b} + \vec{c}}{2} \quad \Rightarrow \quad E \equiv \frac{\vec{b} + \vec{c}}{4}$$

$$\Rightarrow \quad \text{Equation of } BE\text{: } \vec{r} = \vec{b} + \lambda\left(\frac{\vec{b}+\vec{c}}{4} - \vec{b}\right)$$

$$\Rightarrow \quad \vec{r} = \left(1 - \frac{3\lambda}{4}\right)\vec{b} + \frac{\lambda}{4}\vec{c}$$

Also,

$$\text{Equation of } AC\text{:} \qquad \vec{r} = \lambda'\vec{c}$$

Since AC and BE intersect in F, there must be some $\lambda, \lambda' \in \mathbb{R}$ such that

$$\left(1 - \frac{3\lambda}{4}\right)\vec{b} + \frac{\lambda}{4}\vec{c} = \lambda'\vec{c} \quad \Rightarrow \quad \lambda = \frac{4}{3}, \lambda' = \frac{1}{3} \quad \Rightarrow \quad F \equiv \frac{\vec{c}}{3}$$

If E divides BF in the ratio $k : 1$, we have

$$\frac{k(\frac{\vec{c}}{3}) + 1(\vec{b})}{k+1} = \frac{\vec{b}+\vec{c}}{4} \quad \Rightarrow \quad 4k\vec{c} + 12\vec{b} = 3(k+1)\vec{b} + 3(k+1)\vec{c}$$

$$\Rightarrow \quad k = 3$$

Thus,

$$BE : EF = 3 : 1 \quad \Rightarrow \quad EF : BF = 1 : 4$$

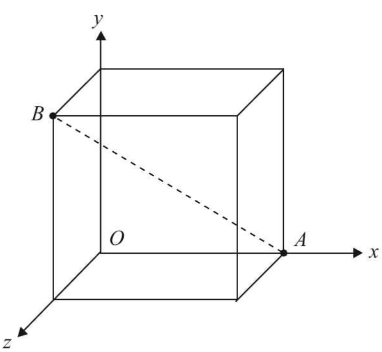
The required answer is 4. ■

Example 13

The perpendicular distance of any corner of a cube of side a from a diagonal not passing through it is

(A) $\frac{2}{3}a$ (B) $\frac{2}{\sqrt{3}}a$ (C) $\frac{\sqrt{2}}{3}a$ (D) $\sqrt{\frac{2}{3}}\,a$ (E) None of these

Solution: Let us take the cube in the following configuration:



The position vectors of A and B are $a\hat{i}$ and $a(\hat{j} + \hat{k})$ respectively

Let us find the perpendicular distance d of O from AB. This will be given by

$$d = \frac{|\overrightarrow{OA} \times \overrightarrow{OB}|}{|\overrightarrow{OA} - \overrightarrow{OB}|} = \frac{|a\hat{i} \times a(\hat{j}+\hat{k})|}{|a\hat{i} - a(\hat{j}-\hat{k})|} = \frac{a\,|\hat{k} - \hat{j}|}{|\hat{i} - \hat{j} - \hat{k}|} = a\sqrt{\frac{2}{3}}$$

The correct option is (D). ■

Example 14

Suppose that $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar unit vectors equally inclined to one another at an angle θ such that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}.$$

(a) Find p, q, r in terms of θ.

(b) The value of $p^2 + \dfrac{q^2}{\cos\theta} + r^2$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: (a) Note that

$$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = \vec{c} \cdot \vec{c} = 1; \quad \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = \cos\theta.$$

We evaluate the (square of) the STP of these three vectors:

$$\left[\vec{a} \;\; \vec{b} \;\; \vec{c}\right]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= 1 - 3\cos^2\theta + 2\cos^3\theta = (1-\cos\theta)^2(1+2\cos\theta)$$

$$\Rightarrow \quad [\vec{a} \;\; \vec{b} \;\; \vec{c}] = (1-\cos\theta)\sqrt{1+2\cos\theta}.$$

The given relation is

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}.$$

Taking the dot product on both sides successively with $\vec{a}, \vec{b}, \vec{c}$, we get the following system of equations:

$$p + q\cos\theta + r\cos\theta = [\vec{a} \;\; \vec{b} \;\; \vec{c}]$$
$$p\cos\theta + q + r\cos\theta = 0$$
$$p\cos\theta + q\cos\theta + r = [\vec{a} \;\; \vec{b} \;\; \vec{c}]$$

Using Cramer's rule (or otherwise, by elimination), p, q, r can be evaluated. For example,

$$p = \frac{\begin{vmatrix} [\vec{a} \;\; \vec{b} \;\; \vec{c}] & \cos\theta & \cos\theta \\ 0 & 1 & \cos\theta \\ [\vec{a} \;\; \vec{b} \;\; \vec{c}] & \cos\theta & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}} = \frac{[\vec{a} \;\; \vec{b} \;\; \vec{c}](1-\cos\theta)}{[\vec{a} \;\; \vec{b} \;\; \vec{c}]^2} = \frac{1}{\sqrt{1+2\cos\theta}}$$

Similarly, $q = \dfrac{-2\cos\theta}{\sqrt{1+2\cos\theta}}$, $\quad r = \dfrac{1}{\sqrt{1+2\cos\theta}}$.

(b) Using the values above, we have $p^2 + \dfrac{q^2}{\cos\theta} + r^2 = 2$.

The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

Example 15

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar vectors. Are the vectors $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ coplanar or non-coplanar?

Solution: Three vectors are coplanar if there exist scalars $\lambda, \mu \in \mathbb{R}$ using which one vector can be expressed as the linear combination of the other two. Let us try to find such scalars:

$$2\vec{a}-\vec{b}+3\vec{c} = \lambda(\vec{a}+\vec{b}-2\vec{c}) + \mu(\vec{a}+\vec{b}-3\vec{c})$$

$$\Rightarrow \quad (2-\lambda-\mu)\vec{a} + (-1-\lambda-\mu)\vec{b} + (3+2\lambda+3\mu)\vec{c} = \vec{0}.$$

Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, we must have

$$2-\lambda-\mu = 0$$

$$-1-\lambda-\mu = 0$$

$$3+2\lambda+3\mu = 0$$

This system, as can be easily verified, does not have a solution for λ and μ. Thus, we cannot find scalars for which one vector can be expressed as the linear combination of the other two, implying the three vectors must be non-coplanar.

As an additional exercise, show that for three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, the vectors $\vec{a}-2\vec{b}+3\vec{c}$, $\vec{a}-3\vec{b}+5\vec{c}$ and $-2\vec{a}-3\vec{b}-4\vec{c}$ are coplanar. ■

Example 16

Show that the vectors $\hat{i}-3\hat{j}+2\hat{k}$, $2\hat{i}-4\hat{j}-4\hat{i}$ and $3\hat{i}+2\hat{j}-\hat{k}$ are linearly independent.

Solution: Let $\lambda, \mu, \gamma \in \mathbb{R}$ be scalars such that

$$\lambda(\hat{i}-3\hat{j}+2\hat{k}) + \mu(2\hat{i}-4\hat{j}-4\hat{k}) + \gamma(3\hat{i}+2\hat{j}-\hat{k}) = \vec{0}$$

$$\Rightarrow \quad (\lambda+2\mu+3\gamma)\hat{i} + (-3\lambda-4\mu+2\gamma)\hat{j} + (2\lambda-4\mu-\gamma)\hat{k} = \vec{0}$$

$$\Rightarrow \quad \left.\begin{aligned} \lambda+2\mu+3\gamma &= 0 \\ -3\lambda-4\mu+2\gamma &= 0 \\ 2\lambda-4\mu-\gamma &= 0 \end{aligned}\right\} \qquad (1)$$

The determined of the coefficient matrix is

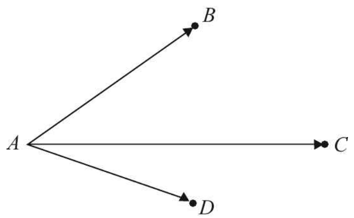
$$\begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -4 & -1 \end{vmatrix} \neq 0.$$

Thus, the system of equations in (1) has no solution for λ, μ and γ apart from the trivial solution $\lambda = \mu = \gamma = 0$. This implies that the three vectors are linearly independent. ■

Example 17

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors. Prove that the points $A(2\vec{a}+3\vec{b}-\vec{c})$, $B(\vec{a}-2\vec{b}+3\vec{c})$, $C(3\vec{a}+4\vec{b}-2\vec{c})$ and $D(\vec{a}-6\vec{b}+6\vec{c})$ are coplanar.

Solution: We first draw a visual picture to determine when four points can be coplanar.



Draw the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and the plane passing thriugh the two vectors· For $\vec{C}$ to lie in this plans, $\overrightarrow{AC}$ must be coplannar with $\overrightarrow{AB}$ and $\overrightarrow{AD} \Rightarrow \overrightarrow{AC}$ must be expressible as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

Thus, as explained in the figure, we must have some scalars $\lambda, \mu \in \mathbb{R}$ for which

$$\overrightarrow{AC} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AD}$$

$$\Rightarrow \quad (\overrightarrow{OC} - \overrightarrow{OA}) = \lambda(\overrightarrow{OB} - \overrightarrow{OA}) + \mu(\overrightarrow{OD} - \overrightarrow{OA}) \qquad \{O \text{ is the origin}\}$$

$$\Rightarrow \quad \vec{a} + \vec{b} - \vec{c} = \lambda(-\vec{a} - 5\vec{b} + 4\vec{c}) + \mu(-\vec{a} - 9\vec{b} + 7\vec{c})$$

$$\Rightarrow \quad (1 + \lambda + \mu)\vec{a} + (1 + 5\lambda + 9\mu)\vec{b} + (-1 - 4\lambda - 7\mu)\vec{c} = \vec{0}$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, we must have

$$1 + \lambda + \mu = 0$$

$$1 + 5\lambda + 9\mu = 0$$

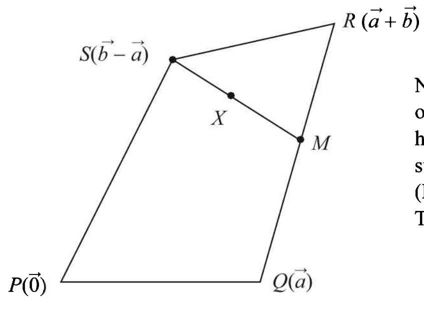
$$1 + 4\lambda + 7\mu = 0$$

As can be easily verified, this system has the solution $\lambda = -2, \mu = 1$, implying $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are indeed coplanar. Thus, the points A, B, C and D are coplanar. ■

Example 18

In a quadrilateral $PQRS$, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$ and $\overrightarrow{SP} = \vec{a} - \vec{b}$. If M is the mid-point of QR and X is a point on SM such that $SX : SM = 4 : 5$, prove that P, X and R collinear.

Solution: Since no position vectors have been specified in the question (only the sides have been specified), there is no loss of generality in assuming that P is the origin $\vec{0}$.



Note, how the positing vectors of the vertices of the quadrilateral have been specified. X is a point such that $SX : SM = 4 : 5$. (Diagram not to scale) Thus, $SX : XM = 4 : 1$.

We have,

$$M \equiv \frac{\vec{a} + (\vec{a} + \vec{b})}{2} = \vec{a} + \frac{\vec{b}}{2}$$

$$\Rightarrow \quad X \equiv \frac{4 \times (\vec{a} + \frac{\vec{b}}{2}) + 1 \times (\vec{b} - \vec{a})}{4 + 1} = \frac{3\vec{a} + 3\vec{b}}{5}$$

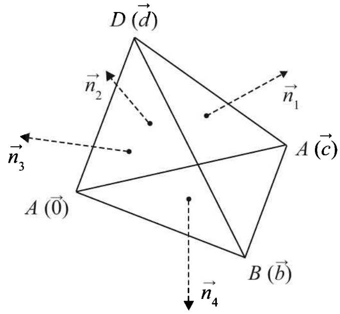
$$\Rightarrow \quad X \equiv \frac{3}{5}(\vec{a}+\vec{b})$$

Thus, $\overrightarrow{PX} = \frac{3}{5}\overrightarrow{PR}$, implying that P, X and R are collinear. ■

Example 19

Let A_i, $i = 1, 2, 3, 4$ be the areas of the faces of a tetrahedron. Let $\vec{n}_i$, $i = 1, 2, 3, 4$ be the outward drawn normals to the respective faces with magnitudes equal to the corresponding areas. Prove that $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}$.

Solution: There is no loss of generality in assuming one vertex A to be the origin $\vec{0}$. Let the other vertices be $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$.



In the following, note carefully how the order of the cross-product is taken in each case. We have:

$$\vec{n}_1 = \frac{1}{2}(\overrightarrow{BC} \times \overrightarrow{BD}) = \frac{1}{2}\{(\vec{c}-\vec{b})\times(\vec{d}-\vec{b})\} = \frac{1}{2}\{\vec{c}\times\vec{d}+\vec{b}\times\vec{c}+\vec{d}\times\vec{b}\}$$

$$\vec{n}_2 = \frac{1}{2}(\overrightarrow{AD} \times \overrightarrow{AC}) = \frac{1}{2}(\vec{d}\times\vec{c}) = -\frac{1}{2}(\vec{c}\times\vec{d})$$

$$\vec{n}_3 = \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AD}) = \frac{1}{2}(\vec{b}\times\vec{d}) = -\frac{1}{2}(\vec{d}\times\vec{b})$$

$$\vec{n}_4 = \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AB}) = \frac{1}{2}(\vec{c}\times\vec{b}) = -\frac{1}{2}(\vec{b}\times\vec{c})$$

From these relations, it is clear that

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}.$$

■

Example 20

For three arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$, prove that

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}.$$

Solution: The relation is most easily proved by assuming $\vec{a}$, $\vec{b}$, $\vec{c}$ in rectangular form:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

There's no loss of generality in this assumption. Now,

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}.$$

■

Example 21

(a) For three arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$, prove that the vectors

$$\vec{r}_1 = \vec{a} \times (\vec{b} \times \vec{c})$$
$$\vec{r}_2 = \vec{b} \times (\vec{c} \times \vec{a})$$
$$\vec{r}_3 = \vec{c} \times (\vec{a} \times \vec{b})$$

are coplanar.

(b) For three arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$, prove that

$$[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2.$$

Solution: (a) Using the expansion rule for the vector triple product, we have

$$\vec{r}_1 = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$
$$\vec{r}_2 = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$$
$$\vec{r}_3 = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

This gives $\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = \vec{0}$, which implies that $\vec{r}_1, \vec{r}_2, \vec{r}_3$ must be the sides of a triangle, and hence must be coplanar.

(b) The left hand side, upon expansion, gives

$$(\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} = (\vec{a}\times\vec{b})\cdot\{((\vec{b}\times\vec{c})\cdot\vec{a})\vec{c} - ((\vec{b}\times\vec{c})\cdot\vec{c})\vec{a}\}$$
$$= (\vec{a}\times\vec{b})\cdot\{[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{c}\} = \{(\vec{a}\times\vec{b})\cdot\vec{c}\}\,[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

This also proves as a consequence that $(\vec{a}\times\vec{b}), (\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ are coplanar iff $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, a fact that can intuitively be expected. ■

Example 22

Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

Solution: Let us first assume that

$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$$

$$\Rightarrow \quad -\vec{c}\times(\vec{a}\times\vec{b}) = \vec{a}\times(\vec{b}\times\vec{c})$$

$$\Rightarrow \quad (-\vec{c}\cdot\vec{b})\vec{a} - (-\vec{c}\cdot\vec{a})\vec{b} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Rightarrow \quad (\vec{b}\cdot\vec{c})\vec{a} = (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Rightarrow \quad \vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{c}}\right)\vec{c} = \lambda\vec{c} \text{ where } \lambda\in\mathbb{R}.$$

Since $\vec{a}$ is a scalar multiple of $\vec{c}$,this proves that $\vec{a}$ and $\vec{c}$ are collinear. To prove the reverse implication is left to the reader as an exercise. ■

Example 23

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that

$$\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

and $\vec{b}$ and $\vec{c}$ are non-collinear, find the angle θ_1 and θ_2 which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ respectively.

Solution:

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{\vec{b}+\vec{c}}{\sqrt{2}} \quad \text{(given)}$$

$$\Rightarrow \quad \left(\vec{a}\cdot\vec{c} - \frac{1}{\sqrt{2}}\right)\vec{b} - \left(\vec{a}\cdot\vec{b} + \frac{1}{\sqrt{2}}\right)\vec{c} = 0.$$

Since $\vec{b}, \vec{c}$ are non-collinear vectors, we must have

$$\vec{a}\cdot\vec{c} = \frac{1}{\sqrt{2}} \text{ and } \vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{1}{\sqrt{2}} \text{ and } \cos\theta_2 = -\frac{1}{\sqrt{2}}$$

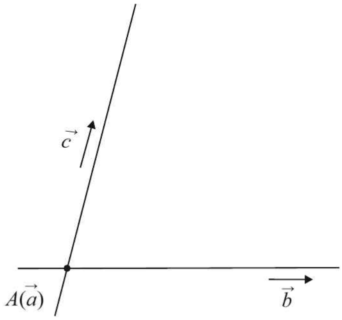
$$\Rightarrow \quad \theta_1 = \frac{\pi}{4} \text{ and } \theta_2 = \frac{3\pi}{4}$$

■

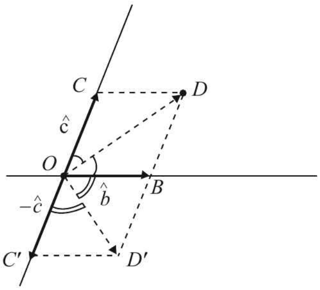
Example 24

Find the equation of the bisector(s) of the angle(s) between the straight lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$.

Solution: Note that the lines pass through $A(\vec{a})$ and are respectively parallel to $\vec{b}$ and $\vec{c}$:



We need to find the vector(s) equally inclined to $\vec{b}$ and $\vec{c}$ to be able to write the equation (s) of the angle bisector(s). For this purpose, consider the co-initial unit vectors $\hat{b}$ and $\hat{c}$



Complete the parallelogram $OBDC$ and $OBD'\ C'$ as indicated.
Since $OB = OC = 1$ unit,
$\Rightarrow \angle COD = \angle DOB$
Since $OB = OC = 1$ unit,
$\Rightarrow \angle C'OD = \angle DOB$

It should be clear from this figure that the two vectors equally inclined to $\vec{b}$ and $\vec{c}$ are simply the diagonals of the two parallelograms drawn, *i.e.*, $\overrightarrow{OD}$ and $\overrightarrow{OD'}$. Thus, the directions of the two angle bisectors are given by the vectors $\hat{b}+\hat{c}$ and $\hat{b}-\hat{c}$. Since the angle bisectors we require both pass through $A(\vec{a})$, we have their equations as

$$\vec{r} = \vec{a} + \lambda(\hat{b} + \hat{c}) \quad \text{and} \quad \vec{r} = \vec{a} + \lambda(\hat{b} - \hat{c}), \quad \lambda \in \mathbb{R}$$

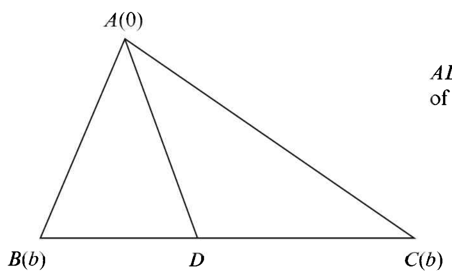
$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\left\{\frac{\vec{b}}{|\vec{b}|} \pm \frac{\vec{c}}{|\vec{c}|}\right\}, \quad \lambda \in \mathbb{R}$$

■

Example 25

Using vector methods, prove this result from plane geometry: in a triangle, the angle bisector of any angle divides the opposite side in the ratio of the sides containing the angle.

Solution: Assume a triangle ABC with position vectors of the vertices as indicated:



AD is the angle disector of angle A

The equation of AD can be written as

$$\vec{r} = \vec{0} + \lambda(\hat{b} + \hat{c}), \quad \lambda \in \mathbb{R}$$

$$\Rightarrow \quad \vec{r} = \lambda\left(\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right), \quad \lambda \in \mathbb{R}$$

Assume that D divides BC in the ratio $\mu : 1$. We wish to determine μ. The position vector of D is

$$D \equiv \frac{\mu\vec{c} + \vec{b}}{\mu + 1}.$$

Thus, for some $\lambda,\ \mu \in \mathbb{R}$, we must have

$$\lambda\left(\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right) = \frac{\mu\vec{c} + \vec{b}}{\mu + 1}$$

$$\Rightarrow \quad \left(\frac{\lambda}{|\vec{b}|} - \frac{1}{\mu + 1}\right)\vec{b} + \left(\frac{\lambda}{|\vec{c}|} - \frac{\mu}{\mu + 1}\right)\vec{c} = \vec{0}.$$

Since $\vec{b}$ and $\vec{c}$ are non collinear, we have

$$\frac{\lambda}{|\vec{b}|} = \frac{1}{\mu + 1} \tag{1}$$

$$\frac{\lambda}{|\vec{c}|} = \frac{\mu}{\mu + 1} \tag{2}$$

Dividing (2) by (1), we have

$$\mu = \frac{|\vec{b}|}{|\vec{c}|} = \frac{AB}{AC}.$$

Thus, D divides BC in the ratio $AB : AC$; this proves the theorem. ■

Example 26

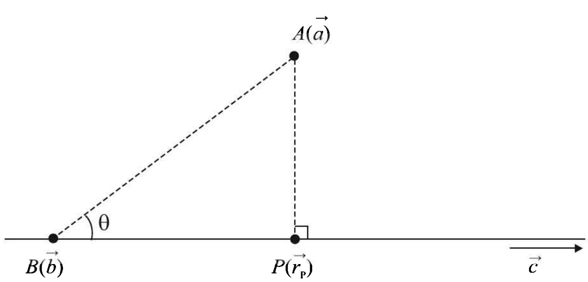
(a) Find the perpendicular distance of the point $A(\vec{a})$ from the line $\vec{r} = \vec{b} + \lambda\vec{c}$.

(b) As we know, two straight lines in space are called skew lines if they are neither parallel nor intersecting. Find the shortest distance between the two skew lines

$$L_1 : \vec{r} = \vec{a} + \lambda\vec{b}$$

$$L_2 : \vec{r} = \vec{c} + \mu\vec{d}$$

Solution: (a)



We need to find AP

Let $\vec{r}_P$ be the position vector of P

$$AP = \sqrt{AB^2 - BP^2} = \sqrt{AB^2 - (\overrightarrow{AB} \cdot \hat{C})^2}$$

$$= \sqrt{|\vec{a} - \vec{b}|^2 - (\vec{a} - \vec{b}) \cdot \vec{c}} = \sqrt{|\vec{a} - \vec{b}|^2 - \frac{(\vec{a} - \vec{b}) \cdot \vec{c}}{|\vec{c}|}}$$

This is one way to specify AP. Another way could be as follows:

$$\text{Area } (\Delta ABP) = \frac{1}{2} \times BP \times AP$$

$$\Rightarrow \quad \frac{1}{2} |(\vec{a} - \vec{b}) \times (\vec{r}_P - \vec{b})| = \frac{1}{2} |\vec{r}_P - \vec{b}| \, (AP)$$

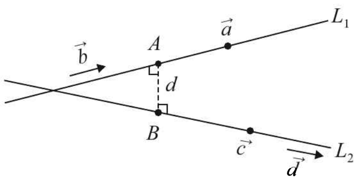
$$\Rightarrow \quad \frac{1}{2} |(\vec{a} - \vec{b}) \times \lambda \vec{c}| = \frac{1}{2} |\lambda \vec{c}| \, (AP) \qquad \left\{ \begin{matrix} \because \vec{r}_P - \vec{b} = \lambda \vec{c} \\ \text{for some } \lambda \in \mathbb{R} \end{matrix} \right\}$$

$$\Rightarrow \quad AP = \frac{|(\vec{a} - \vec{b}) \times \vec{c}|}{|\vec{c}|} = |(\vec{a} - \vec{b}) \times \hat{c}|$$

We could have arrived at this last result even more easily:

$$AP = AB \sin\theta = |\vec{a} - \vec{b}| \sin\theta = |(\vec{a} - \vec{b}) \times \hat{c}|$$

(b) First of all, convince yourself that there will be only one line along which the distance between L_1 and L_2 is minimum. Such a line will be perpendicular to both L_1 and L_2.



Two skew lines, at a distance *d*. Visualise this diagram in three dimensions

It should be obvious that since AB is perpendicular to both L_1 and L_2, *i.e.*, since AB is perpendicular to both $\vec{b}$ and $\vec{d}$, it must be collinear with the vector $\vec{b} \times \vec{d}$. Thus, a unit vector $\hat{n}$ along the direction AB is given by

$$\hat{n} = \frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}|}.$$

The distance AB is now simply the projection of the line segment joining the points $\vec{a}$ and $\vec{c}$ along the (extended) line AB. Again, visualise this in your mind; you must be very clear why this is so. We thus have,

$$d = AB = |(\vec{a} - \vec{c}) \cdot \hat{n}|$$

$$= \left| \frac{(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|} \right|$$

We can deduce a very useful corollary from this result. The two straight lines L_1 and L_2 intersect (in other words, they are coplanar) if

$$d = 0$$

$$\Rightarrow \quad (\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d}) = 0$$

$$\Rightarrow \quad \vec{a} \cdot (\vec{b} \times \vec{d}) = \vec{c} \cdot (\vec{b} \times \vec{d})$$

$$\Rightarrow \quad [\vec{a} \;\; \vec{b} \;\; \vec{d}] = [\vec{c} \;\; \vec{b} \;\; \vec{d}]$$

Note that if L_1 and L_2 are parallel, then the distance between them can be evaluated simply as the perpendicular distance of $\vec{a}$ from L_2 (or $\vec{c}$ from L_1). ■

Example 27

Find the position vector of the point of intersection of the three planes

$$\vec{r} \cdot \vec{n}_1 = d_1, \;\; \vec{r} \cdot \vec{n}_2 = d_2, \;\; \vec{r} \cdot \vec{n}_3 = d_3$$

where $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are non-coplanar vectors.

Solution: The condition that $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are non-coplanar will ensure that the three planes are guaranteed to intersect in a point (think about this carefully). Let $\vec{r}_0$ be the point of intersection of the three planes. We have,

$$\vec{r}_0 \cdot \vec{n}_1 = d_1, \quad \vec{r}_0 \cdot \vec{n}_2 = d_2, \quad \vec{r}_0 \cdot \vec{n}_3 = d_3.$$

We must find a way to express $\vec{r}_0$ in terms of the known vectors/quantities. For this purpose, let us consider as the basis of our three dimensional space the vectors $\{(\vec{n}_1 \times \vec{n}_2)(\vec{n}_2 \times \vec{n}_3)(\vec{n}_3 \times \vec{n}_1)\}$. This can be done since if $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are non-coplanar, so are $\vec{n}_1 \times \vec{n}_2$, $\vec{n}_2 \times \vec{n}_3$ and $\vec{n}_3 \times \vec{n}_1$. It will soon become clear why this should be done. We can now write $\vec{r}_0$ as:

$$\vec{r}_0 = \lambda_1(\vec{n}_1 \times \vec{n}_2) + \lambda_2(\vec{n}_2 \times \vec{n}_3) + \lambda_3(\vec{n}_3 \times \vec{n}_1) \text{ where } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

$$\Rightarrow \quad \vec{r}_0 \cdot \vec{n}_3 = \lambda_1[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3] \quad \Rightarrow \quad d_3 = \lambda_1[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3]$$

$$\Rightarrow \quad \lambda_1 = \frac{d_3}{[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3]}.$$

Similarly, $$\lambda_2 = \frac{d_1}{[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3]} \text{ and } \lambda_3 = \frac{d_2}{[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3]}$$

Thus,

$$\vec{r}_0 = \frac{1}{[\vec{n}_1 \;\; \vec{n}_2 \;\; \vec{n}_3]} \{d_3(\vec{n}_1 \times \vec{n}_2) + d_1(\vec{n}_2 \times \vec{n}_3) + d_2(\vec{n}_3 \times \vec{n}_1)\}.$$

■

Vectors

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. Let $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ be four given points such that

$$4\vec{a} - 3\vec{b} + \lambda\vec{c} - 3\vec{d} = \vec{0}.$$

Suppose that AC and BD meet at P.

(a) The value of λ is

(A) 2 (B) 3 (C) 4 (D) 5

(b) The ratios in which P divides AC and BD are

(A) 1:2, 1:3 (B) 1:2, 1:1 (C) 1:1, 1:3 (D) 1:2, 1:3

P2. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be the sides of a regular polygon inscribed in a circle of unit radius. If

$$|\vec{a}_1 \times \vec{a}_2 + \vec{a}_2 \times \vec{a}_3 + \cdots + \vec{a}_n \times \vec{a}_1| = |\vec{a}_1 \cdot \vec{a}_2 + \vec{a}_2 \cdot \vec{a}_3 + \cdots + \vec{a}_n \cdot \vec{a}_1|,$$

the smallest possible value of n is

(A) 4 (B) 6 (C) 8 (D) 12

P3. *ABCDEF* is a regular hexagon. $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$ equals which of the following?

(A) $3\overrightarrow{AD}$ (B) $3(\overrightarrow{AE} + \overrightarrow{AB})$ (C) $2(\overrightarrow{AC} + \overrightarrow{AE})$ (D) None of these

P4. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$ respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, which of the following can be the possible position vectors of the point E?

(A) $-\hat{i} + 3\hat{j} + 3\hat{k}$ (B) $\hat{i} - 2\hat{j} + 4\hat{k}$ (C) $3\hat{i} + \hat{j} + 2\hat{k}$ (D) $3\hat{i} - \hat{j} - \hat{k}$ (E) None of these

P5. If $\vec{u}, \vec{v}, \vec{w}$ are three non-coplanar unit vectors and α, β, γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, $\vec{w}$ and $\vec{u}$ respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisectors of the angles α, β, γ respectively, what is the value of the following expression?

$$\frac{[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}}{\left[\vec{x}\times\vec{y}\quad \vec{y}\times\vec{z}\quad \vec{z}\times\vec{x}\right]}$$

(A) 4 (B) 8 (C) 16 (D) 32

P6. Let the unit vectors $\vec{A}$ and $\vec{B}$ be perpendicular and the unit vector $\vec{C}$ be inclined at an angle θ to both $\vec{A}$ and $\vec{B}$. If $\vec{C} = \alpha\vec{A} + \beta\vec{B} + \gamma(\vec{A}\times\vec{B})$, then which of the following statements are true?

(A) $\alpha = \beta$ (B) $\gamma^2 = 1 - 2\alpha^2$ (C) $\gamma^2 = -\cos 2\theta$ (D) $\beta^2 = \dfrac{1+\cos 2\theta}{2}$

P7. Let a, b, c be three non-coplanar unit vectors equally inclined to one another at angle θ. The value of $[a, b, c]$ in terms of θ is

(A) $2\cos^2\dfrac{\theta}{2}\sqrt{1+2\cos\theta}$ (B) $2\sin^2\dfrac{\theta}{2}\sqrt{1+2\cos\theta}$

(C) $2\cos^2\dfrac{\theta}{2}\sqrt{1+2\sin\theta}$ (D) $2\sin^2\dfrac{\theta}{2}\sqrt{1+2\sin\theta}$

P8. Let $\vec{p}, \vec{q}, \vec{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector $\vec{x}$ satisfies the equation

$$\vec{p}\times((\vec{x}-\vec{q})\times\vec{p}) + \vec{q}\times((\vec{x}-\vec{r})\times\vec{q}) + \vec{r}\times((\vec{x}-\vec{p})\times\vec{r}) = \vec{0},$$

then the expression for $\vec{x}$ in terms of $\vec{p}, \vec{q}, \vec{r}$ is

(A) $\vec{x} = -\dfrac{1}{4}(\vec{p}+\vec{q}+\vec{r})$ (B) $\vec{x} = -\dfrac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

(C) $\vec{x} = \dfrac{1}{4}(\vec{p}+\vec{q}+\vec{r})$ (D) $\vec{x} = \dfrac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

SUBJECTIVE TYPE EXAMPLES

P10. Consider two non-collinear vectors $\vec{a}$ and $\vec{b}$.

(a) Express $\vec{b}$ as a sum of a vector parallel to $\vec{a}$ and a vector $\vec{v}$ perpendicular to $\vec{a}$ in the plane of $\vec{a}$ and $\vec{b}$, that is, express $\vec{b}$ as $\vec{b} = \alpha\vec{a} + \vec{v}$.

(b) Now consider a third vector $\vec{c}$ such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar. Express $\vec{c}$ as sum of the following form:

$$\vec{c} = \lambda_1\vec{a} + \lambda_2\vec{b} + \vec{v}_1$$

where $\vec{v}_1$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

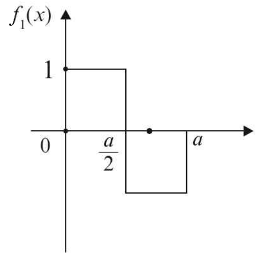
P11. Find the distance of the plane passing through the points $\vec{a}$, $\vec{b}$, $\vec{c}$ from the origin.

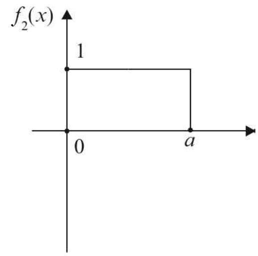
P12. Define vectors $\vec{f}_i = f_i(x)$ on $[0, a]$ and the corresponding modulus and dot product operations as follows:

$$|\vec{f}_i|^2 = \int_0^a f_i^2(x)\,dx$$

$$\vec{f}_i \cdot \vec{f}_j = \int_0^a f_i(x)\,f_j(x)\,dx$$

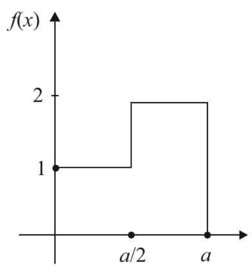
(a)





Show that $\vec{f}_1$ and $\vec{f}_2$ are perpendicular

(b)



Express $\vec{f}$ as a linear combination of $\vec{f}_1$ and $\vec{f}_2$.

(c) Find the angles between $\vec{f}$ and $\vec{f}_1$, and $\vec{f}$ and $\vec{f}_2$.

(d) Find the distances between the points $P(\vec{f})$, $Q(\vec{f}_1)$ and $R(\vec{f}_2)$.

P13. For any two vectors $\vec{u}$ and $\vec{v}$ prove that

(a) $|\vec{u}\cdot\vec{v}|^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2$

(b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2) = |1-\vec{u}\cdot\vec{v}|^2 + |\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

P14. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j},\ t\in[0,1]$ where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all t and $\vec{A}(0) = 2\hat{i}+3\hat{j}, \vec{A}(1) = 6\hat{i}+2\hat{j}, \vec{B}(0) = 3\hat{i}+2\hat{j}$ and $\vec{B}(1) = 2\hat{i}+6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t.

P15. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w}+(\vec{w}\times\vec{u}) = \vec{v}$, then prove that $(\vec{u}\times\vec{v})\cdot\vec{w} \le \frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

P16. Let $\vec{a}, \vec{b}$ be two non-collinear unit vectors inclined to each other at an acute angle. A third unit vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Let

$$\vec{P} = \vec{a}\cos t\cos 2t + \vec{b}\sin t\cos 2t + \vec{c}\sin 2t$$

(a) Find the unit vector along $\overrightarrow{OP}$ when $|\overrightarrow{OP}|$ is maximum. ($O \equiv$ origin)

(b) Find the maximum value of $|\overrightarrow{OP}|$.

P17. Let V be the volume of the parallelopiped formed by the vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
and
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

If a_r, b_r, c_r (where $r = 1, 2, 3$) are non-negative real numbers and $\sum_{r=1}^{3}(a_r+b_r+c_r) = 3Ld$, is it true that $V \le L^3$?

P18. Find 3-dimensional vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ satisfying

$$\vec{v}_1\cdot\vec{v}_1 = 4, \vec{v}_1\cdot\vec{v}_2 = -2, \vec{v}_1\cdot\vec{v}_3 = 6,$$
$$\vec{v}_2\cdot\vec{v}_2 = 2, \vec{v}_2\cdot\vec{v}_3 = -5, \vec{v}_3\cdot\vec{v}_3 = 29.$$

P19. A, B, C, D are any four points in space. Find the value of λ in the following expressions:

(a) $\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD} = \lambda(\overrightarrow{AB}\times\overrightarrow{CA})$

(b) $|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = \lambda\{\text{area}(\Delta\ ABC)\}$

P20. If $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$, find the value of $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})\cdot(\vec{B}+\vec{C})$.

P21. If the vectors $\vec{b}, \vec{c}, \vec{d}$ are not coplanar, then can we say that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$?

P22. A line through the origin has the vector equation $\vec{r} = \lambda\vec{s}$, where λ is a scalar parameter. Consider a family of lines L with

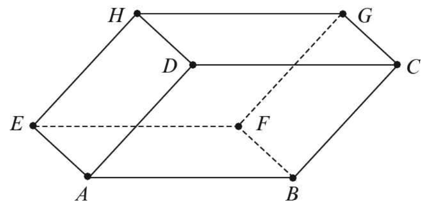
$$\vec{s} = (\cos\theta+\sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta-\sqrt{3})\hat{k}$$

(a) Show that there is a line M through the origin such that the acute angle between M and all the lines of the L-family is $\frac{\pi}{6}$.

(b) The lines in L meet the plane $x - z = 4\sqrt{3}$ at P. Show that as θ varies, P describes a circle on the plane with center on M. Find the radius of this circle.

P23. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points A, B, C to the sides QR, PR, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB respectively are also concurrent.

P24. Consider a parallelopiped as shown below:



The diagonals $\overrightarrow{AC}$, $\overrightarrow{AH}$ and $\overrightarrow{AF}$ are extended to $\overrightarrow{AP}$, $\overrightarrow{AQ}$ and $\overrightarrow{AR}$ such that

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{AC}|} = \frac{|\overrightarrow{AQ}|}{|\overrightarrow{AH}|} = \frac{|\overrightarrow{AR}|}{|\overrightarrow{AF}|} = \frac{3}{2}.$$

Find the distance of the point G from the plane passing through the points P, Q and R.

Vectors

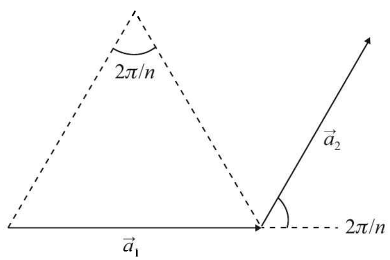
PART-D: Solutions to Advanced Problems

S1. Rearranging the given equality, we have

$$4\vec{a}+\lambda\vec{c}=3(\vec{b}+\vec{d}) \quad \Rightarrow \quad \frac{4\vec{a}+\lambda\vec{c}}{6}=\frac{\vec{b}+\vec{d}}{2}.$$

The section formula implies that the lines will meet if $\lambda = 2$. Consequently, P divides AC in the ratio 1:2 and BD in the ratio 1 : 1. The correct options are (A) for part-(a) and (B) for part (b).

S2.



$$|\vec{a}_1\times\vec{a}_2| = |\vec{a}_1||\vec{a}_2|\sin\frac{2\pi}{n}$$

$$|\vec{a}_1\cdot\vec{a}_2| = |\vec{a}_1||\vec{a}_2|\cos\frac{2\pi}{n}$$

The given equality therefore reduces to:

$$n\sin\frac{2\pi}{n}=n\cos\frac{2\pi}{n}$$

$$\Rightarrow \quad \tan\left(\frac{2\pi}{n}\right)=1 \Rightarrow \quad \frac{2\pi}{n}=m\pi+\frac{\pi}{4} \Rightarrow \quad n=\frac{8}{1+4m}.$$

The smallest and only possible value of n corresponds to $m = 0$, which is $n = 8$. The correct option is (C).

S3. Note that

$$\overrightarrow{AD}=\overrightarrow{2b}=\vec{a}+\vec{b}+\vec{c}$$

$$\Rightarrow \quad \vec{b}=\vec{a}+\vec{c}$$

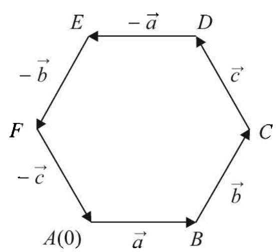
Now,

$$\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$$

$$=\vec{a}+(\vec{a}+\vec{b})+2\vec{b}+(\vec{b}+\vec{c})+\vec{c}$$

$$=2\vec{a}+4\vec{b}+2\vec{c}$$

$$=6\vec{b}$$



Also, we have the following:

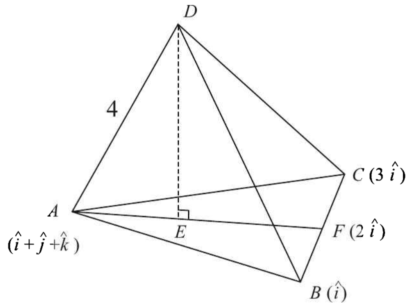
(A) $3\overrightarrow{AD} = 6\vec{b}$

(B) $3(\overrightarrow{AE} + \overrightarrow{AB}) = 3(\vec{b} + \vec{c} + \vec{a}) = 6\vec{b}$

(C) $2(\overrightarrow{AC} + \overrightarrow{AE}) = 2(\vec{a} + \vec{b} + \vec{b} + \vec{c}) = 6\vec{b}$

Therefore, all the three relations in (A), (B) and (C) are correct.

S4. Consider the following diagram:



Note that the area of ΔABC is $\sqrt{2}$ units, so that

$$\frac{1}{3}\times\sqrt{2}\times DE = \text{Vol.} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad AE = 2\sqrt{3}$$

Now, let E divide AF in the ratio $\lambda{:}1$. Thus,

$$E \equiv \frac{2\lambda\hat{i} + \hat{i} + \hat{j} + \hat{k}}{\lambda+1} = \left(\frac{2\lambda+1}{\lambda+1}\right)\hat{i} + \frac{1}{\lambda+1}(\hat{j}+\hat{k})$$

$$AE = 2\sqrt{3} = \left|\frac{\lambda}{\lambda+1}\hat{i} - \frac{\lambda}{\lambda+1}(\hat{j}+\hat{k})\right| \Longrightarrow \quad \lambda = -\frac{2}{3}, -2 \qquad \text{(Verify)}$$

Therefore, E can thus have two positions corresponding to the two values of λ: $-\hat{i} + 3\hat{j} + 3\hat{k}$ or $3\hat{i} - \hat{j} - \hat{k}$. The correct options are (A) and (D).

S5. We note that $\vec{x} = \frac{\vec{u}+\vec{v}}{2\cos\frac{\alpha}{2}}$, $\vec{y} = \frac{\vec{v}+\vec{w}}{2\cos\frac{\beta}{2}}$, $\vec{z} = \frac{\vec{w}+\vec{u}}{2\cos\frac{\gamma}{2}}$, so that

$$[\vec{x}\times\vec{y} \quad \vec{y}\times\vec{z} \quad \vec{z}\times\vec{x}] = [\vec{x} \quad \vec{y} \quad \vec{z}]^2$$

$$= \frac{1}{64}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}+\vec{v} \quad \vec{v}+\vec{w} \quad \vec{w}+\vec{u}]^2.$$

Finally, we note that $[\vec{u}+\vec{v} \quad \vec{v}+\vec{w} \quad \vec{w}+\vec{u}] = 2[\vec{u} \quad \vec{v} \quad \vec{w}]$, from which we immediately deduce that the required value is 16. The correct option is (C).

S6. We let $\vec{A} = \hat{i}$ and $\vec{B} = \hat{j}$, so that

$$\vec{C} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}, \ \alpha^2 + \beta^2 + \gamma^2 = 1$$

Also, $\quad \vec{A}\cdot\vec{C} = \vec{B}\cdot\vec{C} = \cos\theta$, *i.e.*,

$$\alpha = \beta = \cos\theta \quad \Rightarrow \quad \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$$

Thus, all the options provided are true.

S7. Let $\vec{a}=\hat{i}$ and $\vec{b}=\cos\theta\hat{i}+\sin\theta\hat{j}$. Let $\vec{c}=\ell\hat{i}+m\hat{j}+n\hat{k}$. Thus, we have

$$\left.\begin{array}{r}\vec{a}\cdot\vec{c}=\cos\theta\\ \vec{b}\cdot\vec{c}=\cos\theta\\ |\vec{c}|=1\end{array}\right\}\Rightarrow\left.\begin{array}{c}l=\cos\theta\\ l\cos\theta+m\sin\theta=\cos\theta\\ l^2+m^2+n^2=1\end{array}\right\}\Rightarrow\begin{array}{l}l=\cos\theta\\ m=\dfrac{\cos\theta-\cos^2\theta}{\sin\theta}\\ \quad=\tan\dfrac{\theta}{2}\cos\theta\\ n^2=\sin^2\theta-\tan^2\dfrac{\theta}{2}\cos^2\theta\end{array}$$

Now,

$$\left[\vec{a}\;\vec{b}\;\vec{c}\right]=\begin{vmatrix}1&0&0\\ \cos\theta&\sin\theta&0\\ l&m&n\end{vmatrix}=n\sin\theta$$

$$=\sin\theta\sqrt{\sin^2\theta-\tan^2\frac{\theta}{2}\cos^2\theta}=2\sin^2\frac{\theta}{2}\sqrt{4\cos^4\frac{\theta}{2}-\cos^2\theta}$$

$$=2\sin^2\frac{\theta}{2}\sqrt{4\cos^4\frac{\theta}{2}-\left(2\cos^2\frac{\theta}{2}-1\right)^2}=2\sin^2\frac{\theta}{2}\sqrt{4\cos^2\frac{\theta}{2}-1}$$

$$=2\sin^2\frac{\theta}{2}\sqrt{2\left(2\cos^2\frac{\theta}{2}-1\right)+1}=2\sin^2\frac{\theta}{2}\sqrt{1+2\cos\theta}.$$

The correct option is (B).

S8. Assuming the magnitude of the three vectors as l, and $\vec{x}$ to be $a\vec{p}+b\vec{q}+c\vec{r}$ (the reason we *can* make this assumption is that the three vectors $\vec{p},\vec{q},\vec{r}$ are mutually independent and so they form a basis for 3-D space; the reason we *did* make this assumption is for convenience, as will soon become clear), the first term can simplified as:

$$\begin{aligned}&\vec{p}\times((\vec{x}-\vec{q})\times\vec{p})=(\vec{p}\cdot\vec{p})(\vec{x}-\vec{q})-(\vec{p}\cdot(\vec{x}-\vec{q}))\vec{p}\\ &=l^2(\vec{x}-\vec{q})-(\vec{p}\cdot\vec{x})\vec{p}\\ &=l^2(\vec{x}-\vec{q})-al^2\,\vec{p}\\ &=l^2(\vec{x}-\vec{q}-a\vec{p}).\end{aligned}$$

Similarly, the second and third terms can be simplified, and thus from the given condition, we have

$$\begin{aligned}&l^2(\vec{x}-\vec{q}-a\vec{p})+l^2(\vec{x}-\vec{r}-b\vec{q})+l^2(\vec{x}-\vec{p}-c\vec{r})=\vec{0}\\ \Rightarrow\;&3\vec{x}-(\vec{p}+\vec{q}+\vec{r})-(a\vec{p}+b\vec{q}+c\vec{r})=\vec{0}\\ \Rightarrow\;&3\vec{x}-(\vec{p}+\vec{q}+\vec{r})-\vec{x}=\vec{0}\\ \Rightarrow\;&\vec{x}=\frac{1}{2}(\vec{p}+\vec{q}+\vec{r}).\end{aligned}$$

The correct option is (D).

SUBJECTIVE TYPE EXAMPLES

S9. If Δ is 0, then there will exist a triplet of numbers x, y, z not all zero (assume $|x| \geq |y| \geq |z|$ and $|x| \neq 0$) such that

$$a_1x + a_2y + a_3z = b_1x + b_2y + b_3z = c_1x + c_2y + c_3z = 0 \qquad \text{(why?)}$$

Thus,

$$|a_1x| = |a_1||x| = |-a_2y - a_3z| \leq |a_2||y| + |a_3||z|.$$

Since $|x| \geq |y| \geq |z|$, this can only happen if

$$|a_1| \leq |a_2| + |a_3|.$$

Using similar reasonings, we can obtain

$$|b_2| \leq |b_1| + |b_3| \text{ and } |c_3| \leq |c_1| + |c_2|.$$

But this is contradictory to the information provided in the question. So $\Delta \neq 0$.

S10. (a) We have $\vec{b} = \alpha\,\vec{a} + \vec{v}$:

$$\Rightarrow \quad \vec{b}\cdot\vec{a} = \alpha\,\vec{a}\cdot\vec{a} + \vec{v}\cdot\vec{a} = \alpha\,\vec{a}\cdot\vec{a}$$

$$\Rightarrow \quad \alpha = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}} \Rightarrow \quad \vec{v} = \vec{b} - \frac{(\vec{b}\cdot\vec{a})\vec{a}}{\vec{a}\cdot\vec{a}} = \frac{\vec{a}\times(\vec{b}\times\vec{a})}{\vec{a}\cdot\vec{a}}$$

(b) We have

$$\vec{c} = \lambda_1\vec{a} + \lambda_2\vec{b} + \vec{v}_1 \qquad (1)$$

$$= \lambda_1\vec{a} + \lambda_2(\alpha\vec{a} + \vec{v}) + \vec{v}_1$$

$$= (\lambda_1 + \alpha\lambda_2)\vec{a} + \lambda_2\vec{v} + \vec{v}_1$$

$$\Rightarrow \quad \begin{cases} \vec{c}\cdot\vec{a} = (\lambda_1 + \alpha\lambda_2)(\vec{a}\cdot\vec{a}) \\ \vec{c}\cdot\vec{v} = \lambda_2(\vec{v}\cdot\vec{v}) \end{cases}$$

$$\Rightarrow \quad \lambda_2 = \frac{\vec{c}\cdot\vec{v}}{\vec{v}\cdot\vec{v}},\ \lambda_1 + \alpha\lambda_2 = \frac{\vec{c}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}$$

Thus,

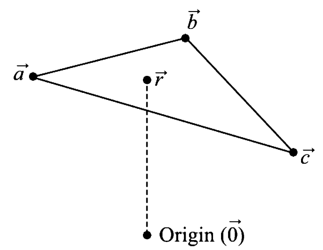
$$\vec{v}_1 = \vec{c} - (\lambda_1 + \alpha\lambda_2)\vec{a} - \lambda_2\vec{v}$$

$$= \vec{c} - \left(\frac{\vec{c}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} - \left(\frac{\vec{c}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}.$$

From (1), we see that all parameters are known for us to be able to express $\vec{c}$ in the desired form.

S11. Let the foot of the perpendicular drawn from the origin to the plane P containing $\vec{a}$, $\vec{b}$ and $\vec{c}$, be the point $\vec{r}$.
We have to find $|\vec{r}|$.

We observe that



$$\vec{r}\cdot(\vec{r} - \vec{a}) = \vec{r}\cdot(\vec{r} - \vec{b}) = \vec{r}\cdot(\vec{r} - \vec{c}) = 0$$

$$\Rightarrow \quad \vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot\vec{c} = |\vec{r}|^2 \qquad (1)$$

Now, the unit vector perpendicular to P can be written as

$$\hat{n} = \pm\frac{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})}{|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|} = \pm\frac{(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{|(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})|}$$

Since $\vec{r} = |\vec{r}|\,\hat{n}$, we have from (1)

$$|\vec{r}|\,\hat{n}\cdot\vec{a} = |\vec{r}|^2 \quad\Rightarrow\quad |\vec{r}| = \vec{a}\cdot\hat{n}$$

Evaluating $\vec{a}\cdot\hat{n}$ and simplifying (and considering only the positive sign since $|\vec{r}|$ must be positive), we have

$$|\vec{r}| = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})|}$$

S12. (a) It is easy to see that $\int_0^a f_1(x)\, f_2(x)dx$ is 0, *i.e.*, $\vec{f_1}\cdot\vec{f_2} = 0$, so these vectors are perpendicular.

(b) Let $\vec{f} = \lambda\vec{f_1} + \mu\vec{f_2}$. Then, observe the following carefully:

$$\text{In}\left(0, \frac{a}{2}\right): 1 = \lambda + \mu$$

$$\text{In}\left(\frac{a}{2}, a\right): 2 = -\lambda + \mu \quad\Rightarrow\quad \lambda = -\frac{1}{2},\ \mu = \frac{3}{2} \Rightarrow \quad \vec{f} = -\frac{1}{2}\vec{f_1} + \frac{3}{2}\vec{f_2}$$

(c)
$$|\vec{f}|^2 = \int_0^a f^2(x)dx = \frac{5a}{2}$$

$$|\vec{f_1}|^2 = \int_0^a f_1^2(x)dx = a, \quad |\vec{f_2}|^2 = \int_0^a f_2^2(x)dx = a$$

Also,
$$\vec{f}\cdot\vec{f_1} = \frac{-a}{2};\ \vec{f}\cdot\vec{f_2} = \frac{3a}{2}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{\vec{f}\cdot\vec{f_1}}{|\vec{f}||\vec{f_1}|} = \frac{\frac{-a}{2}}{\sqrt{\frac{5a}{2}}\sqrt{a}} \equiv -\frac{1}{\sqrt{10}}$$

$$\cos\theta_2 = \frac{\vec{f}\cdot\vec{f_2}}{|\vec{f}||\vec{f_2}|} = \frac{\frac{3a}{2}}{\sqrt{\frac{5a}{2}}\sqrt{a}} \equiv \frac{3}{\sqrt{10}}$$

(d)
$$PQ^2 = |\vec{f}-\vec{f_1}|^2 = \int_0^a (f(x) - f_1(x)^2)\, dx$$

$$= \int_0^a f^2(x)dx + \int_0^a f_1^2(x)dx - 2\int_0^a f(x)f_1(x)\, dx$$

$$= |\vec{f}|^2 + |\vec{f_1}|^2 - 2\vec{f}\cdot\vec{f_1} = \frac{9a}{2}$$

Similarly, $QR^2 = 2a$ and $PR^2 = \frac{a}{2}$.

S13. Part (a) is trivial. To prove the second equality, we expand the right hand side (RHS) by noting two facts:

(a) $\vec{u}\cdot\vec{v}$ is a scalar:

(b) $\vec{u}+\vec{v}$ and $\vec{u}\times\vec{v}$ are perpendicular.

Thus,

$$\text{RHS} = 1 + |\vec{u}\cdot\vec{v}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{u}+\vec{v}|^2 + |\vec{u}\times\vec{v}|^2$$

$$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + (|\vec{u}\cdot\vec{v}|^2 + |\vec{u}\times\vec{v}|^2)$$

$$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + |\vec{u}|^2|\vec{v}|^2 \qquad \text{(From part (a))}$$

$$= (1 + |\vec{u}|^2)(1 + |\vec{v}|^2).$$

S14. $\vec{A}(t)$ and $\vec{B}(t)$ are parallel at some $t = t_0$ if

$$\frac{f_1(t_0)}{g_1(t_0)} = \frac{f_2(t_0)}{g_2(t_0)},$$

that is, if $f_1(t_0)g_2(t_0) - f_2(t_0)g_1(t_0) = 0$. If we consider the function $h(x) = f_1(x)g_2(x) - f_2(x)g_1(x)$, we note that $h(0) = -5$ and $h(1) = 32$. Since $h(x)$ is continuous, it must attain the value of 0 for some $t_0 \in [0, 1]$. Hence, the result.

S15. We just have to use the relation $\vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$ appropriately. Taking the dot product with $\vec{u}$ on both sides gives $\vec{u}\cdot\vec{w} = \vec{u}\cdot\vec{v}$. Taking the dot product with $\vec{w}$ gives $|\vec{w}|^2 = \vec{v}\cdot\vec{w}$. Taking the dot product with $\vec{v}$ gives:

$$\vec{v}\cdot\vec{w} = 1 - (\vec{u}\times\vec{v})\cdot\vec{w}$$

$$\Rightarrow \quad |\vec{w}|^2 = 1 - (\vec{u}\times\vec{v})\cdot\vec{w}.$$

Taking the cross product with $\vec{u}$ gives

$$\vec{u}\times\vec{w} + \vec{w} - (\vec{u}\cdot\vec{w})\vec{u} = \vec{u}\times\vec{v}.$$

Taking the dot product with $\vec{w}$ on both sides of the last relation gives

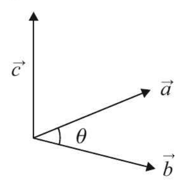
$$(\vec{u}\times\vec{v})\cdot\vec{w} = |\vec{w}|^2 - (\vec{u}\cdot\vec{w})^2$$

$$= 1 - (\vec{u}\times\vec{v})\cdot\vec{w} - (\vec{u}\cdot\vec{v})^2$$

$$\Rightarrow \quad (\vec{u}\times\vec{v})\cdot\vec{w} = \frac{1}{2}(1 - (\vec{u}\cdot\vec{v})^2) \le \frac{1}{2}.$$

The maximum is achieved when $\vec{u}\cdot\vec{v} = 0$, *i.e.*, when $\vec{u}$ and $\vec{v}$ are perpendicular.

S16. Assuming the angle between $\vec{a}$ and $\vec{b}$ to be θ, we have the following situation:



We note that $\vec{a}\cdot\vec{b} = \cos\theta$

Also,

$$\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b} = \vec{c}\cdot\vec{c} = 1$$

Now,

$$|\overrightarrow{OP}|^2 = \overrightarrow{OP}\cdot\overrightarrow{OP}$$

$$= \cos^2 t\cos^2 2t + \sin^2 t\cos^2 2t + \sin^2 2t + 2\cos\theta\sin t\cos t\cos^2 2t$$

$$= 1 + \cos\theta\sin 2t\cos^2 2t \qquad (1)$$

$|\overrightarrow{OP}|$ is maximum when $f(t) = \sin 2t \cos^2 2t$ is maximum, that is, when

$$f'(t) = 0 \Rightarrow \quad 2\sin 2t \cos 2t \cdot (-2\sin 2t) + 2\cos^3 2t = 0$$

$$\Rightarrow \quad \tan 2t = \frac{1}{\sqrt{2}} \Rightarrow \quad t = \frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}$$

It may be verified that this is indeed a point of maximum by demonstrating that $f''(t)\big|_{t=\frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}} < 0$

When $\tan 2t = \frac{1}{\sqrt{2}}$, then $\sin 2t = \frac{1}{\sqrt{3}}, \cos 2t = \frac{\sqrt{2}}{\sqrt{3}}$. Also,

$$\cos 2t = 2\cos^2 t - 1 = \frac{\sqrt{2}}{\sqrt{3}} \quad \Rightarrow \quad \cos t = \sqrt{\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}}$$

$$\Rightarrow \quad \sin t = \sqrt{\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}}.$$

From (1),

$$|\overrightarrow{OP}|_{\max} = \sqrt{1 + \cos\theta \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3}} = \sqrt{1 + \frac{2\cos\theta}{3\sqrt{3}}}.$$

(a) Now, if we represent the unit vector along $\overrightarrow{OP}$ when $|\overrightarrow{OP}|$ is maximum by $\hat{n}$, then

$$\hat{n} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}\left(\text{at } t = \frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\vec{a}\left(\sqrt{\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right) + \vec{b}\left(\sqrt{\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right) + \frac{\vec{c}}{\sqrt{3}}}{\sqrt{1 + \frac{2\cos\theta}{3\sqrt{3}}}}$$

$$= \frac{\vec{a}\sqrt{\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{3}}} + \vec{b}\sqrt{\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{3}}} + \frac{\vec{c}}{\sqrt{3}}}{\sqrt{1 + \frac{2\cos\theta}{3\sqrt{3}}}}$$

$$= \frac{\vec{a}\sqrt{\sqrt{3}+\sqrt{2}} + \vec{b}\sqrt{\sqrt{3}-\sqrt{2}} + \vec{c}\sqrt{\sqrt{3}}}{\sqrt{3\sqrt{3} + 2\cos\theta}}, \quad \text{where } \cos\theta = \vec{a}\cdot\vec{b}$$

(b) $$|\overrightarrow{OP}|_{\max} = \sqrt{1 + \frac{2\vec{a}\cdot\vec{b}}{3\sqrt{3}}}.$$

S17. Note that

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})| \le |\vec{a}||\vec{b}||\vec{c}| = \sqrt{\sum a_r^2}\sqrt{\sum b_r^2}\sqrt{\sum c_r^2}$$

On the other hand,

$$L = \frac{\sum(a_r + b_r + c_r)}{3} \ge \left(\sum a_r \sum b_r \sum c_r\right)^{1/3} \quad \text{(AM-GM inequality)}$$

But $\sum x_r \ge \sqrt{\sum x_r^2}$, which implies that

$$L^3 \ge \sum a_r \sum b_r \sum c_r \ge \sqrt{\sum a_r^2}\sqrt{\sum b_r^2}\sqrt{\sum c_r^2} = V$$

$$\Rightarrow \quad V \le L^3.$$

S18. The problem does not explicitly provide us with an ($\hat{i}, \hat{j}, \hat{k}$) frame of reference. If we are told to *find* three vectors which satisfy the given conditions, it is implied that we have to assume a frame of reference of our own one using which we can conveniently specify the lengths and the *relative* orientations of the three vectors. We start by assuming that the reference frame is such that $\vec{v}_1$ lies along the x-axis. From $\vec{v}_1 \cdot \vec{v}_1 = 4$, this gives $\vec{v}_1 = 2\hat{i}$. Next, we also *orient* our reference frame so that $\vec{v}_2$ lies in the x-y plane. After these two assumptions, the reference plane is uniquely fixed–we cannot make any more assumptions about it. Convince yourself about this point.

Using $\vec{v}_2 \cdot \vec{v}_2 = 2$, we can assume $\vec{v}_2 = \sqrt{2}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$. Now, using $\vec{v}_1 \cdot \vec{v}_2 = -2$, we can show that $\cos\theta = -\frac{2}{\sqrt{2}}$, while $\sin\theta$ can be either $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. Thus, $v_2 = -\hat{i} \pm \hat{j}$. Next, we assume $\vec{v}_3 = a\hat{i} + b\hat{j} + c\hat{k}$ and proceed case wise:

Case 1: $\vec{v}_2 = -\hat{i} - \hat{j}$

$$\vec{v}_1 \cdot \vec{v}_3 = 6 \Rightarrow 2a = 6 \Rightarrow a = 3$$
$$\vec{v}_2 \cdot \vec{v}_3 = -5 \Rightarrow -a + b = -5 \Rightarrow b = -2$$
$$\vec{v}_3 \cdot \vec{v}_3 = 29 \Rightarrow a^2 + b^2 + c^2 = 29 \Rightarrow c = \pm 4$$

We conclude that

$$\vec{v}_3 = 3\hat{i} - 2\hat{j} \pm 4\hat{k}$$

Case 2: $\vec{v}_2 = -\hat{i} - \hat{j}$

$$\vec{v}_1 \cdot \vec{v}_3 = 6 \Rightarrow 2a = 6 \Rightarrow a = 3$$
$$\vec{v}_2 \cdot \vec{v}_3 = -5 \Rightarrow -a - b = -5 \Rightarrow b = 2$$
$$\vec{v}_3 \cdot \vec{v}_3 = 29 \Rightarrow a^2 + b^2 + c^2 = 29 \Rightarrow c = \pm 4$$

We conclude that

$$\vec{v}_3 = 3\hat{i} + 2\hat{j} \pm 4\hat{k}$$

Thus, a total of four sets of vectors satisfying the given conditions are possible:

$$(2\hat{i}, -\hat{i} + \hat{j}, 3\hat{i} - 2\hat{j} \pm 4\hat{k}), \qquad (2\hat{i}, -\hat{i} - \hat{j}, 3\hat{i} + 2\hat{j} \pm 4\hat{k})$$

S19. There is no loss of generality in assuming A to be the origin. The points B, C, D can be represented by $\vec{b}, \vec{c}, \vec{d}$ respectively.

(a) We have

$$\begin{aligned}
&\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD} \\
&= \vec{b} \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times \vec{d} + (-\vec{c}) \times (\vec{d} - \vec{b}) \\
&= \vec{b} \times \vec{d} - \vec{b} \times \vec{c} + \vec{c} \times \vec{d} - \vec{b} \times \vec{d} - \vec{c} \times \vec{d} - \vec{b} \times \vec{c} = -2(\vec{b} \times \vec{c}) \\
&= 2(\vec{b} \times (-\vec{c})) = 2(\overrightarrow{\boldsymbol{AB}} \times \overrightarrow{\boldsymbol{CA}})
\end{aligned}$$

Thus, $\lambda = 2$.

(b) $\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD} = 2(\overrightarrow{AB} \times \overrightarrow{CA}) = 2 \times 2\{\text{area}(\Delta ABC)\} = 4\{\text{area}(\Delta ABC)\}$

Thus, $\lambda = 4$.

S20. If we represent the value of the given expression by λ, we have

$$\begin{aligned}
\text{LHS} &= \left[(\vec{A} + \vec{B}) \times (\vec{A} + \vec{C})\right] \times (\vec{B} \times \vec{C}) \cdot (\vec{B} + \vec{C}) \\
&= (\vec{A} \times \vec{C} + \vec{B} \times \vec{A} + \vec{B} \times \vec{C}) \times (\vec{B} \times \vec{C}) \cdot (\vec{B} + \vec{C}) \\
&= \{(\vec{A} \times \vec{C}) \times (\vec{B} \times \vec{C}) + (\vec{B} \times \vec{A}) \times (\vec{B} \times \vec{C})\} \cdot (\vec{B} + \vec{C}) \\
&= \{-[\vec{A}\ \ \vec{C}\ \ \vec{B}]\vec{C} + [\vec{B}\ \ \vec{A}\ \ \vec{C}]\vec{B}\} \cdot (\vec{B} + \vec{C}) \\
&= [\vec{A}\ \ \vec{C}\ \ \vec{B}](\vec{B} - \vec{C}) \cdot (\vec{B} + \vec{C}) = [\vec{A}\ \ \vec{C}\ \ \vec{B}](|\vec{B}|^2 - |\vec{C}|^2) = 0
\end{aligned}$$

S21. We have

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$=[\vec{c}\ \ \vec{d}\ \ \vec{a}]\vec{b}-[\vec{c}\ \ \vec{d}\ \ \vec{b}]\vec{a}+[\vec{d}\ \ \vec{b}\ \ \vec{a}]\vec{c}-[\vec{d}\ \ \vec{b}\ \ \vec{c}]\vec{a}+[\vec{a}\ \ \vec{d}\ \ \vec{c}]\vec{b}-[\vec{a}\ \ \vec{d}\ \ \vec{b}]\vec{c}$$

$$=-2[\vec{c}\ \ \vec{d}\ \ \vec{b}]\vec{a}=\lambda\vec{a}\ \text{ where } \lambda\in\mathbb{R}$$

It took only two steps to show that the given vector is parallel to the vector $\vec{a}$. The important part of the solution is how we took the cross products in the first step. For the first two terms, the cross products were taken as follows:

$$\begin{aligned}(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})&=-(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})\\&=-\{((\vec{c}\times\vec{d})\cdot\vec{b})\vec{a}-((\vec{c}\times\vec{d})\cdot\vec{a})\vec{b}\}\\&=[\vec{c}\ \ \vec{d}\ \ \vec{a}]\,\vec{b}-[\vec{c}\ \ \vec{d}\ \ \vec{b}]\vec{a}\end{aligned}$$

Similarly, the second cross product was expanded. The third cross product was taken as follows:

$$\begin{aligned}(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})&=((\vec{a}\times\vec{d})\cdot\vec{c})\vec{b}-((\vec{a}\times\vec{d})\cdot\vec{b})\vec{c}\\&=[\vec{a}\ \ \vec{d}\ \ \vec{c}]\vec{b}-[\vec{a}\ \ \vec{d}\ \ \vec{b}]\vec{c}\end{aligned}$$

The reason for these is to obtain some of the terms as multiples of vector $\vec{a}$, while the other terms should cancel out. Try taking the cross products in some other order and see what happens. Also, can you explain why the fact that $\vec{b},\vec{c},\vec{d}$ are not coplanar (given in the problem) is necessary?

S22. (a) Let line M be $\vec{r}=\mu(a\hat{i}+b\hat{j}+c\hat{k})$. Let the angle between this line and a line of the L-family be α. We have,

$$\begin{aligned}\cos\alpha&=\frac{a(\cos\theta+\sqrt{3})+\sqrt{2}b\sin\theta+c(\cos\theta-\sqrt{3})}{2\sqrt{2}\sqrt{a^2+b^2+c^2}}\\&=\frac{(a+c)\cos\theta+\sqrt{2}b\sin\theta+(a-c)\sqrt{3}}{2\sqrt{2}\sqrt{a^2+b^2+c^2}}\end{aligned}$$

Since we want to find M so that α is a constant equal to $\frac{\pi}{6}$, we have

$$a+c=0, b=0\ \Rightarrow\ c=-a$$

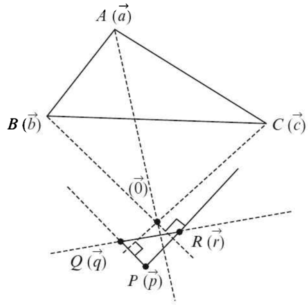
For these values, we further have

$$\cos\alpha=\frac{2a\sqrt{3}}{2\sqrt{2}\times\sqrt{2a^2}}=\frac{\sqrt{3}}{2}\ \Rightarrow\ \alpha=\frac{\pi}{6}$$

Thus, $M=\mu(\hat{i}-\hat{k})$.

(b) Line M meets the given plane in $P_1(2\sqrt{3},0,-2\sqrt{3})$. Line L meets the plane in $P(2\cos\theta+2\sqrt{3}, 2\sqrt{2}\sin\theta, 2\cos\theta-2\sqrt{3})$. Thus, $PP_1^2=8$, which implies that P describes a circle of radius $2\sqrt{2}$ with center at P_1.

S23. We assume the first point of concurrency (the one given to us in the question) to be the origin and the points A, B, C, P, Q, R as $\vec{a}, \vec{b}, \vec{c}, \vec{p}, \vec{q}, \vec{r}$ respectively:



This assumption really simplifies things. We have

$$\vec{a}\cdot(\vec{q}-\vec{r}) = \vec{b}\cdot(\vec{r}-\vec{p}) = \vec{c}\cdot(\vec{p}-\vec{q}) = 0. \tag{1}$$

Assume that the perpendiculars from P to BC and Q to CA meet at $\vec{x}$. Then,

$$(\vec{p}-\vec{x})\cdot(\vec{b}-\vec{c}) = 0,\ (\vec{q}-\vec{x})\cdot(\vec{c}-\vec{a}) = 0. \tag{2}$$

We just need to show that $(\vec{r}-\vec{x})\cdot(\vec{a}-\vec{b}) = 0$. Now, from (2), we have

$$\left.\begin{array}{l}\vec{x}\cdot(\vec{b}-\vec{c}) = \vec{p}\cdot(\vec{b}-\vec{c})\\ \vec{x}\cdot(\vec{c}-\vec{a}) = \vec{q}\cdot(\vec{c}-\vec{a})\end{array}\right\} \overset{\text{Add}}{\Rightarrow} \vec{x}\cdot(\vec{b}-\vec{a}) = (\vec{p}\cdot\vec{b}-\vec{p}\cdot\vec{c}) + (\vec{q}\cdot\vec{c}-\vec{q}\cdot\vec{a})$$

The RHS above can be simplified using (1):

$$\text{RHS} = \{\vec{p}\cdot\vec{b}-\vec{q}\cdot\vec{a}\} + \{(\vec{q}-\vec{p})\cdot\vec{c}\} = \vec{r}\cdot(\vec{b}-\vec{a}) + 0 \quad \text{(verify)}$$

Thus,

$$\vec{x}\cdot(\vec{b}-\vec{a}) = \vec{r}\cdot(\vec{b}-\vec{a}) \quad \Rightarrow \quad (\vec{r}-\vec{x})\cdot(\vec{a}-\vec{b}) = 0$$

This completes our proof.

S24. We let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AE} = \vec{b}$ and $\overrightarrow{AD} = \vec{c}$, so that

$$\overrightarrow{AG} = \vec{a}+\vec{b}+\vec{c}$$

Also,

$$\overrightarrow{AP} = \frac{3}{2}(\vec{a}+\vec{c}),\ \overrightarrow{AQ} = \frac{3}{2}(\vec{b}+\vec{c}),\ \overrightarrow{AR} = \frac{3}{2}(\vec{a}+\vec{b})$$

This gives

$$\overrightarrow{PG} = -\frac{\vec{a}}{2}+\vec{b}-\frac{\vec{c}}{2}$$

$$\overrightarrow{QG} = \vec{a}-\frac{\vec{b}}{2}-\frac{\vec{c}}{2}$$

$$\overrightarrow{RG} = -\frac{\vec{a}}{2}-\frac{\vec{b}}{2}+\vec{c}$$

We can easily observe that

$$\overrightarrow{PG}+\overrightarrow{QG}+\overrightarrow{RG} = 0$$

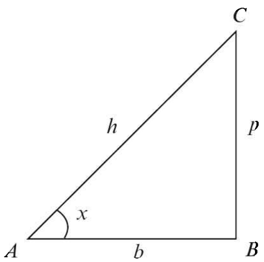
The only way this can happen is if G lies *in* the plane passing through the points P, Q and R, since otherwise, this vector sum must be non-zero. We thus conclude that the required distance is 0.

Trigonometry

PART - A : Summary of Important Concepts

1. Basics of Trigonometry

The subject of Trigonometry is based on six fundamental trignonometric ratios, or trigonometric functions:



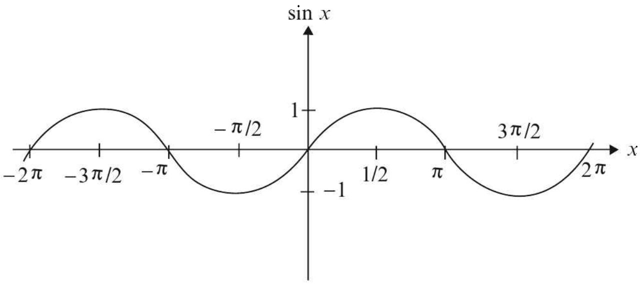
$\sin x = \dfrac{p}{h}$ $\quad \operatorname{cosec} x = \dfrac{h}{p}$

$\cos x = \dfrac{b}{h}$ $\quad \sec x = \dfrac{h}{b}$

$\tan x = \dfrac{p}{b}$ $\quad \cot x = \dfrac{b}{p}$

$\angle A = x$ radians

(a) $f(x) = \sin x$



Domain: $\mathbb{R}$

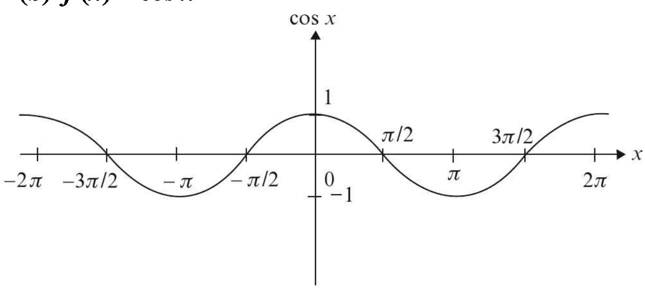
Range: $[-1, 1]$

$\sin x = 0$ if $x = n\pi, n \in \mathbb{Z}$

$\sin x > 0$ if $x \in (2n\pi, (2n+1)\pi), n \in \mathbb{Z}$

$\sin x < 0$ if $x \in ((2n-1)\pi, 2n\pi), n \in \mathbb{Z}$

(b) $f(x) = \cos x$



Domain: $\mathbb{R}$

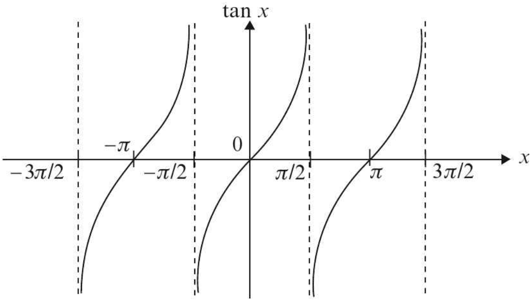
Range: $[-1, 1]$

$\cos x = 0$ if $x = (2n+1)\dfrac{\pi}{2}, \; n \in \mathbb{Z}$

$\cos x > 0$ if $x \in \left((4n-1)\dfrac{\pi}{2}, (4n+1)\dfrac{\pi}{2}\right), n \in \mathbb{Z}$

$\cos x < 0$ if $x \in \left((4n+1)\dfrac{\pi}{2}, (4n+3)\dfrac{\pi}{2}\right), n \in \mathbb{Z}$

(c) $f(x) = \tan x$



Domain: $\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}\right\}, n \in \mathbb{Z}$

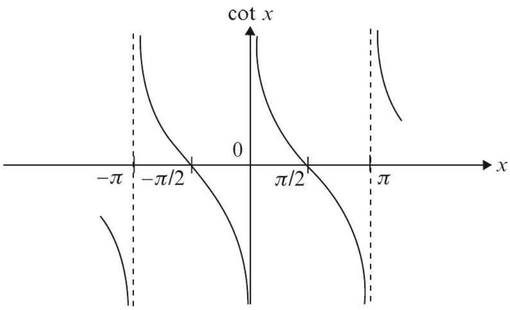
Range: $\mathbb{R}$

$\tan x = 0$ if $x = n\pi,\ n \in \mathbb{Z}$

$\tan x > 0$ if $x \in \left(n\pi, \left(n+\frac{1}{2}\right)\pi\right), n \in \mathbb{Z}$

$\tan x < 0$ if $x \in \left(\left(n-\frac{1}{2}\right)\pi, n\pi\right), n \in \mathbb{Z}$

(d) $f(x) = \cot x$



Domain: $\mathbb{R} \setminus \{n\pi\}, n \in \mathbb{Z}$

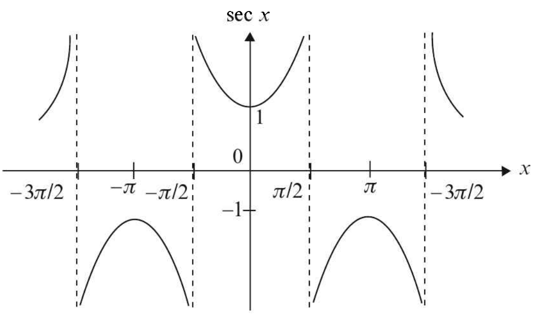
Range: $\mathbb{R}$

$\cot x = 0$ if $x = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$

$\cot x > 0$ if $x \in \left(n\pi, \left(n+\frac{1}{2}\right)\pi\right), n \in \mathbb{Z}$

$\cot x < 0$ if $x \in \left(\left(n-\frac{1}{2}\right)\pi, n\pi\right), n \in \mathbb{Z}$

(e) $f(x) = \sec x$



Domain: $\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}\right\}, n \in \mathbb{Z}$

Range: $(-\infty, -1] \cup [1, \infty)$

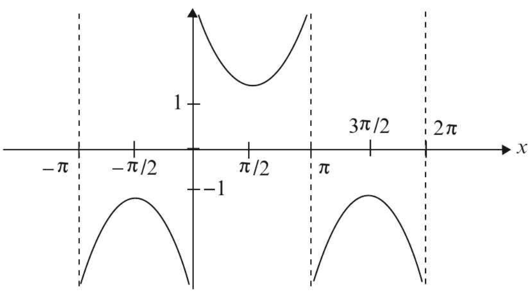
$\sec x = 1$ when $\cos x = 1$

$\Rightarrow \quad x = 2n\pi, n \in \mathbb{Z}$

$\sec x = -1$ when $\cos x = -1$

$\Rightarrow \quad x = (2n+1)\pi, n \in \mathbb{Z}$

(f) $f(x) = \operatorname{cosec} x$



Domain: $\mathbb{R} \setminus \{n\pi\}, n \in \mathbb{Z}$

Range: $(-\infty, -1] \cup [1, \infty)$

$\operatorname{cosec} x = 1$ when $\sin x = 1$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

$\operatorname{cosec} x = -1$ when $\sin x = -1$

$\Rightarrow \quad x = (2n-1)\frac{\pi}{2}, n \in \mathbb{Z}$

Listed below are basic relations satisfied by these functions:

(A)
- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \text{cosec}^2 x$

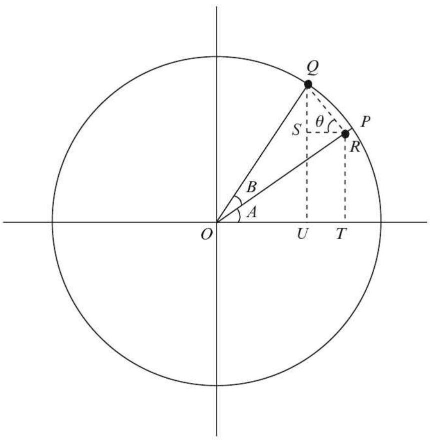
(B)
- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$
- $\cot(-x) = -\cot x$
- $\sec(-x) = \sec x$
- $\text{cosec}(-x) = -\text{cosec}\ x$

(C)
- $\sin\left(\dfrac{\pi}{2} \mp x\right) = \cos x$
- $\sin(\pi - x) = \sin x$
- $\sin(\pi + x) = -\sin x$
- $\cos\left(\dfrac{\pi}{2} \mp x\right) = \pm \sin x$
- $\cos(\pi - x) = -\cos x$
- $\cos(\pi + x) = -\cos x$
- $\tan\left(\dfrac{\pi}{2} \mp x\right) = \pm \cot x$
- $\tan(\pi \pm x) = \pm \tan x$

We now list down the various *multiple and sub-multiple angle properties* followed by the trigonometric functions.

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

It is instructive to understand the justification of this relationship, from which other similar relationships can be analogously justified. Consider a unit circle with angles A and B depicted as shown:



Note that

$$\begin{aligned}\sin(A+B) &= QU \\ &= QS + SU \\ &= QS + RT \\ &= QR \sin\theta + OR \sin A\end{aligned}$$

Now, note further that

$QR = \sin B,\ \theta = 90^\circ - A,\ OR = \cos B$

$$\Rightarrow \quad \sin(A+B) = \sin B \sin(90^\circ - A) + \cos B \sin A$$
$$= \sin A \cos B + \cos A \sin B$$

Similarly, we can prove the relation for $\sin(A - B)$.

- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A} = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$

- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\sin\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{2}}$
- $\cos\dfrac{A}{2} = \sqrt{\dfrac{1+\cos A}{2}}$
- $\tan\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A} = \dfrac{\sin A}{1+\cos A}$
- $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
- $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$
- $\tan C + \tan D = \dfrac{\sin(C+D)}{\cos C\cos D}$
- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
- $\sin^2 A = \dfrac{1}{2}(1-\cos 2A) \quad \cos^2 A = \dfrac{1}{2}(1+\cos 2A) \quad \tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

  $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4} \quad \cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

All these relations are very frequently used and it is advisable to commit them to memory.

2. Conditional Identities

Conditional identities are statements involving trigonometric terms which hold true subject to certain conditions satisfied by the arguments of those trigonometric terms. For example, suppose that $A+B+C=\pi$. Consider the expression $\sin 2A + \sin 2B + \sin 2C$. Let us convert this sum into a product using the constraint specified.

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= 2\sin(A+B)\cos(A-B) + \sin 2C\\ &= 2\sin C\cos(A-B) + 2\sin C\cos C\\ &= 2\sin C(\cos(A-B) - \cos(A+B))\\ &= 4\sin A\sin B\sin C\end{aligned}$$

The identity

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C \quad \text{for } A+B+C=\pi$$

is an example of a conditional identity. It is an identity because it is satisfied for all values of the angles A, B, C which satisfy the constraint $A+B+C=\pi$.
Here are two more examples, all having same constraint $A+B+C=\pi$.

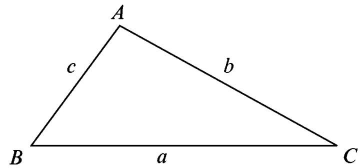
(i) $$\begin{aligned}\sin 2A + \sin 2B - \sin 2C &= 2\sin(A+B)\cos(A-B) - \sin 2C\\ &= 2\sin C\cos(A-B) - 2\sin C\cos C\\ &= 2\sin C(\cos(A-B) + \cos(A+B))\\ &= 4\cos A\cos B\sin C\end{aligned}$$

(ii) $\cos 2A + \cos 2B + \cos 2C = 2\cos(A+B)\cos(A-B) + \cos 2C$

$$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$$
$$= 2\cos C(\cos C - \cos(A-B)) - 1$$
$$= -2\cos C(\cos(A+B) + \cos(A-B)) - 1$$
$$= -1 - 4\cos A\cos B\cos C$$

3. Properties and Solutions of Triangles

Whenever we are discussing a triangle and its properties in general, the notation we'll assume will correspond to the following triangle:



A triangle follows these basic properties:

(a) $A + B + C = \pi$

(b) Triangle inequality: $a + b > c, b + c > a, c + a > b$

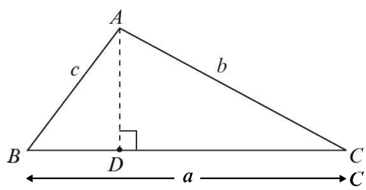
These are the two most basic properties satisfied by any triangle; there are, however, a large number of other very important properties satisifed by triangles, which we will now summarize:

3.1 The Sine Rule

The sides of a triangle are proportional to the sines of the angles opposite to them:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Since the sine rule is so important and so frequently used, we discuss a quick justification of this rule:



$$\sin B = \frac{AD}{c}, \sin C = \frac{AD}{b}$$
$$\Rightarrow c\sin B = b\sin C$$
$$\Rightarrow \frac{\sin B}{b} = \frac{\sin C}{c}$$

The sine rule follows from extending this. Similarly, we can prove the sine rule for an obtuse angled or right angled triangle. Note that the area of the triangle Δ is

$$\Delta = \frac{1}{2} \times BC \times AD = \frac{1}{2}\ ab\sin C = \frac{1}{2} ac\sin B$$

This leads to the expression for area of a triangle:

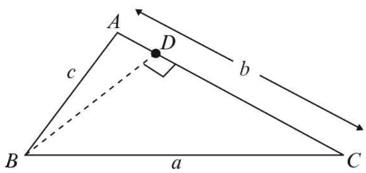
$$\boxed{\Delta = \frac{1}{2} ab\sin C = \frac{1}{2} bc\sin A = \frac{1}{2} ca\sin B}$$

3.2 The Cosine Rule

This rule says that in a ΔABC, the following relations will hold true:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The justification is straightforward. We consider the first part.



We have,

$$\begin{aligned} a^2 &= BD^2 + CD^2 \\ &= (c \sin A)^2 + (AC - AD)^2 \\ &= (c \sin A)^2 + (b - c\cos A)^2 \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

Observe that these relations will hold even when the triangle is obtuse. If the triangle is right angled, say at A, the cosine rule reduces to the Pythagoras theorem: $a^2 = b^2 + c^2$.

3.3 Semi-Perimeter and Half-Angle Formulae

For a ΔABC, with sides a, b, c, its semi perimeter is the quantity

$$\boxed{s = \frac{a+b+c}{2}}$$

Listed below are half-angle formulae which all triangles satisfy:

sin of half-angles

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \qquad \sin\frac{A}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

cos of half - angles

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

tan of half - angles

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

These are widely used expressions and should be memorized. A very important corollary of the half-angle formulae is the Heron's formula for the area of a triangle:

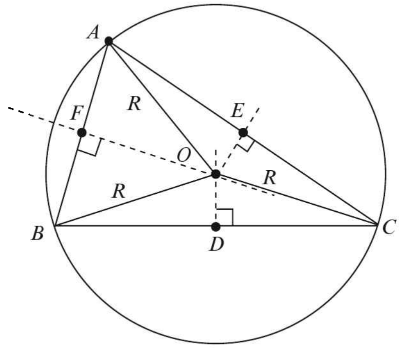
$$\Delta = \frac{1}{2} bc \sin A = bc \sin\frac{A}{2} \cos\frac{A}{2} = bc \cdot \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad \boxed{\Delta = \sqrt{s(s-a)(s-b)(s-c)}} \quad \text{Heron's formula.}$$

3.4 Circumcircle, Incircle and Excircle relations

3.4.1 Circumcircle

The radius of the circumcircle of a triangle ΔABC is generally denoted as R. Recall how we can construct the circumcircle, by first determining its center as the point of concurrency of the perpendicular bisectors of the sides of the triangle.



The following are two important relations satisfied by R:

$$\boxed{\begin{array}{l} R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} \\ R = \dfrac{abc}{4\Delta} \end{array}}$$

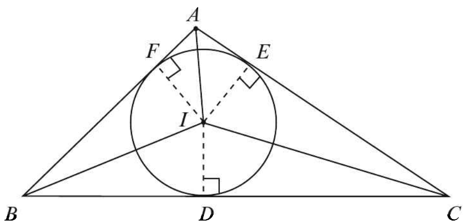
Important!

Incidentally, the sine rule can now be re-written as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

3.4.2 Incircle

The radius of the incircle of a ΔABC is generally denoted by r. The incenter is the point of concurrency of the angle bisectors of the angles of ΔABC, while the perpendicular distance of the incenter from any side is the radius r of the incircle:



$ID = IE = IF = r$

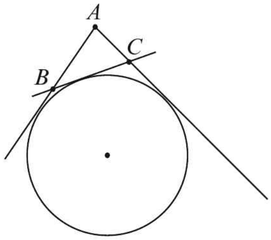
AI, BI, CI are the angle bisectors.

The next four relations relate r with the other parameters of the triangle:

$$\boxed{\begin{array}{l} r = \dfrac{\Delta}{s} \\ r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2} \\ r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}} \\ r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} \end{array}}$$

3.4.3 Excircle

For any triangle ΔABC, there will be three ex-circles. For examples, the ex-circle opposite to the angle A will touch the side BC and two sides AB and AC produced:



The ex-circle opposite to the angle A. The radius of this circle will be denoted by r_1. Similarly, the radii of the other two ex-circles are denoted by r_2 and r_3.

The next four sets of relations relate the ex-radii r_1, r_2, r_3 to the other parameters of the triangle. The reader is urged to prove these relations.

$r_1 = \dfrac{\Delta}{s-a}$,	$r_2 = \dfrac{\Delta}{s-b}$,	$r_3 = \dfrac{\Delta}{s-c}$
$r_1 = s\tan\dfrac{A}{2}$,	$r_2 = s\tan\dfrac{B}{2}$,	$r_3 = s\tan\dfrac{C}{2}$
$r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$,	$r_2 = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$,	$r_3 = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$
$r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$,	$r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$,	$r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

4. Inverse Trigonometry

Recall the following facts about inverse functions:

- For a function $f: A \to B$, we can define f^{-1} on this pair of sets A, B if f is a bijection, that is, f is one-one and onto.
- Given that this constrain is satisfied, we can define f^{-1} as $f^{-1}: B \to A$ such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. For example, for $f:[0,\infty) \to [0,\infty)$ given by $f(x) = x^2$, the inverse function is defined as $f^{-1}:[0,\infty) \to [0,\infty)$ and given by $f^{-1}(x) = \sqrt{x}$.
- The graphs of $f(x)$ and $f^{-1}(x)$ are mirror reflections of each other in the line $y = x$.

Since all the trigonometric functions are periodic, they are not bijections over their entire domains. This means that we have to restrict the domains of definition of these functions when defining their inverses, so that the functions are bijections over the restricted domains. However, there can be many such restricted domains over which a given trigonometric function is a bijection. Conventionally, we choose one of these to define the inverse. For example, $f(x) = \sin x$ is a bijection over both $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{2}]$, but it is the first interval over which we define the inverse of $\sin x$. You could call it a matter of convention and/or convenience. We now define the various trigonometric inverses and plot their graphs:

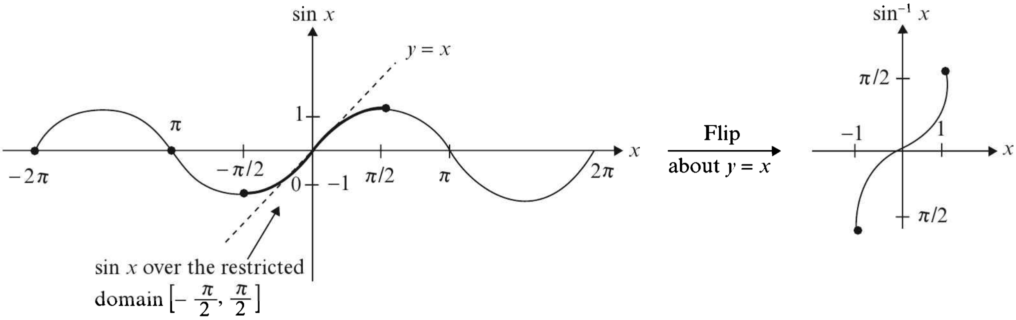
4.1 $f(x) = \sin^{-1} x$

We restrict the domain of $\sin x$ to $[\frac{-\pi}{2}, \frac{\pi}{2}]$, since over this interval, $\sin x$ is a bijection. Thus, we redefine $\sin x$ as

$$g(x) = \sin x, \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1],$$

and so we have

$$f(x)=\sin^{-1}x,\ [-1,1]\to\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$



Note the slopes of $\sin^{-1}x$ carefully. It is vertical at the points $x=\pm 1$ and has a slope of 1 at $x=0$. Also, it is strictly increasing, since $\sin x$ itself is strictly increasing over $[\frac{-\pi}{2},\frac{\pi}{2}]$.

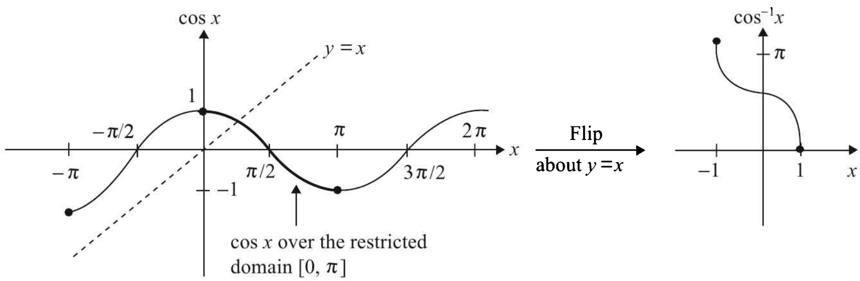
4.2 $f(x)=\cos^{-1}x$

We define $\cos x$ as

$$g(x)=\cos x,[0,\pi]\to[-1,1]$$

so that

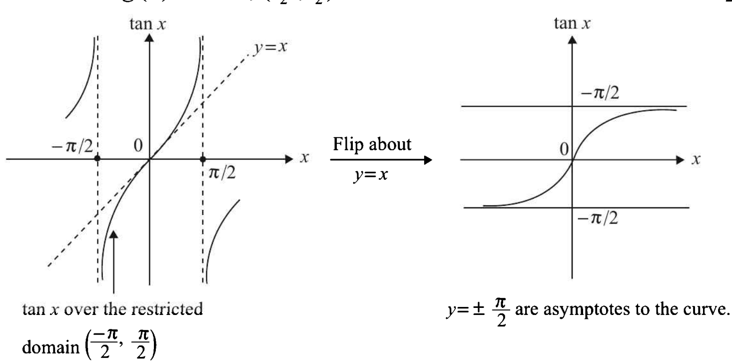
$$f(x)=\cos^{-1}x,[-1,1]\to[0,\pi]$$



Once again, note the slopes of $\cos^{-1}x$. It is strictly decreasing since $\cos x$ is strictly decreasing.

4.3 $f(x)=\tan^{-1}x$

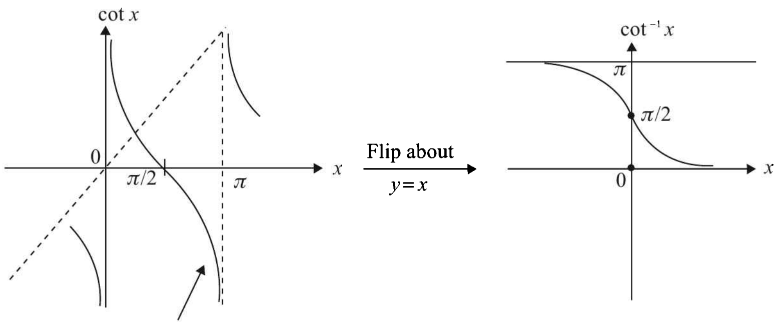
We define $\tan x$ as $g(x)=\tan x,(\frac{-\pi}{2},\frac{\pi}{2})\to\mathbb{R}$ so that $f(x)=\tan^{-1}x,\mathbb{R}\to(\frac{-\pi}{2},\frac{\pi}{2})$:



Note that $\tan^{-1}x$ is strictly increasing. Also, at $x=0$, $(\tan^{-1}x)'=1$.

4.4 $f(x) = \cot^{-1} x$

We define $\cot x$ as $g(x) = \cot x, (0, \pi) \to \mathbb{R}$ so that $f(x) = \cot^{-1} x, \mathbb{R} \to (0, \pi)$.



cot x over the restricted domain $(0, \pi)$ $\quad$ $y = 0$, $y = \pi$ are asymptotes to the curve.

Note that $\cot^{-1} x$ is strictly decreasing, and that $(\cot^{-1} x)'$ at $x = 0$ is -1.

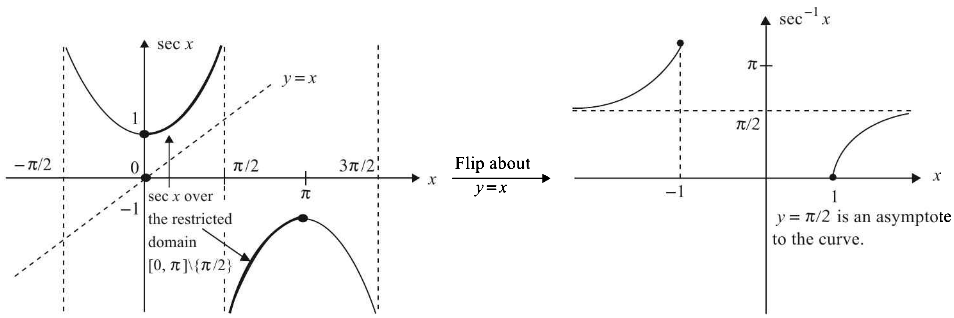
4.5 $f(x) = \sec^{-1} x$

We define $\sec x$ as

$$g(x) = \sec x, [0, \pi] \setminus \left\{\frac{\pi}{2}\right\} \to (-\infty, -1] \cup [1, \infty)$$

so that

$$f(x) = \sec^{-1} x, (-\infty, -1] \cup [1, \infty) \to [0, \pi] \setminus \{\pi/2\}$$



Note that $\sec^{-1} x$ is strictly increasing over $(-\infty, -1]$ and $[1, \infty)$. Also at $x = \pm 1$, $(\sec^{-1} x)'$ is infinite, *i.e.*, the curves are 'vertical'.

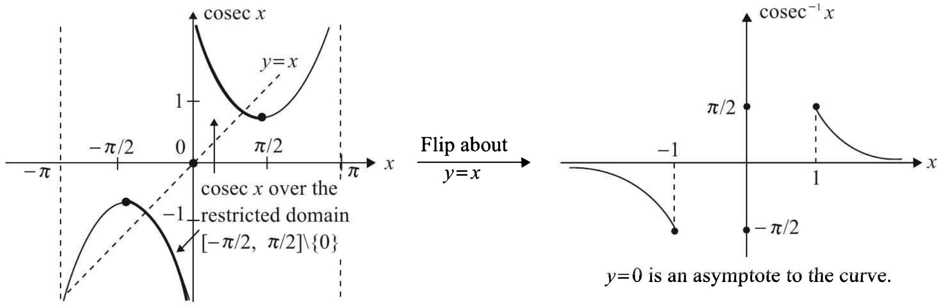
4.6 $f(x) = \operatorname{cosec}^{-1} x$

We define cosec x as

$$g(x) = \operatorname{cosec} x, \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\} \to [-\infty, -1] \cup [1, \infty)$$

so that

$$f(x) = \text{cosec}^{-1} x, (-\infty, -1] \cup [1, \infty) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$



$\text{cosec}^{-1}x$ is strictly decreasing on $(-\infty, -1]$ and on $[1, \infty)$. At $x = \pm 1$, its slopes are vertical.

4.7 Properties of Inverse Trigonometric Functions

From the graphs of the inverse functions, note the following properties:

$$\sin^{-1}(-x) = -\sin^{-1} x, \ \cos^{-1}(-x) = \pi - \cos^{-1}(x), \ \tan^{-1}(-x) = -\tan^{-1} x$$

$$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, \ \sec^{-1}(-x) = \pi - \sec^{-1}(x), \cot^{-1}(-x) = \pi - \cot^{-1} x$$

We have seen that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$. For example,

$$\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \ \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \text{etc.}$$

But this relation must be used with caution, since it is valid only if x lies in the appropriate domain. For example,

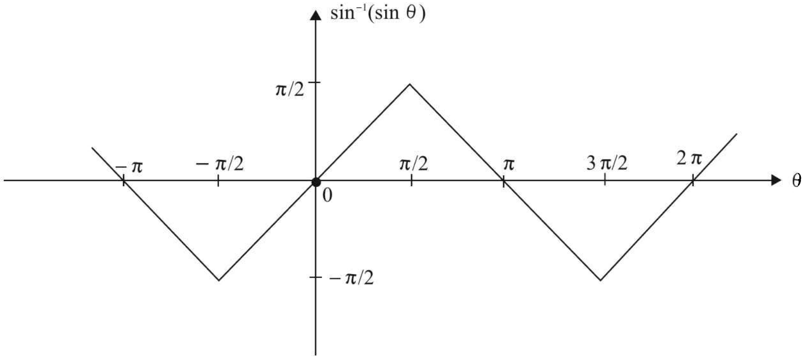
$$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \text{ and } \textbf{not } \frac{5\pi}{6}$$

Therefore, $\sin^{-1}(\sin\theta)$ will not always be θ, since the output of $\sin^{-1}$ will necessarily lie in the range $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

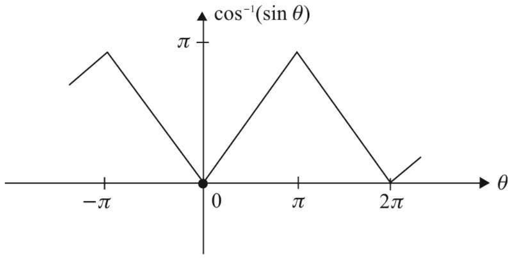
To adjust for this fact, note the following:

$$\sin^{-1}(\sin\theta) = \left\{\begin{array}{c} -\pi-\theta, \quad \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta, \quad \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi-\theta, \quad \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi+\theta, \quad \text{if } \theta \in [3\pi/2, 5\pi/2] \\ \text{and so on} \end{array}\right\} \quad \text{(Verify!)}$$

That is, we have adjusted the output values so that they always fall within the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Here's the graph of $\sin^{-1}(\sin\theta)$.



So, no matter what the value of θ, the output of $\sin^{-1}(\sin\theta)$ will always fall in the range $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Similarly, we have



We note the following important relations satisfied by inverse trigonometric terms:

(1) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x,$ for $x \in (-\infty, -1] \cup [1, \infty)$

(2) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,$ for $x \in (-\infty, -1] \cup [1, \infty)$

(3) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & x > 0 \\ -\pi + \cot^{-1}x, & x < 0 \end{cases}$

(4) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for all $x \in [-1, 1]$

(5) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in \mathbb{R}$

(6) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(7) $\sin^{-1}x + \sin^{-1}y$. We will analyze this case in detail.

Note that $x \in [-1, 1]$ and $y \in [-1, 1]$ for this expression to be defined.

Let $\quad \sin^{-1}x = \theta$ and $\sin^{-1}y = \phi$ so that

$$x = \sin\theta, \quad y = \sin\phi, \quad \theta, \phi \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Now,

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2} = z \text{ (say)}$$

which points to the fact that

$$\theta+\phi = \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

But wait! This is not entirely correct! Take $x = y = 1$, and we have

$$\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

while

$$\sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1}(0) = 0$$

There seems to be some problem in this equality, and we have to modify it appropriately. To do that, we note that

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi = \sqrt{1-x^2}\,\sqrt{1-y^2} - xy$$

The problem is originating from the fact that $\theta+\phi$ may not be in the restricted domain of sin, *i.e.*, in $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Therefore, we have to determine, based on x and y, the value of $\theta+\phi$. We consider the following cases:

(a) $\boldsymbol{x^2 + y^2 \le 1}$

$\Rightarrow \quad \sqrt{1-x^2} \ge |y|$ and $\sqrt{1-y^2} \ge |x|$

$\Rightarrow \quad \sqrt{1-x^2}\,\sqrt{1-y^2} - xy \ge 0$

$\Rightarrow \quad \cos(\theta+\phi) \ge 0 \Rightarrow \theta+\phi \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \theta+\phi = \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

(b) $\boldsymbol{x^2 + y^2 > 1,\ xy < 0}$

$\Rightarrow \quad \sqrt{1-x^2} > |y|, \sqrt{1-y^2} < |x|$

$\Rightarrow \quad \sqrt{1-x^2}\,\sqrt{1-y^2} < |xy|$

Since $xy < 0$,

$\sqrt{1-x^2}\,\sqrt{1-y^2} - xy > 0 \Rightarrow \cos(\theta+\phi) > 0$

Once again

$\theta+\phi = \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

(c) $\boldsymbol{x^2 + y^2 > 1,\ x, y > 0}$

Since $x > 0, \theta \in (0, \frac{\pi}{2}]$. Similarly, $\phi \in (0, \frac{\pi}{2}]$. Thus, $\theta + \phi \in (0, \pi]$ and so $\theta + \phi$ may not be in the restricted domain of sin. Also,

$$\sqrt{1-x^2} < |y|, \sqrt{1-y^2} < |x|$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\sqrt{1-y^2} < xy, \quad \text{since } xy > 0$$

$$\Rightarrow \quad \cos(\theta+\phi) = \sqrt{1-x^2}\,\sqrt{1-y^2} - xy < 0$$

$$\Rightarrow \quad \theta + \phi \in \left(\frac{\pi}{2}, \pi\right]$$

Therefore,

$$\theta + \phi = \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$$

to adjust for the fact that $\theta + \phi$ is in $(\frac{\pi}{2}, \pi]$

(d) $\boldsymbol{x^2 + y^2 > 1, x, y < 0}$

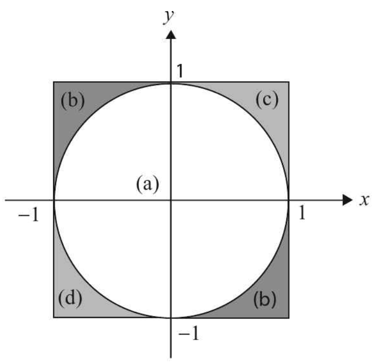
We can show that $\theta + \phi \in [-\pi, 0)$ and $\cos(\theta+\phi) < 0$, which means that

$$\theta + \phi \in \left[-\pi, \frac{-\pi}{2}\right)$$

Thus,

$$\theta + \phi = \sin^{-1} x + \sin^{-1} y = -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$$

Graphically, the four cases can be correlated with four distinct regions in a unit square:



One way to remember these relations is by making the following observations

(i) In region (c), both $\sin^{-1} x$ and $\sin^{-1} y$ will be large, so $\sin^{-1} x + \sin^{-1} y$ will be large. Precisely speaking, it will be above $\frac{\pi}{2}$, while the value of $\sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ will not be. Thus, an extra term of π will be needed to 'compensate'.

(ii) In region (d), both $\sin^{-1} x$ and $\sin^{-1} y$ will be large, but negative in sign, so $\sin^{-1} x + \sin^{-1} y$ will be large and negative. Therefore, we need an extra term of $-\pi$ to compensate.

(iii) In the region (b) in the second quadrant, $\sin^{-1} x$ will be negative while $\sin^{-1} y$ is positive, so $\sin^{-1} x + \sin^{-1} y$ falls in the restricted domain of sin, *i.e.*, in $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Similar arguments hold for region (d) in the fourth quadrant.

(iv) In region (a), both $\sin^{-1} x$ and $\sin^{-1} y$ are such that their sum always falls in $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

We have discussed the case of $\sin^{-1}x + \sin^{-1}y$ elaborately. In the following, we are just stating the results, and the reader is urged to use a similar line of reasoning to justify each one of them.

(8) $$\cos^{-1} x + \cos^{-1} y = \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, & x+y \geq 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, & x+y \leq 0 \end{cases}$$

(9) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0, xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0, xy > 1 \end{cases}$$

(10) $$\sin^{-1} x - \sin^{-1} y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & x^2+y^2 \leq 1 \text{ or } xy > 0 \text{ and } x^2+y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & x^2+y^2 > 1,\ x > 0,\ y < 0 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & x^2+y^2 > 1,\ x < 0,\ y > 0 \end{cases}$$

(11) $$\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, & x \leq y \\ -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, & x \geq y \end{cases}$$

(12) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & x < 0, y > 0, xy < -1 \end{cases}$$

(13) $$2\sin^{-1} x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & x \in \left[\dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right] \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & x \in \left[\dfrac{1}{\sqrt{2}}, 1\right] \\ -\pi - \sin(2x\sqrt{1-x^2}), & x \in \left[-1, \dfrac{-1}{\sqrt{2}}\right] \end{cases}$$

$$(14)\quad 3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), & x \in \left[\dfrac{-1}{2}, \dfrac{1}{2}\right] \\ \pi - \sin^{-1}(3x-4x^3), & x \in \left(\dfrac{1}{2}, 1\right] \\ -\pi - \sin^{-1}(3x-4x^3), & x \in \left[-1, \dfrac{-1}{2}\right) \end{cases}$$

$$(15)\quad 2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1), & x \in [0, 1] \\ 2\pi - \cos^{-1}(2x^2-1), & x \in [-1, 0) \end{cases}$$

$$(16)\quad 3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), & x \in \left[\dfrac{1}{2}, 1\right] \\ 2\pi - \cos^{-1}(4x^3-3x), & x \in \left[\dfrac{-1}{2}, \dfrac{1}{2}\right] \\ 2\pi + \cos^{-1}(4x^3-3x), & x \in \left[-1, \dfrac{-1}{2}\right] \end{cases}$$

$$(17)\quad 2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & x \in (-1, 1) \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & x < 1 \end{cases}$$

$$(18)\quad 3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & x \in \left(\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right) \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & x > \dfrac{1}{3} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & x < \dfrac{-1}{\sqrt{3}} \end{cases}$$

The following relations help us interconvert between the various inverse trigonometric functions. You are urged to prove each one of them.

$$(19)\quad \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$(20)\quad \cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$(21)\quad \tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

(22) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & x \in [-1, 1] \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & x < -1 \end{cases}$$

(23) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & x \in [0, \infty) \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & x \in (-\infty, 0] \end{cases}$$

It might not always be possible to keep all this relations in mind, and in fact it is not a good idea to try doing so. However, what is of the utmost importance is to understand how these relations are obtained.

5. Trigonometric Equations

The fact that trigonometric functions are periodic means that there can be multiple (or even infinite) solutions to any trigonometric equation. Here is an example:

Ilustration: Solve for $\theta : \sqrt{2}\sec\theta + \tan\theta = 1$

Working: $$\frac{\sqrt{2}}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1 \quad \Rightarrow \quad \sqrt{2} + \sin\theta = \cos\theta$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = 1 \quad \Rightarrow \quad \cos\left(\frac{\pi}{4} + \theta\right) = 1$$

$$\Rightarrow \quad \frac{\pi}{4} + \theta = 2n\pi, n \in \mathbb{Z} \text{ (How ?)} \quad \Rightarrow \quad \theta = 2n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$$

We see that θ can have an infinite number of values.

IMPORTANT IDEAS AND TIPS

1. *Pitfalls in Inverse Trigonometry.* A common problem in the subject of Trigonometry pertains to inverse trigonometric relations. For example, most students, when asked to write the relation for the sum of $\sin^{-1}x$ and $\sin^{-1}y$, will write the following relation:

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \tag{1}$$

However, as we have discussed in the preceding pages, this is not always necessarily correct. This is be cause the range of the inverse sine function is the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and thus any output from the inverse sine function *must* lie in this interval. Thus, in (1) above, the right hand side will always lie in this interval, but the left hand side may not. Suppose for example that $x = 1, y = 1$, so that

$$\sin^{-1}x = \frac{\pi}{2}, \ \sin^{-1}y = \frac{\pi}{2}, \ \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \sin^{-1}(0) = 0$$

For this case, the left hand side of (1) is equal to π which does not lie in the range of the inverse sine function. The right hand side, on the other hand, is equal to 0.

Thus, whenever you are combining inverse trigonometric terms, always keep in mind that the range of each inverse term must be considered.

2. *Multiple Solutions of Trigonometric Equations.* In general, a trigonometric equation will have multiple (or even infinite) solutions, due to the inherent periodic nature of the trigonometric terms which compose the equations. Suppose that you are asked to evaluate the solutions to a trigonometric equation in a specified interval *I.* A common mistake is to consider only the most obvious solutions which come to mind, ignoring the other solutions which might exist. For example, suppose that we have find the solutions to the equation

$$(\sin\theta - 1)\left(\cos\theta - \frac{1}{2}\right) = 0 \tag{2}$$

in the interval $I = [-2\pi, 2\pi]$. Many students would only specify the most obvious solutions which come to mind quickly:

$$\begin{cases} \sin\theta = 1 & \Rightarrow \quad \theta = \dfrac{\pi}{2} \\ \cos\theta = \dfrac{1}{2} & \Rightarrow \quad \theta = \dfrac{\pi}{3} \end{cases}$$

However, there are other values of θ in the interval I for which $\sin\theta = 1$ or $\cos\theta = \frac{1}{2}$, and we cannot ignore those values. If you were asked to find the *number of solutions* to (2) in the interval *I*, and you ignored these other values, your answer would turn out to be completely incorrect.

3. *Remembering Formulae.* Trigonometry is one of those subjects where remembering formulae holds a lot of importance in problem solving, as you will not have the time to derive every formula from first principles in an actual exam. The best way to ensure that you remember all the formulae of this chapter is to

(i) Understand the justification behind all of them.

(ii) Use *flash cards* or *charts:* write down these formulaes and relations (categorized by type) on cards or a large chart. Looking at the cards or the chart now and then, continuously for a few days, will aid in the retention of this material in your memory.

Trigonometry

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

Let $\tan\alpha = \frac{p}{q}$, and let $\beta = \frac{\alpha}{6}$. If α is acute, then the value of $p \operatorname{cosec} 2\beta - q\sec 2\beta$ is

(A) $\frac{1}{2}\sqrt{p^2+q^2}$ (B) $\frac{1}{3}\sqrt{p^2+q^2}$ (C) $\sqrt{p^2+q^2}$ (D) $2\sqrt{p^2+q^2}$

Solution: We have,

$$\sin\alpha = \frac{p}{\sqrt{p^2+q^2}}, \cos\alpha = \frac{q}{\sqrt{p^2+q^2}}$$

Now,

$$p \operatorname{cosec}2\beta - q\sec 2\beta = \frac{p}{\sin 2\beta} - \frac{q}{\cos 2\beta}$$

$$= \sqrt{p^2+q^2}\left(\frac{\frac{p}{\sqrt{p^2+q^2}}}{\sin 2\beta} - \frac{\frac{q}{\sqrt{p^2+q^2}}}{\cos 2\beta}\right) = \sqrt{p^2+q^2}\left(\frac{\sin\alpha}{\sin 2\beta} - \frac{\cos\alpha}{\cos 2\beta}\right)$$

$$= \sqrt{p^2+q^2}\,\frac{\sin(\alpha - 2\beta)}{\frac{1}{2}\sin 4\beta}$$

Since $\alpha = 6\beta$, this reduces to $2\sqrt{p^2+q^2}$. Thus, the correct option is (D). ■

Example 2

If $\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a+b}$, the value of $\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3}$ is

(A) $\frac{1}{4(a+b)^3}$ (B) $\frac{1}{2(a+b)^3}$ (C) $\frac{1}{(a+b)^3}$ (D) $\frac{2}{(a+b)^3}$

Solution: Rearranging the given relation gives us

$$\left(1+\frac{b}{a}\right)\sin^4\alpha+\left(1+\frac{a}{b}\right)\cos^4\alpha=1$$

$$\Rightarrow\quad \sin^4\alpha+\cos^4\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha=1$$

Since $\sin^4\alpha+\cos^4\alpha=(\sin^2\alpha+\cos^2\alpha)^2-2\sin^2\alpha\cos^2\alpha=1-2\sin^2\alpha\cos^2\alpha$, we have,

$$\Rightarrow\quad \frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha-2\sin^2\alpha\cos^2\alpha=0$$

$$\Rightarrow\quad \left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2=0$$

$$\Rightarrow\quad \frac{\sin^2\alpha}{a}=\frac{\cos^2\alpha}{b}=\frac{\sin^2\alpha+\cos^2\alpha}{a+b}=\frac{1}{a+b}$$

$$\Rightarrow\quad \sin^2\alpha=\frac{a}{a+b},\quad \cos^2\alpha=\frac{b}{a+b}$$

$$\Rightarrow\quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

The correct option is (C). ■

Example 3

The value of $S=\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$ is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{5}{2}$

Solution: Since $\sin\theta=\sin(\pi-\theta)$, we have

$$\sin\frac{5\pi}{8}=\sin\frac{3\pi}{8}\text{ and }\sin\frac{7\pi}{8}=\sin\frac{\pi}{8}$$

$$\Rightarrow\quad S=2\left(\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}\right)$$

Now, $$\sin^4\frac{\pi}{8}=\left(\sin^2\frac{\pi}{8}\right)^2=\left(\frac{1-\cos(\pi/4)}{2}\right)^2=\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)^2$$

Similarly, $$\sin^4\frac{3\pi}{8}=\frac{1}{4}\left(1+\frac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow\quad S=\frac{3}{2}\quad\text{(verify!)}$$

Thus, the correct option is (B). ■

Example 4

The value of $\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}$ is

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

Solution: If $\theta = \frac{\pi}{7}$, then $7\theta = \pi \quad \Rightarrow \quad 4\theta = \pi - 3\theta \quad \Rightarrow \quad \sin 4\theta = \sin 3\theta$
Upon expansion and rearrangement of this relation, we will obtain

$$8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0$$

Note carefully that each of the terms $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$ will satisfy this cubic (Why?). These are therefore the three roots of the cubic, and their product is:

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8} \tag{1}$$

Finally,

$$\cos\frac{5\pi}{7} = \cos\left(\pi - \frac{2\pi}{7}\right) = -\cos\frac{2\pi}{7}.$$

Replacing this in (1) gives

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$$

The correct option is (B).
As an exercise, show that $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$. ■

Example 5

Let A, B, C be the three angles of a triangle such that $A = \frac{\pi}{4}$ and $\tan B \cdot \tan C = \lambda$. In which of the following intervals can λ lie?

(A) $(-\infty, -3-2\sqrt{2}]$ (C) $[3+2\sqrt{2}, \infty)$ (E) None of these
(B) $(-\infty, -1-\sqrt{2}]$ (D) $[1+\sqrt{2}, \infty)$

Solution: For A, B, C to represent the angles of a triangle, we must have

$$A + B + C = \pi \quad \Rightarrow \quad B + C = \frac{3\pi}{4}$$

Now,

$$\tan B \cdot \tan C = \tan B \cdot \tan\left(\frac{3\pi}{4} - B\right) = \lambda$$

$$\Rightarrow \quad \tan B\left(\frac{-1-\tan B}{1-\tan B}\right) = \lambda \quad \Rightarrow \quad \tan^2 B + (1-\lambda)\tan B + \lambda = 0$$

Since $\tan B$ is real, the discriminant of this quadratic must be non-negative:

$$(1-\lambda)^2 - 4\lambda \geq 0$$
$$\Rightarrow \quad \lambda \in (-\infty, -3-2\sqrt{2}] \cup [3+2\sqrt{2}, \infty)$$

The correct options are (A) and (C). ■

Example 6

What is the ratio $a : b : c$ in a ΔABC if the following holds?

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

(A) $1:1:2$ (B) $1:1:\frac{1}{2}$ (C) $1:1:\frac{1}{\sqrt{2}}$ (D) $1:1:\sqrt{2}$

Solution: Note that

$$\sin C = \frac{1-\cos A\cos B}{\sin A\sin B} \le 1$$

$$\Rightarrow \quad \sin A\sin B + \cos A\cos B = \cos(A-B) \ge 1$$

This necessarily implies that

$$\cos(A-B) = 1 \quad \Rightarrow \quad \angle A = \angle B$$

$$\Rightarrow \quad \sin C = \frac{1-\cos^2 A}{\sin^2 A} = 1 \quad \Rightarrow \quad C = \frac{\pi}{2} \quad \Rightarrow \quad A = B = \frac{\pi}{4}$$

Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ from the sine rule, we have

$$\frac{a}{1/\sqrt{2}} = \frac{b}{1/\sqrt{2}} = \frac{c}{1}$$

$$\Rightarrow \quad a : b : c = 1 : 1 : \sqrt{2}$$

Thus, the correct option is (D). ■

Example 7

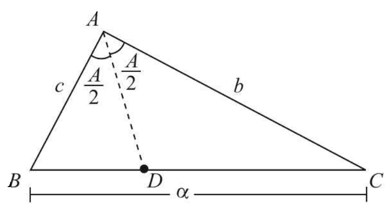
Let α, β, γ be the lengths of the internal angle bisectors of angles A, B, C respectively in a ΔABC. Consider S given by the expression

$$S = \frac{1}{\alpha}\cos\frac{A}{2} + \frac{1}{\beta}\cos\frac{B}{2} + \frac{1}{\gamma}\cos\frac{C}{2}$$

Which of the following is true?

(A) $S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (B) $\frac{1}{S} = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ (C) $S = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ (D) $\frac{1}{S} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Solution: Let us first find an expression for α:



Note that

$$\text{area }(\Delta ABD) + \text{area }(\Delta ACD) = \text{area }(\Delta ABC)$$

$$\Rightarrow \quad \frac{1}{2}c\alpha\sin\frac{A}{2} + \frac{1}{2}b\alpha\sin\frac{A}{2} = \frac{1}{2}bc\sin A$$

$$\Rightarrow \quad \alpha = \frac{bc\sin A}{(b+c)\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}$$

Similar expressions hold for β and γ. We note that

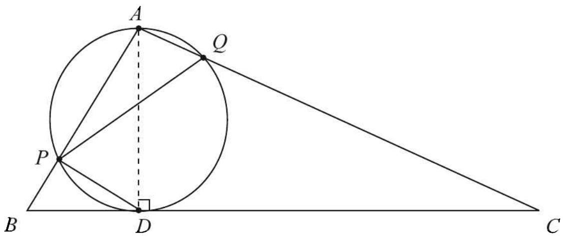
$$\frac{1}{\alpha}\cos\frac{A}{2} = \frac{b+c}{2bc}$$

$$\Rightarrow \quad S = \frac{b+c}{2bc} + \frac{c+a}{2ca} + \frac{a+b}{2ab} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \qquad \blacksquare$$

Thus, the correct option is (A).

Example 8

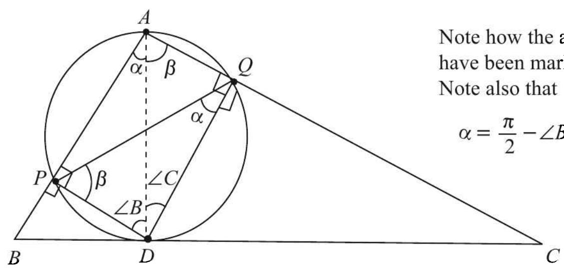
In an acute-angled triangle ABC, the circle on the altitude AD as diameter cuts AB at P and AC at Q.



The length PQ is equal to

(A) $\dfrac{2\Delta}{R}$ (B) $\dfrac{\Delta}{R}$ (C) $\dfrac{\Delta}{2R}$ (D) $\dfrac{\Delta}{4R}$

Solution: Let us redraw the figure above with all the angles marked:



Note how the angles have been marked. Note also that

$$\alpha = \frac{\pi}{2} - \angle B, \ \beta = \frac{\pi}{2} - \angle C$$

Applying the sine rule in $\angle PQD$, we have

$$\frac{PQ}{\sin(B+C)} = \frac{DQ}{\cos C} = \frac{PD}{\cos B} \quad \text{(how?)}$$

But, $$\frac{DQ}{\cos C} = \frac{PQ}{\cos B} = AD, \text{ so that}$$

$$PQ = AD\sin(B+C) = AD\sin A = c\sin B\sin A \qquad \text{(how?)}$$

$$= c\left(\frac{b}{2R}\right)\left(\frac{a}{2R}\right) = \frac{abc}{4R^2} = \frac{\Delta}{R}$$

Thus, the correct option is (B). $\blacksquare$

SUBJECTIVE TYPE EXAMPLES

Example 9

Find the range of

(a) $f(x) = \dfrac{\tan 3x}{\tan x}$ (b) $f(x) = \dfrac{\tan 3x}{\tan 2x}$

Solution: (a) $y = f(x) = \dfrac{\left(\frac{3\tan x - \tan^3 x}{1-3\tan^2 x}\right)}{\tan x} = \dfrac{3-\tan^2 x}{1-3\tan^2 x} \Rightarrow \tan^2 x = \dfrac{y-3}{3y-1}$

Since $\tan^2 x$ is necessarily non-negative, we have

$$\frac{y-3}{3y-1} \ge 0 \Rightarrow y \notin \left[\frac{1}{3}, 3\right)$$

Also, note that $y = 3 \Rightarrow \tan x = 0$, for which the ratio $\frac{\tan 3x}{\tan x}$ is non-defined. So technically speaking, it is incorrect to include 3 in the range, even though

$$\lim_{x\to 0} \frac{\tan 3x}{\tan x} = 3$$

Therefore, the range is $\mathbb{R} \setminus [\frac{1}{3}, 3]$.

(b) $y = f(x) = \dfrac{\tan 3x}{\tan 2x} = \dfrac{\left(\frac{3\tan x - \tan^3 x}{1-3\tan^2 x}\right)}{\left(\frac{2\tan x}{1-\tan^2 x}\right)}$

Rearranging this yields

$$\tan^4 x - 2(2-3y)\tan^2 x + 3 - 2y = 0$$

This is a quadratic in $\tan^2 x$, which means that the equation

$$z^2 - 2(2-3y)z + 3 - 2y = 0, \quad z = \tan^2 x$$

should have at least one non-negative root. This implies that

(I) $D \ge 0 \Rightarrow y \le -1$ or $y \ge 1$

and

(II) Either there's just one positive root $\Rightarrow f(0) < 0 \Rightarrow y > \frac{3}{2}$

OR

Both roots are positive

$$\Rightarrow f(0) > 0 \text{ and } \frac{-b}{2a} > 0 \Rightarrow y < \frac{2}{3} \Rightarrow y \le -1$$

Combining the above results, the possible values that y can take are given by

$$y > \frac{3}{2} \text{ or } y \le -1$$

■

Example 10

Eliminate θ from the following relation:

$$\frac{\cos(\alpha-3\theta)}{\cos^3\theta}=\frac{\sin(\alpha-3\theta)}{\sin^3\theta}=m$$

Solution: Expanding the given relations, we have

$$\cos\alpha\cos 3\theta+\sin\alpha\ \sin 3\theta=\frac{m}{4}(3\cos\theta+\cos 3\theta) \qquad (1)$$

$$-\cos\alpha\ \sin 3\theta+\sin\alpha\cos 3\theta=\frac{m}{4}(3\sin\theta-\sin 3\theta) \qquad (2)$$

Our approach is motivated by an attempt to generate $\cos 4\theta$ and $\sin 4\theta$ terms from (1) and (2), from which θ can then be eliminated. $(1)\times\cos 3\theta-(2)\times\sin 3\theta$ gives

$$\cos\alpha=\frac{m}{4}(3\cos\theta\cos 3\theta-3\sin\theta\sin 3\theta+1)=\frac{m}{4}(3\cos 4\theta+1)$$

$$\Rightarrow\quad 3\cos 4\theta=\frac{4\cos\alpha}{m}-1 \qquad (3)$$

$(1)\times\sin 3\theta+(2)\times\cos 3\theta$ similarly yields

$$3\sin 4\theta=\frac{4\sin\alpha}{m} \qquad (4)$$

$(3)^2+(4)^2$ eliminates θ:

$$\left(\frac{4\cos\alpha}{m}-1\right)^2+\left(\frac{4\sin\alpha}{m}\right)^2=9$$

$$\Rightarrow\quad m^2+m\cos\alpha-2=0$$

■

Example 11

A right angle is divided into three positive parts α,β,γ. Prove that for all the possible divisions, $\tan\alpha+\tan\beta+\tan\gamma>1+\tan\alpha\tan\beta\tan\gamma$.

Solution: The trick lies in manipulating the term

$$T=\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma$$

to

$$T=\frac{1}{\cos\alpha\ \cos\beta\cos\gamma} \qquad \text{(Verify)}$$

and then realizing that T will be minimum when the denominator $\cos\alpha\cos\beta\cos\gamma$ is maximum, which by symmetry occurs at

$$\alpha=\beta=\gamma=\frac{\pi}{6}\quad\Rightarrow\quad(\cos\alpha\cos\beta\cos\gamma)_{\max}=\frac{3\sqrt{3}}{8}$$

$$\Rightarrow\quad T_{\min}=\frac{8}{3\sqrt{3}}>1.$$

From this, the result follows. If you are not convinced with using symmetry arguments, here's a more rigorous approach:

$$T = \frac{1}{\cos\alpha\cos\beta\cos\gamma} = \frac{2}{(\cos(\beta+\gamma)+\cos(\beta-\gamma))\cos\alpha}$$
$$= \frac{2}{(\sin\alpha+\cos(\beta-\gamma))\cos\alpha}$$

Assuming α fixed for a moment, the minimum value of T for a given α occur when $\beta=\gamma$:

$$T_{\min}(\alpha) = \frac{2}{(\sin\alpha+1)\cos\alpha}$$

On the other hand, if we had assumed β fixed, the minimum value of T would occur for $\alpha=\gamma$. This proves that the overall minimum of T occurs when $\alpha=\beta=\gamma=\frac{\pi}{6}$. ■

Example 12

If $\alpha,\beta,\gamma\in(0,\frac{\pi}{2})$, prove that

$$\sin\alpha+\sin\beta+\sin\gamma > \sin(\alpha+\beta+\gamma)$$

Solution: The solution is in two easy steps, if we can recall that

Step I: $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\gamma+\alpha}{2}\right) \quad \text{(Verify)}$$

Step II: Since $\alpha,\beta,\gamma\in(0,\frac{\pi}{2})$, we have

$$\frac{\alpha+\beta}{2},\ \frac{\beta+\gamma}{2},\ \frac{\gamma+\alpha}{2} \in \left(0,\frac{\pi}{2}\right)$$

so their sines are positive.

From these two steps, the result follows. ■

Example 13

Prove that ΔABC is equilateral if and only if

$$\tan A+\tan B+\tan C = 3\sqrt{3}$$

Solution: This is a two-way implication of which one is very easy to prove. Namely, if ΔABC is equilateral, then

$$\tan A+\tan B+\tan C = \sqrt{3}+\sqrt{3}+\sqrt{3} = 3\sqrt{3}$$

The main part of this problem is proving the converse. We know that given the fact that $A+B+C=\pi$, the following identity holds true:

$$\tan A+\tan B+\tan C = \tan A\tan B\tan C$$

On the other hand, by the $\text{AM}-\text{GM}$ inequality,

$$\tan A+\tan B+\tan C \ge (\tan A\tan B\tan C)^{1/3}$$

The AM of $\tan A, \tan B, \tan C$ is $\sqrt{3}$, whereas the GM is $(3\sqrt{3})^{1/3}=\sqrt{3}$, *i.e.*, the same as the AM. This can happen only if

$$\tan A = \tan B = \tan C \Rightarrow A = B = C.$$

■

Example 14

Find the value of

(a) $S_1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$ (b) $S_2 = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}$

Solution: (a) $S_1 = \left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5}\right) + \tan^{-1}\frac{1}{8} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right) + \tan^{-1}\frac{1}{8}$

$$= \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} = \tan^{-1}\left\{\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right\} = \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

(b) $S_2 = \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}\right) + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = S_1 = \frac{\pi}{4}.$ ■

Example 15

Find a closed form expression for the product

$$P = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\cdots(2\cos 2^{n-1}\theta - 1).$$

Solution: We have to do some form of 'collapsing', a technique typical to such problems. This basically means that we've to introduce a term that will lead to a cascade of combinations of successive terms, as shown below, by multiplying and dividing P by the term $(2\cos\theta + 1)$

$$P = \frac{1}{2\cos\theta + 1}(4\cos^2\theta - 1)(2\cos 2\theta - 1)...$$

Now, since $(4\cos^2\theta - 1) = 2(1+\cos 2\theta) - 1 = 2\cos 2\theta + 1$, this term multiplies with the next in a similar way:

$$(2\cos 2\theta + 1)(2\cos 2\theta - 1) = 4\cos^2 2\theta - 1 = 2\cos 2^2\theta + 1$$

which multiplies similarly with the next term, and so on. Eventually, we have

$$S = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$$ ■

Example 16

Evaluate a closed form expression for the product.

$$P = \cos A\cos 2A\cos 2^2 A\cos 2^3 A\cdots\cos 2^{n-1} A$$

Solution: Once again, we have to introduce a term that causes a cascade multiplication of successive terms. That term, as should be evident, is $\sin A$.

$$P = \frac{1}{\sin A}(\sin A\cos A)\cos 2A\cos 2^2 A... = \frac{1}{2\sin A}(\sin 2A\cos 2A)\cos 2^2 A...$$

$$= \frac{1}{2^2\sin A}(\sin 2^2 A\cos 2^2 A)...$$

$$\vdots$$

$$= \frac{\sin 2^n A}{2^n\sin A}$$ ■

Example 17

Find a closed form expression for the series

$$S = \tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots + 2^{n-1}\tan 2^{n-1}\alpha$$

Solution: Yet again, we have to produce a cascade. Note that

$$\tan\alpha - \cot\alpha = \frac{\sin\alpha}{\cos\alpha} - \frac{\cos\alpha}{\sin\alpha} = -2\cot 2\alpha$$

Thus,

$$\begin{aligned} S - \cot\alpha &= (\tan\alpha - \cot\alpha) + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots \\ &= (2\tan 2\alpha - 2\cot 2\alpha) + 2^2\tan 2^2\alpha + \cdots \\ &= (2^2\tan 2^2\alpha - 2^2\cot 2^2\alpha) + \cdots \\ &= -2^n\cot 2^n\alpha \end{aligned}$$

$$\Rightarrow \quad S = \cot\alpha - 2^n\cot 2^n\alpha$$

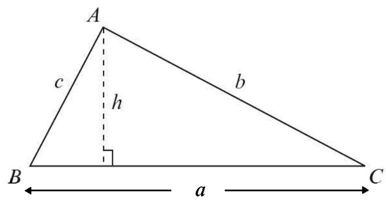
■

Example 18

In a triangle of base a, the ratio of the other two sides is r (<1).

(a) What is the maximum possible altitude of the triangle in terms of the given parameters?
(b) Find the vertical angle of the triangle in case the altitude has this maximum value.

Solution: (a)



Let $\frac{c}{b} = r$

$\Rightarrow \quad c = rb$

Assuming a fixed, what we've to do here is express h as a function of a, r and the angles of the triangle, and then find the maximum value that h can take. Note that

$$\Delta = \frac{1}{2}ah = \frac{1}{2}bc\sin A$$

$$\Rightarrow \quad h = \frac{bc\sin A}{a} = \frac{abc\sin A}{a^2} = \frac{abc\sin A(\sin^2 B - \sin^2 C)}{a^2(\sin^2 B - \sin^2 C)}$$

$$= \frac{abc\sin^2 A\sin(B-C)}{(b^2-c^2)\sin^2 A} \quad \text{(how?)}$$

$$= \frac{ar\sin(B-C)}{1-r^2}$$

$$\Rightarrow \quad h_{\max} = \frac{ar}{1-r^2}$$

(b) In this case, we have

$$B-C=\frac{\pi}{2} \quad \Rightarrow \quad B=\frac{\pi}{2}+C$$

$$\Rightarrow \quad \frac{c}{b}=\frac{\sin C}{\sin B}=\frac{\sin C}{\cos C}=\tan C=r \quad \Rightarrow \quad C=\tan^{-1} r$$

$$\Rightarrow \quad A=\pi-(B+C)=\frac{\pi}{2}-2C=\frac{\pi}{2}-2\tan^{-1} r$$

■

Example 19

Consider the following statements concerning ΔABC:

I. The sides a, b, c and Δ are rational.
II. $a, \tan\frac{B}{2}, \tan\frac{C}{2}$ are rational.
III. $a, \sin A, \sin B, \sin C$ are rational.

Show that I $\Rightarrow$ II $\Rightarrow$ III $\Rightarrow$ I.

Solution: Assuming I is true, then $s=\frac{a+b+c}{2}$ is rational. Since Δ is also rational,

$$\tan\frac{B}{2}=\frac{\Delta}{s(s-b)} \text{ and } \tan\frac{C}{2}=\frac{\Delta}{s(s-c)}$$

are also rational. Thus, I $\Rightarrow$ II. Now, assuming II, we see that

$$\sin B=\frac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}} \text{ and } \sin C=\frac{2\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}$$

are rational. Also,

$$\tan\frac{A}{2}=\cot\left(\frac{B+C}{2}\right)=\frac{1-\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}$$

becomes rational, which implies that $\sin A$ is rational. Thus, II $\Rightarrow$ III. Finally, assuming III, we have

$$b=a\frac{\sin B}{\sin A} \text{ and } c=a\frac{\sin C}{\sin A}$$

as rational, and further, $\Delta=\frac{1}{2}bc\sin A$ as rational. Thus, III $\Rightarrow$ I. ■

Example 20

Let h_1, h_2, h_3 be the lengths of the altitudes in a ΔABC. Prove that

(a) $\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}=\frac{1}{r}$ (b) $h_1\, h_2\, h_3=\frac{2\Delta^2}{R}$ (c) $\frac{\cos A}{h_1}+\frac{\cos B}{h_2}+\frac{\cos C}{h_3}=\frac{1}{R}$

Solution: Using the fact that

$$\Delta=\frac{1}{2}ah_1=\frac{1}{2}bh_2=\frac{1}{2}ch_3$$

these results are easy to justify.

(a) $\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}=\frac{a+b+c}{2\Delta}=\frac{s}{\Delta}=\frac{1}{r}$

(b) $h_1\, h_2\, h_3 = \dfrac{8\Delta^3}{abc} = \dfrac{8\Delta^3}{4R\Delta} = \dfrac{2\Delta^2}{R}$

(c) $\dfrac{\cos A}{h_1} + \dfrac{\cos B}{h_2} + \dfrac{\cos C}{h_3} = \dfrac{1}{2\Delta}(a\cos A + b\cos B + c\cos C) = \dfrac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C)$ (how?)

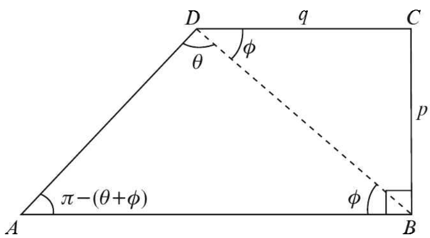
$= \dfrac{R}{2\Delta}(4\sin A \sin B \sin C) = \dfrac{2R}{\Delta}\cdot\dfrac{a}{2R}\cdot\dfrac{b}{2R}\cdot\dfrac{c}{2R} = \dfrac{1}{R}$ ■

Example 21

ABCD is a trapezium such that *AB* and *CD* are parallel and *CB* is perpendicular to them. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, show that

$$AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Solution:



Applying the sine rule to ΔABD, we have

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta+\phi)}$$

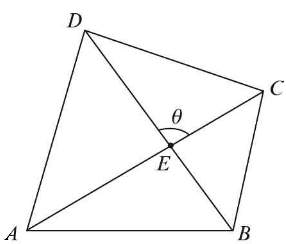
$$\Rightarrow \quad AB = \frac{BD\sin\theta}{\sin\theta\cos\phi + \cos\theta\sin\phi} = \frac{(\sqrt{p^2+q^2})\sin\theta}{\sin\theta\left(\dfrac{q}{\sqrt{p^2+q^2}}\right) + \cos\theta\left(\dfrac{p}{\sqrt{p^2+q^2}}\right)}$$

$$= \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

■

Example 22

Let *ABCD* be a quadrilateral, and let the angle between the diagonals be θ.



(a) Show that the area of $ABCD$ can be written as $\Delta = \frac{1}{2} AC \cdot BD \cdot \sin\theta$
(b) If $ABCD$ is a convex cyclic quadrilateral, show that

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Solution: (a) Note that

$$\text{area } (\Delta ACD) = \frac{1}{2} \cdot AC \cdot ED \sin\theta$$

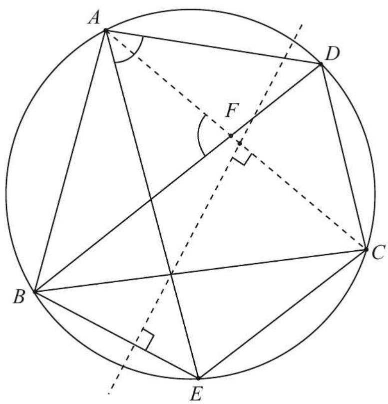
and

$$\text{area } (\Delta ABC) = \frac{1}{2} \cdot AC \cdot EB \sin(\pi - \theta) = \frac{1}{2} \cdot AC \cdot EB \sin\theta$$

so that

$$\Delta = \frac{1}{2} \cdot AC \cdot (ED + EB) \cdot \sin\theta = \frac{1}{2} AC \cdot BD \cdot \sin\theta$$

(b) We undertake a construction by reflecting B along the perpendicular bisector of the diagonal AC.



B is reflected to E, so that $ABEC$ becomes an isosceles trapezium, where

$$AC \parallel BE, AB = CE, AE = BC$$

Let us denote by $\angle(\text{arc})$ the angle subtended by an arc of the circle at the center of the circle

Note that

$$\begin{aligned}\angle EAD &= \frac{1}{2} \angle(ED) \\ &= \frac{1}{2} \angle(EC + CD) \\ &= \frac{1}{2} \angle(AB + CD) \quad \text{How?} \\ &= \angle ADB + \angle CAD \\ &= \angle AFB = \theta \text{ (say)}\end{aligned}$$

Make sure you are clear about how this result was obtained. Also note by symmetry that area ($ABCD$) = area ($AECD$).

$$\text{area } (ABCD) = \frac{1}{2} AC \cdot BD \cdot \sin\theta \quad \text{(by the result of part-(a))}$$

$$\begin{aligned}\text{area } (AECD) &= \text{area}(\Delta AED) + \text{area}(\Delta ECD) \\ &= \frac{1}{2} AE \cdot AD \cdot \sin\theta + \frac{1}{2} EC \cdot CD \cdot \sin(\angle ECD)\end{aligned}$$

But $\angle ECD = \pi - \theta$ due to $AECD$ being cyclic. Thus,

$$\begin{aligned}\text{area } (AECD) &= \frac{1}{2} \sin\theta \, (AE \cdot AD + EC \cdot CD) \\ &= \frac{1}{2} \sin\theta (AD \cdot BC + AB \cdot CD) \quad \text{(from the remarks in the figure)}\end{aligned}$$

Comparing the two areas, the required equality is established. ■

Example 23

Let α, β, γ be the lengths of the altitudes in ΔABC. Prove that

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\cot A+\cot B+\cot C}{\Delta}$$

Solution: Note that $\Delta=\frac{1}{2}a\alpha=\frac{1}{2}bc\sin A$

$$\Rightarrow \quad \frac{1}{\alpha}=\frac{a}{2\Delta} \quad \text{and} \quad \sin A=\frac{2\Delta}{bc} \quad \text{etc}$$

$$\Rightarrow \quad \text{LHS}=\frac{a^2+b^2+c^2}{4\Delta^2}$$

Now, we have

$$\text{RHS}=\frac{\cot A+\cot B+\cot C}{\Delta}=\frac{\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}}{\Delta}$$

$$=\frac{\frac{b^2+c^2-a^2}{(2bc)\cdot(\frac{2\Delta}{bc})}+\frac{c^2+a^2-b^2}{(2ca)\cdot(\frac{2\Delta}{ca})}+\frac{a^2+b^2-c^2}{(2ab)\cdot(\frac{2\Delta}{ab})}}{\Delta}$$

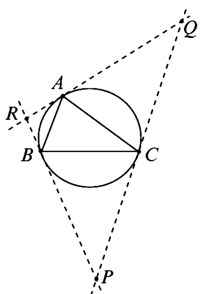
$$=\frac{a^2+b^2+c^2}{4\Delta^2}=\text{LHS}$$

■

Example 24

Consider a triangle ΔABC. Let S be the circumcircle of ΔABC. At each of the three points A, B, C, tangents are drawn to S to form another triangle, ΔPQR. Find the sides of ΔPQR.

Solution:



Let us evaluate the side PQ. Note that

$$\angle PCB=\angle PBC=\angle A \quad \text{(why?)}$$

so that

$$\angle P=\pi-2\angle A$$

Applying the sine rule in ΔPBC, we have

$$\frac{PB}{\sin A}=\frac{PC}{\sin A}=\frac{BC}{\sin 2A}$$

$$\Rightarrow \quad PC=\frac{BC\sin A}{\sin 2A}=\frac{a}{2\cos A}$$

Similarly, $QC=\dfrac{b}{2\cos B}$. Thus,

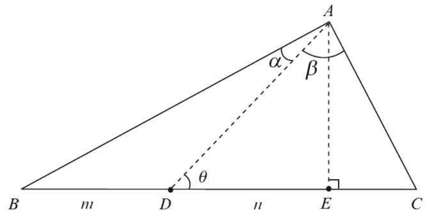
$$PQ=PC+CQ=\frac{a}{2\cos A}+\frac{b}{2\cos B}=\frac{1}{2}\frac{(a\cos B+b\cos A)}{\cos A\cos B}=\frac{\text{c}}{2\cos A\cos B}$$

Similar expressions will hold for QR and RP. ■

Example 25

In a ΔABC, if D is any point on BC such that $BD{:}DC=m:n$, $\angle BAD=\alpha$, $\angle CAD=\beta$, $\angle CDA=\theta$, prove that $(m+n)\cot\theta=m\cot\alpha-n\cot\beta=n\cot B-m\cot C$.

Solution:



Using the sine rule in ΔABD, we have

$$\frac{BD}{\sin\alpha} = \frac{AD}{\sin B} = \frac{AD}{\sin(\theta-\alpha)} \quad \text{(Why?)}$$

$$\Rightarrow \quad \frac{BD}{AD} = \frac{\sin\alpha}{\sin(\theta-\alpha)}$$

Similarly, $\quad \dfrac{AD}{DC} = \dfrac{\sin(\theta+\beta)}{\sin\beta}$ (Verify!)

Now, $\quad \dfrac{BD}{DC} = \dfrac{BD}{AD}\cdot\dfrac{AD}{DC} = \dfrac{m}{n} \quad \Rightarrow \quad \dfrac{\sin\alpha}{\sin(\theta-\alpha)}\cdot\dfrac{\sin(\theta+\beta)}{\sin\beta} = \dfrac{m}{n}$

Cross multiplying and simplifying yields

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

To prove the other equality, we need to find expressions for $\cot B$ and $\cot C$:

$$\cot B = \frac{BE}{AE} = \frac{BD+DE}{AE} = \frac{BD}{AE} + \cot\theta$$

$$\cot C = \frac{CE}{AE} = \frac{CD-DE}{AE} = \frac{CD}{AE} - \cot\theta$$

Now,

$$\frac{BD}{DC} = \frac{BD}{AE}\cdot\frac{AE}{DC} = \frac{m}{n} \quad \Rightarrow \quad \frac{(\cot B - \cot\theta)}{(\cot C + \cot\theta)} = \frac{m}{n}$$

Cross-multiplying and simplifying yields the desired result. ■

Example 26

Find the

(a) minimum value of $\frac{R}{r}$

(b) maximum value of $S = \frac{a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}}{a+b+c}$

Solution: (a) $\quad \dfrac{r}{R} = 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = 2\left\{\cos\dfrac{A-B}{2} - \cos\dfrac{A+B}{2}\right\}\sin\dfrac{C}{2}$

$$= 2\left\{\cos\left(\frac{A-B}{2}\right) - \cos\frac{C}{2}\right\}\sin\frac{C}{2} \le 2\left(1-\sin\frac{C}{2}\right)\sin\frac{C}{2} \le \frac{1}{2}$$

where the maximum holds when

$$\cos\left(\frac{A-B}{2}\right) = 1 \text{ and } \sin\frac{C}{2} = \frac{1}{2} \quad \Rightarrow \quad A = B \quad \Rightarrow \quad C = \frac{\pi}{3}$$

$$\Rightarrow \quad A = B = C = \frac{\pi}{3}$$

Thus, $\frac{R}{r}$ has a minimum value of 2, when the triangle is equilateral. This could have also been proven using symmetry arguments.

(b) $S = \dfrac{\frac{a}{2}(1+\cos A)+\frac{b}{2}(1+\cos B)+\frac{c}{2}(1+\cos C)}{a+b+c} = \dfrac{1}{2}+\dfrac{a\cos A+b\cos B+c\cos C}{a+b+c}$

Using $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C,$ this can be reduced to (verify)

$$S = \tfrac{1}{2}+\tfrac{r}{2R} \le 3/4 \quad \text{(from the result of part a)}$$

Therefore, the maximum value of S is $\frac{3}{4}$. ∎

Example 27

In a ΔABC, the sides are in AP and the greatest angle exceeds the least by $\frac{\pi}{2}$. Prove that the squares of two its sides will be proportional to the roots of the equation $x^2-16x+36=0$.

Solution: Let $\angle C$ be the greatest angle and $\angle A$ the least, so that *a*, *b*, *c* is an increasing AP. Thus, we have

$$\angle C = \angle A+\frac{\pi}{2}, \; 2b = a+c$$

Using the sine rule on the second expression,

$$2\sin B = \sin A+\sin C = \sin A+\cos A$$

Squaring, and noting that $2\angle A = \frac{\pi}{2}-\angle B,$ we have $\cos B = \frac{3}{4}$ (verify this). Since $2\angle A = \frac{\pi}{2}-\angle B,$

$$\cos 2A = \sin B \;\Rightarrow\; 1-2\sin^2 A = \sin B = \sqrt{1-\cos^2 B} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow\; \sin^2 A = \frac{8-2\sqrt{7}}{16} \;\Rightarrow\; \sin^2 C = \cos^2 A = \frac{8+2\sqrt{7}}{16}$$

$$\Rightarrow\; c^2 : a^2 = 8+2\sqrt{7} : 8-2\sqrt{7}$$

If we consider the quadratic $x^2-16x+36=0$, it roots α and β are given by

$$\alpha, \beta = \frac{16\pm 4\sqrt{7}}{2} = 8\pm 2\sqrt{7} \;\Rightarrow\; \alpha : \beta = 8+2\sqrt{7} : 8-2\sqrt{7}$$

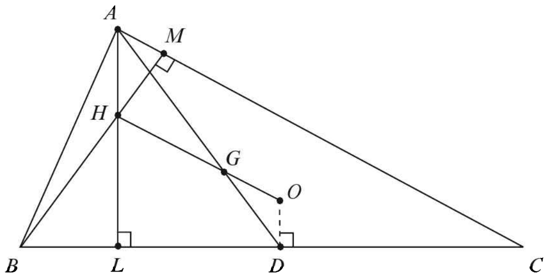
$$\Rightarrow\; c^2 : a^2 = \alpha : \beta$$

∎

Example 28

Let G, H, O denote the centroid, orthocenter and circumcentre in ΔABC. Prove that $OG : GH = 1:2$

Solution:



Note that $OD = R\cos A$ (how? what is the value of angle BOD). We can also show that $HA = 2R\cos A$ (this is an important step in the solution and the reader is urged to justify this). Therefore,

$$HA = 2OD$$

Also, since G is the centroid,

$$AG = 2GD$$

Thus, ΔAGH and ΔDGO are similar with the sides ratio 2 : 1. This implies that $OG : GH = 1 : 2$. Note that we have not proved explicitly that O, G and H are collinear. This fact is not obvious; it requires a rigorous proof. ■

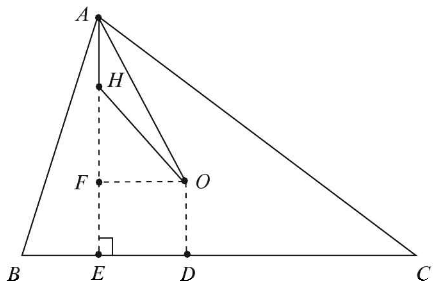
Example 29

In a given ΔABC, find the distance between

(a) circumcenter and orthocenter (b) circumcenter and incenter
(c) incenter and orthocenter (d) circumcenter and centroid

in terms of the relevant parameters.

Solution: (a)



Note that

$$OA = R$$

$$\angle BAH = \frac{\pi}{2} - \angle B$$

$$\angle CAO = \frac{1}{2}(\pi - 2\angle B) \quad \text{(How?)}$$

$$= \frac{\pi}{2} - \angle B$$

$$\Rightarrow \quad \angle HAO = \angle A - 2\left(\frac{\pi}{2} - \angle B\right) = \angle A + 2\angle B - \pi$$

$$= \angle A + 2\angle B - (\angle A + \angle B + \angle C)$$

$$= \angle B - \angle C$$

Also, recall that $AH = 2R\cos A$. Now, applying the cosine rule in ΔOAH, we have

$$OH^2 = AH^2 + OA^2 - 2(AH)(OA)\cos(B - C)$$

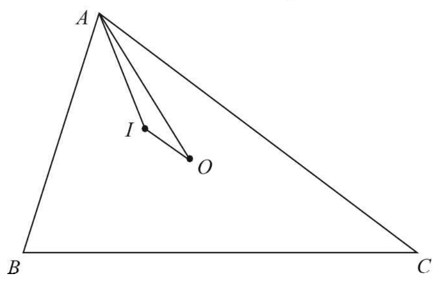
$$= 4R^2\cos^2 A + R^2 - 4R^2\cos A\cos(B - C)$$

$$= R^2(1 + 4\cos A(\cos A + \cos(B - C)))$$

$$= R^2(1 - 8\cos A\cos B\cos C) \quad \text{(Verify!)}$$

$$\Rightarrow \quad OH = R\sqrt{1 - 8\cos A\cos B\cos C}$$

(b)



Note that

$$\angle BAI = \frac{1}{2}\angle A$$

$$\angle CAO = \frac{1}{2}(\pi - 2\angle B) = \frac{\pi}{2} - \angle B$$

$$OA = R$$

$$IA = r\ \text{cosec}\frac{A}{2}$$

We have $\angle OAI = \angle A - \left(\frac{1}{2}\angle A + \frac{\pi}{2} - \angle B\right) = \left|\frac{(\angle B - \angle C)}{2}\right|$

Using the cosine rule in ΔOAI, we have

$$OI^2 = R^2 + r^2 \operatorname{cosec}^2 \frac{A}{2} - 2Rr \operatorname{cosec} \frac{A}{2} \cos\left(\frac{C-B}{2}\right)$$

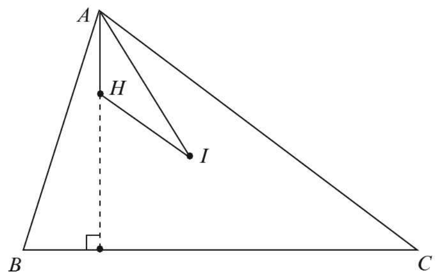
Using $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, we have

$$OI^2 = R^2 + 4R^2 \sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

Upon simplification, this reduces to

$$OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = R\sqrt{1 - \frac{2r}{R}} = \sqrt{R^2 - 2Rr}$$

(c)



Once again,

$$\angle HAI = \angle A - (\angle BAH + \angle CAI)$$

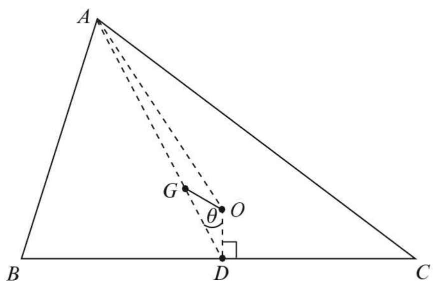
$$= \angle A - \left(\frac{\pi}{2} - \angle B + \frac{\angle A}{2}\right)$$

$$= \frac{\angle C - \angle B}{2}$$

Also, using $HA = 2R\cos A$, $IA = r \operatorname{cosec}\frac{A}{2} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$ and applying the cosine rule to ΔIAH and simplifying, we have

$$IH = \sqrt{2r^2 - 4R^2\cos A\cos B\cos C} \quad \text{(Verify!)}$$

(d)



Note that

$$OA = R, OD = R\cos A$$

$$AD^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

$$AG : GD = 2 : 1$$

In ΔOAD, we have the three sides, and we know the ratio in which G divides AD. To find OG, we apply the cosine rule, once on ΔODG and once on $\angle ODA$:

$$\cos\theta = \frac{OD^2 + GD^2 - OG^2}{2OD \cdot DG} = \frac{OD^2 + AD^2 - OA^2}{2\,OD \cdot AD}$$

From this equality, we can separate OG^2:

$$\frac{R^2\cos^2 A + \frac{AD^2}{9} - OG^2}{2R\cos A \cdot \frac{AD}{9}} = \frac{R^2\cos^2 A + \frac{2b^2 + 2c^2 - a^2}{4} - R^2}{2R\cos A \cdot AD}$$

Cross-multiplying and simplifying, we have

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$ ■

Example 30

Solve for x : $\sin^2 x + \frac{\sin^2 3x}{4} = \sin x \sin^2 3x$

Solution:

$$\sin^2 x - \sin x \sin^2 3x + \frac{\sin^2 3x}{4} = 0$$

$$\Rightarrow \quad \left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{\sin^2 3x}{4} - \frac{\sin^4 3x}{4} = 0 \quad \text{(How?)}$$

$$\Rightarrow \quad \left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \left(\frac{\sin 6x}{4}\right)^2 = 0$$

$$\Rightarrow \quad \sin x = \frac{1}{2}\sin^2 3x \quad \text{and} \quad \sin 6x = 0$$

$$\Rightarrow \quad 6x = n\pi, n \in \mathbb{Z} \quad \Rightarrow \quad x = n\pi/6, n \in \mathbb{Z}$$

If $x = \frac{n\pi}{6}$, then $3x = \frac{n\pi}{2}$. Thus, we have

$$\sin 3x = \begin{cases} 0, \text{ if } n \text{ is even} \\ 1, \text{ if } n \text{ is odd} \end{cases} \quad \Rightarrow \quad \sin x = 0 \text{ or } \frac{1}{2}$$

$$\Rightarrow \quad x = n\pi \text{ or } n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$$

This is the required solution. Notice that the solution set is infinite, as can be expected. ■

Example 31

Solve for α and β:

$$12\sin\alpha + 5\cos\alpha = 2\beta^2 - 8\beta + 21$$

Solution: The LHS can be written as

$$\text{LHS} = 13\left(\frac{12}{13}\sin\alpha + \frac{5}{13}\cos\alpha\right) = 13\sin(\alpha + \phi), \text{ where } \tan\phi = \frac{5}{12}$$

$$\Rightarrow \quad (\text{LHS})_{\max} = 13$$

The RHS can be written as

$$\text{RHS} = 2(\beta - 2)^2 + 13 \geq 13$$

$$\Rightarrow \quad (\text{RHS})_{\min} = 13$$

The only way the two can be equal are if both are equal to 13:

(i) $\sin(\alpha + \phi) = 1 \;\Rightarrow\; \alpha + \phi = 2n\pi + \frac{\pi}{2} \;\Rightarrow\; \alpha = 2n\pi + \frac{\pi}{2} - \phi$ where $\phi = \tan^{-1}\frac{5}{12}$

(ii) RHS $= 13 \;\Rightarrow\; \beta = 2$ ■

Example 32

Solve for θ: $\cos 4\theta + \sin 5\theta = 2$

Solution: It should be immediately evident that since the maximum value of both the sin and cos functions is 1, the given equality can only hold if

$$\cos 4\theta = 1 \quad \text{and} \quad \sin 5\theta = 1$$

$$\Rightarrow \quad 4\theta = 2n\pi \quad \text{and} \quad 5\theta = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{n\pi}{2} \quad \text{and} \quad \theta = \frac{2m\pi}{5} + \frac{\pi}{10} \quad \text{for } m, n \in \mathbb{Z}$$

Therefore, we have to find m, n such that

$$\frac{n\pi}{2} = \frac{2m\pi}{5} + \frac{\pi}{10}$$

$$\Rightarrow \quad 5n = 4m + 1$$

$$\Rightarrow \quad 5(n-1) = 4(m-1) \quad \text{(say)}$$

$$\Rightarrow \quad \frac{n-1}{4} = \frac{m-1}{5} = \lambda \quad (\text{say; } \lambda \in \mathbb{Z})$$

$$\Rightarrow \quad n = 4\lambda + 1, m = 5\lambda + 1, \lambda \in \mathbb{Z}$$

The solutions to the given equation can thus be written as

$$\theta = (4\lambda + 1)\frac{\pi}{2}, \quad \lambda \in \mathbb{Z}$$

■

Example 33

Solve for x: $\sec x + \operatorname{cosec} x = c$

Solution: Rearranging to $\sin x$ and $\cos x$, we have:

$$\sin x + \cos x = c \sin x \cos x$$

Squaring both sides, we obtain

$$1 + \sin 2x = \frac{c^2}{4}\sin^2 2x \quad \Rightarrow \quad \sin 2x = \frac{2 \pm 2\sqrt{1+c^2}}{c^2}$$

For a real solution to exist,

$$\left|\frac{2 \pm 2\sqrt{1+c^2}}{c^2}\right| \in [-1, 1]$$

$$\Rightarrow \quad c \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, \infty) \quad \text{(Verify!)}$$

Thus, given that c lies in the appropriate interval, the solution is

$$x = \frac{1}{2}\sin^{-1}\left(\frac{2 \pm 2\sqrt{1+c^2}}{c^2}\right) + n\pi, \quad n \in \mathbb{Z}$$

■

Trigonometry

PART-C: Advanced Problems

P1. Let A and B be angles such that

$$\sqrt{2}\cos A = \cos^3 B + \cos B \tag{1}$$
$$\sqrt{2}\sin A = -\sin^3 B + \sin B \tag{2}$$

The value of $\sin(A-B)$ can be

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{3}$

P2. Let θ_1 and θ_2 be two angles such that

$$\sin\theta_1 + \sin\theta_2 = \frac{1}{\sqrt{2}}$$

$$\cos\theta_1 + \cos\theta_2 = \sqrt{\frac{3}{2}}$$

The value of $\sin(\theta_1 + \theta_2)$ is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{4}$ (D) $\frac{\sqrt{3}}{2}$

P3. The value of $\cot 7\frac{1}{2}°$ is

(A) $\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (B) $-\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

(C) $\sqrt{2}-\sqrt{3}+\sqrt{4}+\sqrt{6}$ (D) $\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{6}$

P4. Let $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$. What is the value of the following expressions?

(a) $\cos^2\alpha + \cos^2\alpha + \cos^2\gamma$

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) None of these

(b) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) None of these

P5. Two circles of radii a and b intersect at angle θ. What is the length of the common chord?

(A) $\dfrac{ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$ (B) $\dfrac{ab\cos\theta}{\sqrt{a^2+b^2+2ab\sin\theta}}$

(C) $\dfrac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$ (D) $\dfrac{2ab\cos\theta}{\sqrt{a^2+b^2+2ab\sin\theta}}$

P6. Tangents are drawn to the incircle of ΔABC which are parallel to its sides. Let x, y, z be the lengths of the tangents opposite to a, b, c respectively. What is the value of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2

P7. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)}=\frac{1-m}{1+m}$, what is the value of $P=\tan(\frac{\pi}{4}-\theta)\tan(\frac{\pi}{4}-\alpha)$?

(A) m (B) $m+1$ (C) $m-1$ (D) $2m$

P8. Let $a, b, c, d \in [0, \pi]$ such that

$$\sin a + 7\sin b = 4(\sin c + 2\sin d)$$

$$\cos a + 7\cos b = 4(\cos c + 2\cos d)$$

What is the value of $\frac{\cos(b-c)}{\cos(a-d)}$?

(A) $\dfrac{1}{7}$ (B) $\dfrac{2}{7}$ (C) $\dfrac{3}{7}$ (D) $\dfrac{4}{7}$

P9. (a) The value of $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}$ is

(A) $\sqrt{7}$ (B) $\sqrt{14}$ (C) $\sqrt{21}$ (D) $\sqrt{28}$

(b) The value of $(\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7})(\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7})$ is

(A) 70 (B) 105 (C) 140 (D) 175

P10. What is the minimum value of $\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha}$ for $\alpha, \beta \neq \frac{k\pi}{2}, k \in \mathbb{Z}$?

(A) 2 (B) 4 (C) 8 (D) 16

P11. The value of

$$S=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

is

(A) $\dfrac{1}{16}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{1}{4}$

P12. What is the value of the following expression?

$$P=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

(A) $\dfrac{1}{256}$ (B) $\dfrac{1}{128}$ (C) $\dfrac{1}{64}$ (D) $\dfrac{1}{32}$

P13. What is the value of $S = 2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8}$?

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

P14. What is the sum to n terms of the following series?

$$S = \frac{1}{\sin\theta\sin 2\theta} + \frac{1}{\sin 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\sin 4\theta} + \cdots$$

(A) $\dfrac{\cot\theta - \cot(n+1)\theta}{\cos\theta}$ (B) $\dfrac{\cot\theta - \cot(n+1)\theta}{\sin\theta}$

(C) $\dfrac{\tan\theta - \tan(n+1)\theta}{\cos\theta}$ (D) $\dfrac{\tan\theta - \tan(n+1)\theta}{\sin\theta}$

P15. What are the expressions for the sums of following series?

(a) $S_1 = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+(n-1)\beta)$

(A) $\dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+(n-1)\beta}{2}\right)$ (B) $\dfrac{2\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+(n-1)\beta}{2}\right)$

(C) $\dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+n\beta}{2}\right)$ (D) $\dfrac{2\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+n\beta}{2}\right)$

(b) $S_2 = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$

(A) $\dfrac{2\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+n\beta}{2}\right)$ (B) $\dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\dfrac{2\alpha+(n-1)\beta}{2}\right)$

(C) $\dfrac{2\cos\frac{n\beta}{2}}{\cos\frac{\beta}{2}}\sin\left(\dfrac{2\alpha+n\beta}{2}\right)$ (D) $\dfrac{\cos\frac{n\beta}{2}}{\cos\frac{\beta}{2}}\cos\left(\dfrac{2\alpha+(n-1)\beta}{2}\right)$

P16. What is the sum of the series $S = \sqrt{1+\cos\alpha} + \sqrt{1+\cos 2\alpha} + \sqrt{1+\cos 3\alpha} + \cdots$ to n terms?

(A) $\dfrac{1}{\sqrt{2}}\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\left((n+1)\dfrac{\alpha}{4}\right)$ (B) $\dfrac{1}{\sqrt{2}}\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\sin\left((n+1)\dfrac{\alpha}{4}\right)$

(C) $\sqrt{2}\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\left((n+1)\dfrac{\alpha}{4}\right)$ (D) $\sqrt{2}\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\sin\left((n+1)\dfrac{\alpha}{4}\right)$

P17. What is the sum of the following series?

$$S = \cos^4\theta + \cos^4\left(\theta + \frac{2\pi}{n}\right) + \cos^4\left(\theta + \frac{4\pi}{n}\right) + \cdots \text{ to } n \text{ terms}$$

(A) $\dfrac{n}{8}$ (B) $\dfrac{3n}{8}$ (C) $\dfrac{5n}{8}$ (D) $\dfrac{7n}{8}$

P18. For $a = \frac{2\pi}{1999}$, what is the value of $\cos a\cos 2a \ldots \cos 999a$?

(A) $\dfrac{1}{2^{998}}$ (B) $\dfrac{1}{2^{999}}$ (C) $\dfrac{1}{2^{1000}}$ (D) $\dfrac{1}{2^{1001}}$

P19. What is the smallest positive integer n for which the following relation is satisfied?

$$\frac{1}{\sin 45^\circ \sin 46^\circ}+\frac{1}{\sin 47^\circ \sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ \sin 134^\circ}=\frac{1}{\sin n^\circ}$$

(A) 1 (B) 2 (C) 3 (D) Greater than 3

P20. If $(1+\tan 1^\circ)(1+\tan 2^\circ)\cdots(1+\tan 45^\circ)=2^n$, the value of n is

(A) 22 (B) 23 (C) 24 (D) 25

P21. Let $x_0=2003$, and let $x_{n+1}=\frac{1+x_n}{1-x_n}$ for $n\geq 1$. The value of x_{2004} is

(A) 2003 (B) 2004 (C) 2005 (D) 2006

P22. What is the value of the sum $S=\sum_{r=1}^{\infty}\tan^{-1}(\frac{2r}{r^4+r^2+2})$?

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

P23. In ΔABC, $\frac{a}{b}=2+\sqrt{3}$ and $\angle C=60^\circ$. The value of $\angle A$ in degrees is

(A) 60 (B) 75 (C) 90 (D) 105

P24. Let there be a triangle ABC such that

$$3\sin A+4\cos B=6$$
$$4\sin B+3\cos A=1$$

The value of $\angle C$ in degrees is

(A) 30 (B) 60 (C) 120 (D) 150

P25. In a ΔABC, suppose that $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in harmonic progression.

(a) What is the minimum possible value of $\cot\frac{B}{2}$?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

(b) What is the maximum possible value of angle B?

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

P26. In a ΔABC, $a, c, \angle A$ are fixed. The third side may have two possible values, say b_1 and b_2. It is given that $b_2=2b_1$. The value of $\frac{c}{a}\sqrt{1+8\sin^2 A}$ is

(A) 1 (B) 2 (C) 3 (D) 4

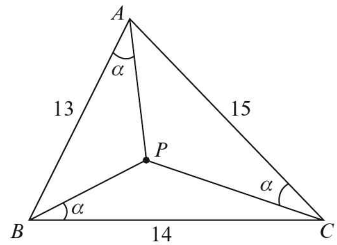
P27. In triangle $ABC, \angle ABC=45^\circ$. Point D is on segment BC such that $2|BD|=|CD|$ and $\angle DAB=15^\circ$. The value of $\angle ACB$ in degrees is

(A) 30 (B) 45 (C) 60 (D) 75

P28. In the trapezoid $ABCD$ (shown in the figure below), $AB \parallel CD$, $|AB| = 4$ and $|CD| = 10$. Suppose that lines AC and BD intersect at right angles, and that lines BC and DA, when extended to point Q, form an angle of 45°. What is the area of trapezoid $ABCD$?

(A) $\frac{130}{3}$ (B) $\frac{140}{3}$ (C) $\frac{154}{3}$ (D) $\frac{170}{3}$

P29. Refer to the figure below. If $\tan\alpha = \frac{m}{n}$, where m, n are relatively prime positive integers, the value of $m + n$ is



(A) 429 (B) 463 (C) 485 (D) 501

P30. Let S be the sum of all x in the interval $[0, 2\pi]$ such that

$$3\cot^2 x + 8\cot x + 3 = 0$$

The value of $\frac{S}{\pi}$ is

(A) 3 (B) 4 (C) 5 (D) 6

SUBJECTIVE TYPE EXAMPLES

P31. Let a and b be non-negative real numbers.

(a) Prove that there is a real number x such that $\sin x + a\cos x = b$ if and only if $a^2 - b^2 + 1 \geq 0$.

(b) If $\sin x + a\cos x = b$, express $|a\sin x - \cos x|$ in terms of a and b.

P32. Is the following inequality true?

$$\left(1+\frac{a}{\sin x}\right)\left(1+\frac{b}{\cos x}\right) \geq (1+\sqrt{2ab})^2$$

P33. Prove that for $a, b, c \in \mathbb{R}$,

$$(ab+bc+ca-1)^2 \leq (1+a^2)\,(1+b^2)\,(1+c^2)$$

P34. Prove that

$$(\sin x + a\cos x)\,(\sin x + b\cos x) \leq 1+\left(\frac{a+b}{2}\right)^2$$

P35. Let x, y, z be positive real numbers such that $x+y+z=1$. What is the minimum value of $\frac{1}{x}+\frac{4}{y}+\frac{9}{z}$?

(A) 18 (B) 24 (C) 30 (D) 36

P36. A single stream of cars, each of width a and exactly in line, is passing along a straight road of breadth b, with uniform speed v. The gap between successive cars is c.

(a) A squirrel crosses the road safely at a constant speed u in a straight line, making an angle θ with the direction of the traffic. Find the minimum possible value of u.

(b) If the squirrel chooses θ and u so that it crosses the road at the least possible (constant) speed, find the time it takes to cross the road.

P37. Prove that a ΔABC is isosceles if and only if

$$a\cos B + b\cos C + c\cos A = \frac{a+b+c}{2}$$

P38. In ΔABC, prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1$$

P39. Evaluate the sum of the following series:

$$S=\tan^{-1}\frac{1}{2\cdot 1^2}+\tan^{-1}\frac{1}{2\cdot 2^2}+\tan^{-1}\frac{1}{2\cdot 3^2}+\cdots\infty$$

P40. Find the sum $S=\sum_{r=1}^{\infty}\cot^{-1}(2^{r+1}+\frac{1}{2^r})$.

P41. Find the sum of the series

$$S=\sin\theta\sec 3\theta+\sin 3\theta\sec 3^2\theta+\sin 3^2\theta\sec 3^3\theta+\cdots \text{ to } n \text{ terms}$$

P42. Find the average of the following numbers:

$$2\sin 2^\circ,\ 4\sin 4^\circ,\ 6\sin 6^\circ,\ \ldots,\ 180\sin 180^\circ$$

P43. Three circles whose radii are a, b, c touch each other externally, and the tangents at their points of contact meet in a point. Find the distance of this point from any of the points of contact.

P44. In $\Delta ABC, \sin A + \sin B + \sin C \le 1$. Prove that

$$\min\{A+B,\ B+C,\ C+A\} < 30°$$

P45. In a ΔABC, suppose that

$$\cos A \cos B \cos C = \frac{\sqrt{3}-1}{8},\ \sin A \sin B \sin C = \frac{3+\sqrt{3}}{8}$$

Find the angles of the triangle.

P46. Let O be a point inside ΔABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$. Prove the following assertions:

(a) $\cot\omega = \cot A + \cot B + \cot C$

(b) $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

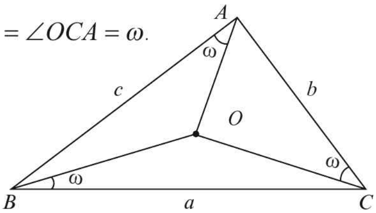
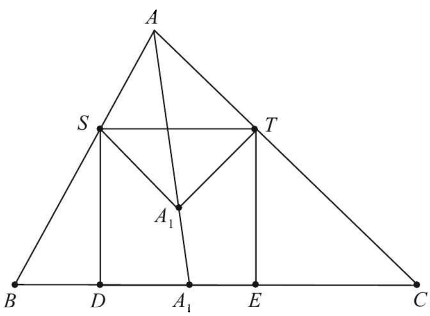


Diagram not to scale

P47. Suppose that the medians of a ΔABC make angles α, β, γ with each other. Find the value of

$$S = \cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C$$

P48. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC (as shown in the figure below). Thus, one of the two remaining vertices of the square lies on side AB and the other on side AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

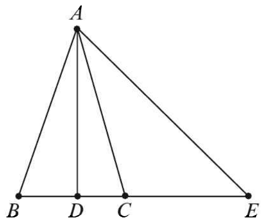


P49. If I_n is the area of an n sided regular polygon inscribed in a circle of unit radius and O_n is the area of the polygon circumscribing the given circle, find the value of

$$\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

in terms of I_n.

P50. In triangle ABC, $|AB| = |AC|$ (as shown in the figure below). Points D and E lie on ray BC such that $|BD| = |DC|$ and $|BE| > |CE|$. Suppose that $\tan \angle EAC$, $\tan \angle EAD$ and $\tan \angle EAB$ form a geometric progression, and that $\cot \angle DAE$, $\cot \angle CAE$ and $\cot \angle DAB$ form an arithmetic progression. If $|AE| = 10$, evaluate the area of triangle ABC.



P51. Solve for $x : |\cos x - 2\sin 2x - \cos 3x| = 1 - 2\sin x - \cos 2x$

P52. For what values of λ does the equation

$$\frac{\lambda \cos 2x}{2\cos 2x - 1} = \frac{\lambda + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x}$$

have real solutions?

P53. Let $S = \{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$ and $T = \{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}$. If S and T are permutations of each other, find α.

P54. Find all pairs (x, y) of real numbers $(x \in (0, \frac{\pi}{2}))$ such that

$$\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} = \sin 2x$$

P55. Find the smallest positive number p for which the equation $\cos(p\sin x) = \sin(p\cos x)$ has a solution for $x \in [0, 2\pi]$.

Trigonometry

PART-D: Solutions to Advanced Problems

S1. We have $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \frac{1}{\sqrt{2}}(-\sin^3 B + \sin B)\cos B - \frac{1}{\sqrt{2}}(\cos^3 B + \cos B)\sin B$$

$$= \frac{-1}{2\sqrt{2}}\sin 2B \quad \text{(Verify)} \tag{1}$$

By $(1)^2 + (2)^2$, we have

$$2 = \{\cos^6 B + \sin^6 B\} + \cos^2 B + \sin^2 B + \{2(\cos^4 B - \sin^4 B)\}$$

$$= \{(\cos^2 B + \sin^2 B)^3 - 3\cos^2 B\sin^2 B(\cos^2 B + \sin^2 B)\} + 1$$

$$+ 2\{\cos^2 B - \sin^2 B\}$$

$$= 1 - 3\cos^2 B\sin^2 B + 1 + 2\cos 2B$$

$$\Rightarrow \quad 0 = -\frac{3}{4}\sin^2 2B + 2\cos 2B$$

$$\Rightarrow \quad 8\cos 2B = 3\sin^2 2B = 3(1-\cos^2 2B)$$

$$\Rightarrow \quad \cos 2B = \frac{1}{3} \quad \text{(Verify)}$$

$$\Rightarrow \quad \sin 2B = \pm\frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \sin(A-B) = \pm\frac{1}{3} \quad \text{from (3)}$$

Thus, the correct options are (C) and (D).

S2. Square and add the two given relations to obtain $\cos(\theta_1 - \theta_2) = 0$. Multiply the two results to obtain

$$2\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2) + 2\sin(\theta_1+\theta_2) = \sqrt{3}$$

Thus,

$$\sin(\theta_1+\theta_2) = \frac{\sqrt{3}}{2}$$

Alternatively, take $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$. Thus,

$$\arg(z_1 + z_2) = \frac{\theta_1 + \theta_2}{2} \quad \text{since } |z_1| = |z_2| = 1$$

Now, we have $z_1 + z_2 = \sqrt{2}\, e^{i\pi/6}$ (how?)

$$\Rightarrow \quad \theta_1 + \theta_2 = \frac{\pi}{3} \quad \Rightarrow \quad \sin(\theta_1 + \theta_2) = \frac{\sqrt{3}}{2}.$$

The correct option is (D).

S3. $S = \cot 7\dfrac{1^\circ}{2} = \dfrac{\cos 7\frac{1}{2}^\circ}{\sin 7\frac{1}{2}^\circ}$

Multiplying both the numerator and denominator by $2\cos 7\frac{1}{2}^\circ$, we have

$$S = \frac{2\cos^2 7\frac{1}{2}^\circ}{2\sin 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ} = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$$

Now,

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\Rightarrow \quad S = \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6} \text{ upon simplification}$$

The correct option is (A).

S4. It should be evident to you that if we are able to evaluate part (a), the answer to part (b) is automatically determined since $\cos^2\alpha = 1 - \sin^2\alpha$, and so on. We proceed to evaluate part (a):

$$\cos\alpha + \cos\beta = -\cos\gamma \tag{1}$$

$$\sin\alpha + \sin\beta = -\sin\gamma \tag{2}$$

Square and add these:

$$2 + 2\cos(\alpha - \beta) = 1 \quad \Rightarrow \quad \cos(\alpha - \beta) = -\frac{1}{2}$$

Similarly,

$$\cos(\beta - \gamma) = -\frac{1}{2}, \quad \cos(\gamma - \alpha) = -\frac{1}{2}$$

Now, if we square (1) and (2) and subtract, we have

$$\cos 2\alpha + \cos 2\beta + 2\cos(\alpha + \beta) = \cos 2\gamma$$

$$\Rightarrow \quad 2\cos(\alpha + \beta)\cos(\alpha - \beta) + 2\cos(\alpha + \beta) = \cos 2\gamma$$

Since $\cos(\alpha-\beta)=-\frac{1}{2}$, we have

$$\cos(\alpha+\beta)=\cos 2\gamma \tag{3}$$

and similarly, two other relations. Now,

$$(\cos\alpha+\cos\beta+\cos\gamma)^2-(\sin\alpha+\sin\beta+\sin\gamma)^2=0$$

$$\Rightarrow \quad \cos 2\alpha+\cos 2\beta+\cos 2\gamma+2[\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)]=0$$

Using (3), this reduces to

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$$

Since $\cos 2\alpha=2\cos^2\alpha-1$ and so on, we finally have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$$

This further implies that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}$$

Thus, the correct option is (C) for both the parts.

Alternative: An equivalent solution can be developed very effectively using complex numbers. Recall the Euler's form of a complex number:

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Using the Euler's form, the given relations reduce to

$$e^{i\alpha}+e^{i\beta}+e^{i\gamma}=0$$

$$e^{-i\alpha}+e^{-i\beta}+e^{-i\gamma}=0$$

The solution is now 3 steps away:

Step 1 $\begin{matrix} e^{i\alpha}+e^{i\beta}=-e^{i\gamma} \\ e^{-i\alpha}+e^{-i\beta}=-e^{-i\gamma}\end{matrix} \xRightarrow{\text{Multiply}} e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}=-1 \qquad (1)$

Step 2 $e^{i\alpha}+e^{i\beta}=-e^{i\gamma} \xRightarrow{\text{Square}} e^{i2\alpha}+e^{i2\beta}+2e^{i(\alpha+\beta)}=e^{2i\gamma}$

$$\Rightarrow \quad e^{i(\alpha+\beta)}\left[e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}\right]+2e^{i(\alpha+\beta)}=e^{2i\gamma} \tag{2}$$

$$\text{Using(1),} \Rightarrow \quad e^{i(\alpha+\beta)}=e^{2i\gamma}$$

Step 3 $e^{i\alpha}+e^{i\beta}+e^{i\gamma}=0 \xRightarrow{\text{Square}} e^{i2\alpha}+e^{i2\beta}+e^{i2\gamma}+2[e^{i(\alpha+\beta)}+e^{i(\beta+\gamma)}+e^{i(\gamma+\alpha)}]=0$

Using (2), this reduces to

$$e^{i2\alpha}+e^{i2\beta}+e^{i2\gamma}=0$$

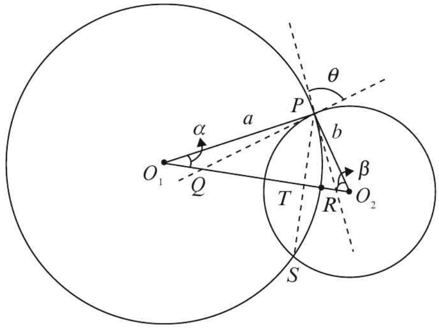
Finally, we take the real parts on both the sides above:

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

$$\Rightarrow \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{3}{2}$$

For students comfortable with complex numbers, many of the preceding steps can be done mentally.

S5.



We need to find the length PS

Note that

$\angle QPR = \theta$

$\angle QPO_2 = \angle O_1PR = \frac{\pi}{2}$

We have $\angle O_1PQ = \frac{\pi}{2} - \theta, \angle O_2PR = \frac{\pi}{2} - \theta$, so that

$$\angle O_1PO_2 = \left(\frac{\pi}{2} - \theta\right) + \theta + \left(\frac{\pi}{2} - \theta\right) = \pi - \theta$$

Applying the cosine rule in ΔO_1PO_2 yields

$$O_1O_2^2 = a^2 + b^2 + 2ab\cos\theta$$

Now, if Δ is the area of ΔO_1PO_2, then

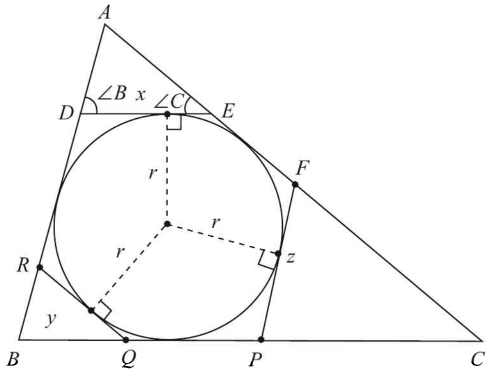
$$\Delta = \frac{1}{2}(O_1 O_2)\,(PT) = \frac{1}{2}ab\sin\theta$$

$$\Rightarrow \quad PT = \frac{ab\sin\theta}{O_1O_2}$$

$$\Rightarrow \quad PS = 2PT = \frac{2ab\sin\theta}{O_1O_2} = \frac{2ab\sin\theta}{\sqrt{a^2 + b^2 + 2ab\cos\theta}}$$

Thus, the correct option is (C).

S6.



In ΔADE

$$\frac{x}{\sin A} = \frac{AE}{\sin B} = \frac{AD}{\sin C}$$

$$\Rightarrow \quad AD = \frac{x\sin C}{\sin A} = \frac{cx}{a}$$

and $AE = \dfrac{bx}{a}$

Now, observe that the incircle of ΔABC is the ex-circle opposite to A for ΔADE. The semi-perimeter of ΔADE is

$$s_{\Delta ADE} = \frac{1}{2}\left(x + \frac{bx}{a} + \frac{cx}{a}\right) = \frac{sx}{a}$$

$$\Rightarrow \quad r = s_{\Delta ADE}\tan\frac{A}{2} = \frac{sx}{a}\tan\frac{A}{2}$$

Since r itself equals $(s-a)\tan\frac{A}{2}$, we have

$$s - a = \frac{sx}{a} \Rightarrow \frac{x}{a} = 1 - \frac{a}{s}$$

Analogous relations will hold for $\frac{y}{b}$ and $\frac{z}{c}$. Summing these three relations, we will have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 - \left(\frac{a+b+c}{s}\right) = 3 - 2 = 1$$

Thus, the correct option is (C).

S7. In the given relation, the arguments of the trigonometric functions contain both θ and α together, while in the expression we need to evaluate, this is not the case—an indication that we should use combination formulae to try and separate these two parameters. Writing $\cos(\theta-\alpha)$ as $\sin(\frac{\pi}{2}-(\theta-\alpha))$ and applying *componendo and dividendo* to the given relation, we have:

$$\frac{\sin(\theta+\alpha)+\sin\left(\dfrac{\pi}{2}-(\theta-\alpha)\right)}{\sin(\theta+\alpha)-\sin\left(\dfrac{\pi}{2}-(\theta-\alpha)\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\dfrac{\pi}{4}+\alpha\right)\cos\left(-\dfrac{\pi}{4}+\theta\right)}{\sin\left(\dfrac{-\pi}{4}+\theta\right)\cos\left(\dfrac{\pi}{4}+\alpha\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\theta-\frac{\pi}{4}\right) = -\frac{1}{m}$$

$$\Rightarrow \quad \cot\left(\frac{\pi}{4}-\alpha\right)\cot\left(\frac{\pi}{4}-\theta\right)=\frac{1}{m} \qquad \text{(How?)}$$

$$\Rightarrow \quad P = m$$

Thus, the correct option is (A).

S8. Rearranging the equations slightly,

$$\sin a - 8\sin d = 4\sin c - 7\sin b$$
$$\cos a - 8\cos d = 4\cos c - 7\cos b$$

Squaring, adding and then simplifying, we will obtain the desired value as $\frac{2}{7}$. Doing this is left to the reader as an exercise. The correct option is (B).

S9. (a) If

$$\theta = \frac{\pi}{7}, \text{ then } 7\theta = \pi$$
$$\Rightarrow \quad 4\theta = \pi - 3\theta$$
$$\Rightarrow \quad \tan 4\theta = -\tan 3\theta$$
$$\Rightarrow \quad \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} = \frac{\tan^3\theta - 3\tan\theta}{1-3\tan^2\theta}$$

Simplifying this to a polynomial equation in $\tan\theta$, we have

$$y^6 - 21y^4 + 35y^2 - 7 = 0 \qquad (1)$$

which has the roots $\tan\frac{\pi}{7}, \tan\frac{2\pi}{7}, \ldots, \tan\frac{6\pi}{7}$ (why?). Since

$$\tan\frac{6\pi}{7} = -\tan\frac{\pi}{7},\ \tan\frac{5\pi}{7} = -\tan\frac{2\pi}{7},\ \tan\frac{4\pi}{7} = -\tan\frac{3\pi}{7},$$

the product of roots P becomes

$$P = \tan^2\frac{\pi}{7}\ \tan^2\frac{2\pi}{7}\ \tan^2\frac{3\pi}{7} = 7$$
$$\Rightarrow \quad \tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}$$

Thus, the correct option is (A).

(b) Note that the 6th degree polynomial equation in (1) can be reduced to the 3rd degree polynomial equation

$$z^3 - 21z^2 + 35z - 7 = 0 \qquad (2)$$

which has the roots $\tan^2\frac{\pi}{7},\ \tan^2\frac{2\pi}{7}, \tan^2\frac{3\pi}{7}$. If we let $z = \frac{1}{u}$, (2) changes to the cubic

$$7u^3 - 35u + 21u - 1 = 0 \qquad (3)$$

with roots $\cot^2\frac{\pi}{7}, \cot^2\frac{2\pi}{7}, \cot^2\frac{3\pi}{7}$. From (2),

$$\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} = 21$$

and from (3),

$$\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=5$$

$\Rightarrow$ The required product is 105.

Thus, the correct option is (B).

S10. Use the substitutions $\tan^2\alpha \to a$ and $\tan^2\beta \to b$ to reduce the given expression to

$$\frac{(a+1)^2}{b}+\frac{(b+1)^2}{a}=\left(\frac{a^2}{b}+\frac{1}{b}+\frac{b^2}{a}+\frac{1}{a}\right)+2\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$\geq 4\left(\frac{a^2}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\right)^{1/4}+2\cdot 2\left(\frac{a}{b}\cdot\frac{b}{a}\right)^{1/2}=8$$

In the last step, we have made use of the AM-GM inequality. The equality holds when $a=b=1$, that is, when $\alpha=\beta=\frac{\pi}{4}$. Thus, the minimum value is 8, which means that the correct option is (C).

S11. We make the following observations:

$$\cos\frac{5\pi}{8}=-\cos\frac{3\pi}{8},\ \cos\frac{7\pi}{8}=-\cos\frac{\pi}{8}$$

$$\Rightarrow\quad S=\left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)=\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$=\frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)=\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)=\frac{1}{8}$$

Thus, the correct option is (B).

S12. Using the fact that $\cos\frac{7\pi}{15}=\cos(\pi-\frac{8\pi}{15})=-\cos\frac{8\pi}{15}$ and $\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$, P can be rearranged to

$$P=-\frac{1}{2}\left(\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)$$

We make use of the following result, which we have worked out elsewhere:

$$P=\cos A\cos 2A\cos 2^2A\cos 2^3A\cdots\cos 2^{n-1}A=\frac{\sin 2^nA}{2^n\sin A}$$

This gives

$$P=-\frac{1}{2}\left(\frac{\sin(2^4\times\frac{\pi}{15})}{2^4\times\sin(\frac{\pi}{15})}\right)\left(\frac{\sin(2^2\times\frac{3\pi}{15})}{2^2\times\sin(\frac{3\pi}{15})}\right)=-\frac{1}{2}\frac{\sin(\frac{16\pi}{15})}{16\sin(\frac{\pi}{15})}\times\frac{\sin(\frac{12\pi}{15})}{4\sin(\frac{3\pi}{15})}$$

Finally, since $\sin(\frac{16\pi}{15})=-\sin(\frac{\pi}{15})$ and $\sin(\frac{12\pi}{15})=\sin(\frac{3\pi}{15})$ (why?), we have

$$P=\frac{-1}{2}\times\frac{-1}{16}\times\frac{1}{4}=\frac{1}{128}$$

Thus, the correct option is (B).

S13. We combine the tan inverse terms in the first step:

$$2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}=2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right)=2\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-(\frac{1}{3})^2}\right)=\tan^{-1}\frac{3}{4}$$

Also, $$\sec^{-1}\frac{5\sqrt{2}}{7}=\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1}=\tan^{-1}\frac{1}{7}$$

Thus,

$$S=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)=\tan^{-1}1=\frac{\pi}{4}$$

The correct option is (B).

S14. The trick is to convert this series of reciprocals into a sum-series where alternate terms cancel out, as shown below:

$$S=\frac{1}{\sin\theta}\left\{\frac{\sin\theta}{\sin\theta\sin 2\theta}+\frac{\sin\theta}{\sin 2\theta\sin 3\theta}+\frac{\sin\theta}{\sin 3\theta\sin 4\theta}+\cdots\right\}$$

Now, the rth term in this series T_r can be manipulated as shown below:

$$\begin{aligned}T_r&=\frac{\sin\theta}{\sin r\theta\sin(r+1)\theta}=\frac{\sin((r+1)\theta-r\theta)}{\sin r\theta\,\sin(r+1)\theta}\\&=\frac{\sin(r+1)\theta\cos r\theta-\cos(r+1)\theta\sin r\theta}{\sin r\theta\,\sin(r+1)\theta}\\&=\cot r\theta-\cot(r+1)\theta\end{aligned}$$

So we now have,

$$\begin{aligned}S&=\frac{1}{\sin\theta}\{(\cot\theta-\cot 2\theta)+(\cot 2\theta-\cot 3\theta)+(\cot 3\theta-\cot 4\theta)+\cdots\}\\&=\frac{\cot\theta-\cot(n+1)\theta}{\sin\theta}\end{aligned}$$

Thus, the correct option is (B).

S15. We will use an approach based on complex numbers. Why? As you'll see, that will modify the series into a GP:

$$\begin{aligned}S_1&=\text{Im}\{e^{i\alpha}+e^{i(\alpha+\beta)}+e^{i(\alpha+2\beta)}+\cdots+e^{i(\alpha+(n-1)\beta)}\}\\&=\text{Im}\{e^{i\alpha}(1+e^{i\beta}+e^{i2\beta}+\cdots+e^{i(n-1)\beta})\}\\&=\text{Im}\left\{e^{i\alpha}\left(\frac{e^{in\beta}-1}{e^{i\beta}-1}\right)\right\}\\&=\text{Im}\left\{(\cos\alpha+i\sin\alpha)\left(\frac{\cos n\beta+i\sin n\beta-1}{\cos\beta+i\sin\beta-1}\right)\right\}\end{aligned}$$

$$= \text{Im}\left\{(\cos\alpha + i\sin\alpha)\left(\frac{-2\sin^2\frac{n\beta}{2} + 2i\sin\frac{n\beta}{2}\cos\frac{n\beta}{2}}{-2\sin^2\frac{\beta}{2} + 2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}}\right)\right\}$$

$$= \text{Im}\left\{(\cos\alpha + i\sin\alpha)\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\left(\frac{\cos\frac{n\beta}{2} + i\sin\frac{n\beta}{2}}{\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}}\right)\right\}$$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\text{Im}\left\{\frac{\cos(\alpha + \frac{n\beta}{2}) + i\sin(\alpha + \frac{n\beta}{2})}{\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}}\right\}$$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\frac{2\alpha + (n-1)\beta}{2}\right). \quad \text{(Verify)}$$

The correct option is (A).

(b) Using an exactly analogous procedure,

$$S_2 = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\text{Re}\left\{\frac{\cos(\alpha + \frac{n\beta}{2}) + i\sin(\alpha + \frac{n\beta}{2})}{\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}}\right\} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\frac{2\alpha + (n-1)\beta}{2}\right)$$

The correct option is (C).

We urge you to note the following special cases:

(a) $\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \dfrac{\sin(\frac{n\alpha}{2})\sin(\frac{(n+1)\alpha}{2})}{\sin\frac{\alpha}{2}}$

(b) $\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \dfrac{\sin(\frac{n\alpha}{2})\cos(\frac{(n+1)\alpha}{2})}{\sin\frac{\alpha}{2}}$

S16. $1 + \cos r\alpha = 2\cos^2\frac{r\alpha}{2}$

$$\Rightarrow \quad S = \sqrt{2}\left(\cos\frac{\alpha}{2} + \cos\alpha + \cos\frac{3\alpha}{2} + \cdots \text{to } n \text{ terms}\right)$$

$$= \sqrt{2}\,\frac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\left((n+1)\frac{\alpha}{4}\right) \qquad \left\{\begin{matrix}\text{using the result of the}\\ \text{previous example}\end{matrix}\right\}$$

Thus, the correct option is (C).

S17. First, let us express $\cos^4 x$ as a linear term:

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right) = \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

Thus,

$$S = \sum_{r=1}^{n}\cos^4\left(\theta + (r-1)\frac{2\pi}{n}\right) = \frac{1}{8}\sum_{r=1}^{n}\left(3 + 4\cos\left(2\theta + (r-1)\frac{4\pi}{n}\right) + \cos\left(4\theta + (r-1)\frac{8\pi}{n}\right)\right)$$

Now we sum the two cosine series using the general relation for the sum of a cosine series. You can verify that both these series sum to zero. Therefore,

$$S = \frac{3n}{8}$$

The correct option is (B).

S18. Let $P = \cos a \cos 2a \ldots \cos 999a$ and $Q = \sin a \sin 2a \ldots \sin 999a$. We consider the product PQ as follows:

$$\begin{aligned}
2^{999} PQ &= \sin 2a \sin 4a \ldots \sin 1998a \quad \text{(how?)} \\
&= (\sin 2a \sin 4a \ldots \sin 998a)\,(-\sin(2\pi - 1000a) \times -\sin(2\pi - 1002a) \times \ldots -\sin(2\pi - 1998a)) \\
&= (\sin 2a \sin 4a \ldots \sin 998a)\,(\sin 997a \sin 995a \ldots \sin a) \\
&= Q \\
\Rightarrow \quad P &= \frac{1}{2^{999}}
\end{aligned}$$

Thus, the correct option is (B).

S19. Multiply both sides of the equality by sin 1°. The general term on the left side can be manipulated as follows:

$$\frac{\sin 1^\circ}{\sin x^\circ \sin(x+1)^\circ} = \frac{\sin\left((x+1)-x\right)^\circ}{\sin x^\circ \sin(x+1)^\circ} = \cot x^\circ - \cot(x+1)^\circ$$

Thus, on the left side we will be left with

$$\begin{aligned}
S &= (\cot 45^\circ - \cot 46^\circ) + (\cot 47^\circ - \cot 48^\circ) + \cdots + (\cot 133^\circ - \cot 134^\circ) \\
&= \cot 45^\circ - (\cot 46^\circ + \cot 134^\circ) + (\cot 47^\circ + \cot 133^\circ) - \cot 90^\circ \\
&= 1
\end{aligned}$$

$$\Rightarrow \quad 1 = \frac{\sin 1^\circ}{\sin n^\circ} \quad \Rightarrow \quad n = 1$$

The correct option is (A).

S20. We note that

$$(1+\tan k^\circ)\,(1+\tan(45-k)^\circ) = (1+\tan k^\circ)\left(1+\frac{1-\tan k^\circ}{1+\tan k^\circ}\right) = 2$$

We can use this fact, and tan 45° = 1, to show that the product is 2^{23}, giving $n = 23$. Doing this is left to the reader as an exercise. The correct option is (B).

S21. If we use $x_n = \tan \alpha_n$, it is easy to see that $x_{n+1} = \tan(\alpha_n + 45^\circ)$, which means that the sequence $\{x_n\}$ is periodic with period 4 (how?). Thus, $x_{2004} = x_0 = 2003$. The correct option is (A).

S22. The rth term is

$$\begin{aligned}
T_r &= \tan^{-1}\left(\frac{2r}{r^4 + r^2 + 2}\right) = \tan^{-1}\left(\frac{(r^2+r+1)-(r^2-r+1)}{1+(r^2+r+1)(r^2-r+1)}\right) \\
&= \tan^{-1}(r^2+r+1) - \tan^{-1}(r^2-r+1)
\end{aligned}$$

Therefore,

$$S = (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}7 - \tan^{-1}3) + \cdots + (\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1))$$

where $n \to \infty$

$$\Rightarrow \quad S = \lim_{n\to\infty} \tan^{-1}(n^2+n+1) - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The correct option is (B).

S23. We have

$$\frac{a}{b} = 2+\sqrt{3} \quad \Rightarrow \quad \frac{a-b}{a+b} = \frac{1+\sqrt{3}}{3+\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = 1$$

Thus, $A - B = 90°$, and since $A + B = 120°$, we have $A = 105°$ and $B = 15°$. The correct option is (D).

S24. Square and add to obtain $\sin(A+B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2}$. Thus, $\angle C$ can be 30° or 150°. If $\angle C = 150°$, $\angle A < 30°$, and this implies that $3\sin A + 4\cos B < \frac{3}{2} + 4 < 6$, which is contradictory. Thus, the only possible value for $\angle C$ is 30°. The correct option is (A).

S25. Given that A, B, C are the angles of a triangle, we make use of the following conditional identity (you are urged to justify this quickly):

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \tag{1}$$

Now, since $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP, their reciprocals, *i.e.*, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP. So,

$$\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2} \tag{2}$$

Using (2) in (1), we obtain

$$\cot\frac{A}{2}\cot\frac{C}{2} = 3$$

Finally, we use the AM-GM inequality on $\cot\frac{A}{2}, \cot\frac{C}{2}$:

$$\frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} = \cot\frac{B}{2} \geq \sqrt{\cot\frac{A}{2}\cot\frac{C}{2}} = \sqrt{3}$$

$$\Rightarrow \quad \cot\frac{B}{2} \geq \sqrt{3}$$

Therefore, the minimum value of $\cot\frac{B}{2}$ is $\sqrt{3}$. The correct option for part-(a) is (C). From this result, we can further infer that the maximum value of $\tan\frac{B}{2}$ is $\frac{1}{\sqrt{3}}$ which implies that $\frac{B}{2} \leq \frac{\pi}{6} \Rightarrow B \leq \frac{\pi}{3}$. Thus, the correct option for part (b) is (C).

S26. Using the cosine rule, we get a quadratic in b.

$$b^2 - 2bc\cos A + (c^2 - a^2) = 0$$

$$\Rightarrow \quad b_1 + b_2 = 2c\cos A,\ b_1 b_2 = c^2 - a^2$$

Using $b_2 = 2b_1$, we have

$$b_1 = \frac{2c}{3}\cos A, \quad b_1^2 = \frac{c^2 - a^2}{2}$$

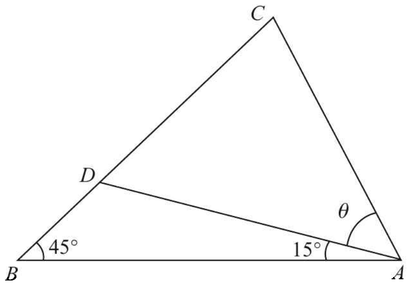
Thus,

$$\frac{4c^2}{9}\cos^2 A = \frac{c^2 - a^2}{2} \quad \Rightarrow \quad 9(c^2 - a^2) = 8c^2\cos^2 A$$

$$\Rightarrow \quad 9a^2 = 9c^2 - 8c^2\cos^2 A = c^2(1 + 8\sin^2 A) \quad \Rightarrow \quad \frac{c}{a}\sqrt{1 + 8\sin^2 A} = 3$$

The correct option is (C).

S27. We note that $\angle CDA = 60°$.



Applying the sine rule in ΔABC and ΔACD, we have

$$\frac{BC}{\sin(\theta + 15°)} = \frac{CA}{\sin 45°} \text{ and } \frac{CD}{\sin\theta} = \frac{CA}{\sin 60°}$$

Dividing these two relations,

$$\frac{CD\sin(\theta + 15)}{BC\sin\theta} = \frac{\sin 45°}{\sin 60°} \qquad (1)$$

Now comes the important step which will greatly simplify the solution. We observe that

$$\frac{CD}{BC} = \frac{2}{3} = \left(\frac{\sin 45°}{\sin 60°}\right)^2, \text{ so that (1) becomes}$$

$$\frac{\sin\theta}{\sin(\theta + 15°)} = \frac{\sin 45°}{\sin 60°}$$

In addition, we observe that only one value of θ is possible which will satisfy the above (why?), which implies $\theta = 45°$. Thus, $\angle ACB = 75°$. The correct option is (D).

S28. $[ABCD]$, the required area, will be equal to $\frac{1}{2}(AC)(BD)$, since AC and BD are perpendicular to each other (can you see how?). Now, since AB and CD are parallel, ΔAPB and ΔCPD are similar, with the corresponding sides being in the ratio 2 : 5. Thus, if we assume $AP = 2x$ and $BP = 2y$, then $CP = 5x$ and $DP = 5y$.

$$\Rightarrow \quad [ABCD] = \frac{1}{2} \times 7x \times 7y = \frac{49xy}{2}$$

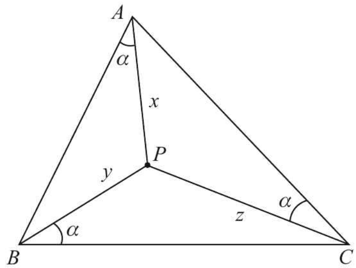
We therefore need to evaluate xy. For this, we make the observation that since $\angle CPD = 90°$, $\angle CQD = 45°$, we must have $\angle QCP + \angle QDP = 45°$, so that

$$1 = \tan(\angle QCD + \angle QDP) = \frac{\frac{2y}{5x} + \frac{2x}{5y}}{1 - \frac{2y}{5x}\cdot\frac{2x}{5y}} = \frac{10(x^2 + y^2)}{21xy}$$

Finally, we have $AP^2 + PB^2 = AB^2$, giving $x^2 + y^2 = 4$, *i.e.*, $xy = \frac{40}{21}$.
Thus, $[ABCD] = \frac{140}{3}$. The correct option is (B).

S29. Let $PA = x$, $PB = y$, $PC = z$. We apply the law of cosines to ΔPCA, ΔPAB and ΔPBC.

$$\Rightarrow \begin{cases} x^2 = z^2 + b^2 - 2bz\cos\alpha \\ y^2 = x^2 + c^2 - 2cx\cos\alpha \\ z^2 = y^2 + a^2 - 2ay\cos\alpha \end{cases}$$



Add these three relations to obtain

$$2(cx + ay + bz)\cos\alpha = a^2 + b^2 + c^2$$

Now, since the area of ΔABC can be written as $\Delta = \frac{1}{2}(cx + ay + bz)\sin\alpha$, we have

$$\tan\alpha = \frac{4\Delta}{a^2 + b^2 + c^2} = \frac{168}{295} \quad \Rightarrow \quad m + n = 463$$

The correct option is (B).

S30. The equation $3y^2 + 8y + 3 = 0$ has two real roots y_1, y_2 such that $y_1 + y_2 = -\frac{8}{3}$ and $y_1\ y_2 = 1$.

Now, observe that $y = \cot x$ is a bijection on $(0, \pi) \to \mathbb{R}$, and also on $(\pi, 2\pi) \to \mathbb{R}$. In these intervals, $\cot x < 0$ when $x \in (\frac{\pi}{2}, \pi)$ or $(\frac{3\pi}{2}, 2\pi)$. Therefore, there exists x_1, x_2 in $(\frac{\pi}{2}, \pi)$ and (x_3, x_4) in $(\frac{3\pi}{2}, 2\pi)$ such that

$$y_1 = \cot x_1 = \cot x_3, \quad y_2 = \cot x_2 = \cot x_4$$

Now,

$$y_1 y_2 = 1 \quad \Rightarrow \quad \cot x_1\ \cot x_2 = \cot x_3\ \cot x_4 = 1$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 5\pi \qquad \text{(how can we say this?)}$$

The required value is 5. The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

S31. (a) We have $\sin x + a\cos x = \sqrt{1+a^2}\,\sin(x+\phi)$ for some appropriate value of ϕ, and since $\sin\theta \in [-1, 1]$, we must have $b \in [-\sqrt{1+a^2}, \sqrt{1+a^2}]$, *i.e.*,

$$|b| \le \sqrt{1+a^2} \quad \Rightarrow \quad a^2 - b^2 + 1 \ge 0$$

The reverse implication is similarly easily proved.

(b) Since $(\sin x + a\cos x)^2 + (a\sin x - \cos x)^2 = 1 + a^2$, we have

$$|a\sin x - \cos x| = \sqrt{a^2 - b^2 + 1}$$

S32. We have, by the AM-GM inequality,

$$\frac{a}{\sin x} + \frac{b}{\cos x} \ge 2\sqrt{\frac{ab}{\sin x\cos x}} = \frac{2\sqrt{2ab}}{\sqrt{\sin 2x}} \ge 2\sqrt{2ab}$$

Therefore,

$$1 + \frac{a}{\sin x} + \frac{b}{\cos x} + \frac{ab}{\sin x\cos x} \ge 1 + 2\sqrt{2ab} + 2ab$$

$$\Rightarrow \quad \left(1 + \frac{a}{\sin x}\right)\left(1 + \frac{b}{\cos x}\right) \ge (1 + \sqrt{2ab})^2$$

S33. Using the substitutions $a \to \tan x$, $b \to \tan y$ and $c \to \tan z$, and multiplying both sides by $\cos^2 x\cos^2 y\cos^2 z$, the LHS can be manipulated as follows:

$$\begin{aligned}
\text{LHS} &= (ab + bc + ca - 1)^2 \cos^2 x\cos^2 y\cos^2 z \\
&= ((ab + bc)\cos x\cos y\cos z + (ca - 1)\cos x\cos y\cos z)^2 \\
&= (\sin y\sin(x+z) - \cos y(x+z))^2 \\
&= \cos^2(x + y + z)
\end{aligned}$$

On the other hand, the RHS now becomes (after multiplying with the same term as the LHS)

$$\begin{aligned}
\text{RHS} &= (1 + a^2)(1 + b^2)(1 + c^2)\cos^2 x\,\cos^2 y\,\cos^2 z \\
&= \sec^2 x\,\sec^2 y\,\sec^2 z\,\cos^2 x\,\cos^2 y\,\cos^2 z \\
&= 1
\end{aligned}$$

Thus,

$$\text{LHS} \le \text{RHS}$$

S34. For $\cos x = 0$, the given inequality holds. Assuming $\cos x \ne 0$, we divide both sides of the given inequality by $\cos^2 x$, and use the substitution $\tan x \to t$ so that the inequality reduces to

$$(t + a)\,(t + b) \le \left(1 + \left(\frac{a+b}{2}\right)^2\right)(1 + t^2)$$

$$\Rightarrow \quad \left(\left(\frac{a+b}{2}\right)t - 1\right)^2 + \left(\frac{a-b}{2}\right)^2 \ge 0, \text{ which is obviously true}$$

Therefore, the given inequality holds.

S35. We would like to point out that this problem can be solved without any usage of trigonometry. For example, applying the Cauchy Schwarz inequality on

$$\vec{X} = \{\sqrt{x}, \sqrt{y}, \sqrt{z}\} \quad \text{and} \quad \vec{Y} = \left\{\frac{1}{\sqrt{x}}, \frac{2}{\sqrt{y}}, \frac{3}{\sqrt{z}}\right\},$$

we obtain

$$|\vec{X}\cdot\vec{Y}|^2 \le |\vec{X}|^2|\vec{Y}|^2 \quad \Rightarrow \quad 6^2 \le (x+y+z)\left(\frac{1}{x}+\frac{4}{y}+\frac{9}{z}\right)$$

Since $x+y+z=1$, we immediately obtain

$$\frac{1}{x}+\frac{4}{y}+\frac{9}{z} \ge 36$$

It is easy to see that the equality will hold when $x=\frac{1}{6}$, $y=\frac{1}{3}$, $z=\frac{1}{2}$. The correct option is (D).

Now, we develop a solution involving trigonometry. Assuming $z=\sin^2 a$ for some angle a, we have $x+y=1-z=1-\sin^2 a=\cos^2 a$, *i.e.*,

$$\frac{x}{\cos^2 a}+\frac{y}{\cos^2 a}=1 \quad \Rightarrow \quad x=\cos^2 a\cos^2 b, y=\cos^2 a\sin^2 b \text{ for some angle } b.$$

Thus, we need to obtain the minimum value of

$$Z = \sec^2 a\sec^2 b + 4\sec^2 a\,\text{cosec}^2 b + 9\text{cosec}^2 a$$

$$= (1+\tan^2 a)(1+\tan^2 b) + 4(1+\tan^2 a)(1+\cot^2 b) + 9(1+\cot^2 a)$$

A little manipulation is now required:

$$Z = 14 + 5\tan^2 a + 9\cot^2 a + \overbrace{(\tan^2 b + 4\cot^2 b)}^{(*)}(1+\tan^2 a)$$

$$\ge 14 + 5\tan^2 a + 9\cot^2 a + 4(1+\tan^2 a) \quad \{\text{We applied the AM-GM inequality on } (*)\}$$

$$= 18 + 9\overbrace{(\tan^2 a + \cot^2 a)}^{(**)}$$

$$\ge 18 + 9\cdot 2 \quad \text{(Again, we applied the AM-GM inequality on (**))}$$

$$= 36$$

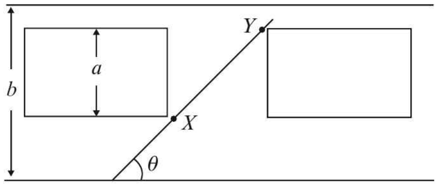
We find that $Z_{\min} = 36$, and this value of Z is attained when:

$$\tan a = \cot a \quad \text{and} \quad \tan b = 2\cot b$$

$$\Rightarrow \quad \cos^2 a = \sin^2 a \quad \text{and} \quad 2\cos^2 b = \sin^2 b$$

$$\Rightarrow \quad \cos^2 a = \frac{1}{2} \text{ and } \cos^2 b = \frac{1}{3} \quad \Rightarrow \quad x=\frac{1}{6}, y=\frac{1}{3}, z=\frac{1}{2}$$

S36. The adjacent figure depicts the situation approximately. The segment XY depicts the 'risky' part of the squirrel's trip, that is, that path of the trip during which a collision with a car is possible.



Let t be the time taken by the squirrel to cross the risky part.

(a) We have

$$a = ut\sin\theta, \quad ut\cos\theta \geq vt - c \qquad \text{(why?)}$$

Using these two relations to eliminate t, we obtain

$$u \geq \frac{av}{c\sin\theta + a\cos\theta}$$

(b) $u_{\text{min}} = \frac{av}{(c\sin\theta + a\cos\theta)_{\text{max}}}$, which happens when $\theta = \tan^{-1}\frac{c}{a}$. Thus, if T is the time taken to cross the road, we have

$$T = \frac{b}{u\sin\theta} = \left.\frac{b(c\sin\theta + a\cos\theta)}{v\sin\theta}\right|_{\theta = \tan^{-1}\frac{c}{a}}$$

$$= \frac{b(c + a^2/c)}{av} = \frac{b}{v}\left(\frac{c}{a} + \frac{a}{c}\right)$$

S37. $a\cos B + b\cos C + c\cos A = \dfrac{a+b+c}{2}$

$\Leftrightarrow\ 2\sin A\cos B + 2\sin B\cos C + 2\sin C\cos A = \sin A + \sin B + \sin C$

$\Leftrightarrow\ \sin(A+B) + \sin(A-B) + \sin(B+C) + \sin(B-C) + \sin(C+A) + \sin(C-A)$

$\quad = \sin A + \sin B + \sin C$

$\Leftrightarrow\ \sin(A-B) + \sin(B-C) + \sin(C-A) = 0 \qquad$ (how?)

$\Leftrightarrow\ 4\sin\left(\dfrac{A-B}{2}\right)\sin\left(\dfrac{B-C}{2}\right)\sin\left(\dfrac{C-A}{2}\right) = 0$

$\Leftrightarrow$ Triangle ABC is isosceles.

S38. Using $x = \sin\frac{A}{2}$, $y = \sin\frac{B}{2}$, $z = \sin\frac{C}{2}$, we need to show that

$$x^2 + y^2 + z^2 + 2xyz = 1$$

$$\Rightarrow\ x = -yz + \sqrt{(1-y^2)(1-z^2)} \qquad \text{(how?)}$$

$$= -\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{B}{2}\cos\frac{C}{2} = \cos\left(\frac{B+C}{2}\right)$$

$$= \sin\left(\frac{A}{2}\right).$$

Since this is true, the given equality holds.

S39. As we know, the approach in such questions is the method of differences. We write each term in the series as a difference, so that terms in successive expressions cancel out. The general, rth term in S is

$$T_r = \tan^{-1}\frac{1}{2\cdot r^2}$$

We know that $\tan^{-1}(\frac{x-y}{1+xy}) = \tan^{-1}x - \tan^{-1}y$. Somehow, we have to use this fact to express T_r as a difference. To do that, we express the denominator as $1+xy$:

$$\frac{1}{2\cdot r^2} = \frac{2}{4\cdot r^2} = \frac{2}{1+(4r^2-1)} = \frac{2}{1+(2r+1)(2r-1)} = \frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}$$

This means that

$$T_r = \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$$

Expressing T_r this way solves our problem, since S now becomes

$$S = (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) +$$
$$(\tan^{-1}7 - \tan^{-1}5) + \cdots + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1))$$

where $n \to \infty$. Thus,

$$S = \lim_{n\to\infty}\tan^{-1}(2n+1) - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

S40. We modify the general $\cot^{-1}$ term into a term containing $\tan^{-1}$:

$$T_r = \cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right) = \tan^{-1}\left(\frac{2^r}{2^{2r+1}+1}\right)$$

$$= \tan^{-1}\left(\frac{2^{r+1}-2^r}{1+2^r\cdot 2^{r+1}}\right) = \tan^{-1}2^{r+1} - \tan^{-1}2^r$$

$$\Rightarrow \quad S = \frac{\pi}{2} - \tan^{-1}2 \text{ (how?) } = \cot^{-1}2$$

S41. As always, our approach should be to try to express the general rth term as a difference:

$$T_r = \sin 3^{r-1}\theta \sec 3^r\theta$$

$$= \left.\frac{\sin 3^{r-1}\theta}{\cos 3^r\theta} \times \frac{\cos 3^{r-1}\theta}{\cos 3^{r-1}\theta}\right\} \text{New term introduced}$$

$$= \frac{1}{2}\frac{\sin 2\cdot 3^{r-1}\theta}{\cos 3^r\theta \cos 3^{r-1}\theta} = \frac{1}{2}\frac{\sin(3^r - 3^{r-1})\theta}{\cos 3^r\theta \cos 3^{r-1}\theta}$$

$$= \frac{1}{2}\frac{\sin 3^r\theta \cos 3^{r-1}\theta - \cos 3^r\theta \sin 3^{r-1}\theta}{\cos 3^r\theta \cos 3^{r-1}\theta} = \frac{1}{2}(\tan 3^r\theta - \tan 3^{r-1}\theta)$$

This solves our problem, since now

$$S = \frac{1}{2}\{(\tan 3\theta - \tan\theta) + (\tan 3^2\theta - \tan 3\theta) + \cdots + (\tan 3^n\theta - \tan 3^{n-1}\theta)\}$$

$$\Rightarrow \quad S = \frac{1}{2}(\tan 3^n\theta - \tan\theta)$$

S42. We have the required average A as

$$A = \frac{1}{180}(2\sin 2^\circ + 4\sin 4^\circ + 6\sin 6^\circ + \cdots + 178\sin 178^\circ)$$

We have omitted the last term since sin 180° = 0. Now, we make the observation that

$$2\sin 2k^\circ \sin 1^\circ = \cos(2k-1)^\circ - \cos(2k+1)^\circ$$

Thus,

$$A(\sin 1^\circ) = \frac{1}{90}\{(\cos 1^\circ - \cos 3^\circ) + 2(\cos 3^\circ - \cos 5^\circ) + \cdots + 89(\cos 177^\circ - 179^\circ)\}$$

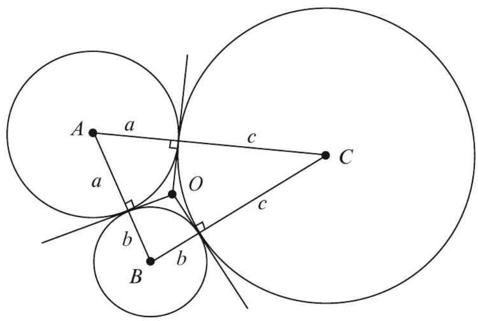
$$= \frac{1}{90}(\cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cdots + \cos 177^\circ - 89\cos 179^\circ)$$

$$= \frac{1}{90}\left(\cos 1^\circ + \underbrace{(\cos 3^\circ + \cos 177^\circ)}_{\text{This is 0}} + \cdots + \underbrace{(\cos 89^\circ + \cos 91^\circ)}_{\text{This is 0}} + 89\cos 1^\circ\right)$$

$$= \frac{1}{90}(90\cos 1^\circ) \quad = \cos 1^\circ$$

$$\Rightarrow \quad A = \cot 1^\circ$$

S43. Observe that the distance of this intersection point will be the same for all the three points of contact:



Note that O is the incentre of ΔABC. The distance of O from any of the points of contact is simply the inradius r of ΔABC. We have

$$s = a + b + c$$

$$\Rightarrow \quad \Delta = \sqrt{s(s-(a+b))(s-(b+c))(s-(c+a))} = \sqrt{abc(a+b+c)}$$

$$\Rightarrow \quad r = \frac{\Delta}{s} = \sqrt{\frac{abc}{a+b+c}}$$

S44. Without loss of generality, we can assume that $A \geq B \geq C$. We therefore need to prove that $B + C < 30°$. Now,

$$b + c > a \Rightarrow \sin B + \sin C > \sin A.$$

Using this reduces the given inequality to $\sin A < \frac{1}{2}$, which implies that $A > 150°$. Thus, $B + C < 30°$.

S45. From the given relations,

$$\tan A \tan B \tan C = \frac{3+\sqrt{3}}{\sqrt{3}-1}$$

We make use of the following fact: if $A+B+C=\pi$, then

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Thus, we have

$$\tan A + \tan B + \tan C = \frac{3+\sqrt{3}}{\sqrt{3}-1}$$

Now, since $A+B+C=\pi$

$$\cos(A+B+C) = -1$$

$$\Rightarrow \quad \cos A \cos B \cos C\{1-\Sigma(\tan A \tan B)\} = -1 \quad \text{(how?)}$$

$$\Rightarrow \quad \Sigma\left(\tan A \tan B\right) = 1 + \frac{8}{\sqrt{3}-1} = 5+4\sqrt{3} \quad \text{(how?)}$$

The last two steps are not straightforward, and the reader is urged to work them out in detail. Finally, for $\tan A, \tan B, \tan C$, we have their sum, their sum taken pairwise and their product, and thus we can say that these three are the roots of the equation

$$x^3 - \left(\frac{3+\sqrt{3}}{\sqrt{3}-1}\right)x^2 + (5+4\sqrt{3})x - \left(\frac{3+\sqrt{3}}{\sqrt{3}-1}\right) = 0$$

$$\Rightarrow \quad (x-1)(x-\sqrt{3})(x-(2+\sqrt{3})) = 0$$

$$\Rightarrow \quad x = 1, \sqrt{3}, 2+\sqrt{3}$$

This means that the angles are $\dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{12}$

S46. (a) In ΔOAB,

$$\frac{OB}{\sin\omega} = \frac{c}{\sin\angle AOB}$$

But $\quad \angle AOB = \pi - (\omega + \angle ABO) = \pi - (\omega + \angle B - \omega) = \pi - \angle B$

$$\Rightarrow \quad \frac{OB}{\sin\omega} = \frac{c}{\sin B} \Rightarrow OB = \frac{c\sin\omega}{\sin B}$$

Similarly, from ΔOBC, we have

$$OB = \frac{a\sin(C-\omega)}{\sin C} \quad \text{(Verify!)}$$

$$\Rightarrow \quad \frac{c\sin\omega}{\sin B} = \frac{a\sin(C-\omega)}{\sin C} \tag{1}$$

Further, from the sine rule, $c = \lambda \sin C$, $a = \lambda \sin A$. Substituting this in (1) gives

$$\frac{\sin C \sin\omega}{\sin B} = \frac{\sin A \sin(C-\omega)}{\sin C}$$

$$\Rightarrow \quad \sin^2 C \sin\omega = \sin A \sin B \sin(C-\omega)$$
$$\Rightarrow \quad \sin C \sin(A+B) \sin\omega = \sin A \sin B \sin(C-\omega)$$

Simplifying this yields the desired result.

(b) Recall that in any ΔABC,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

Now, in part (a), we proved that

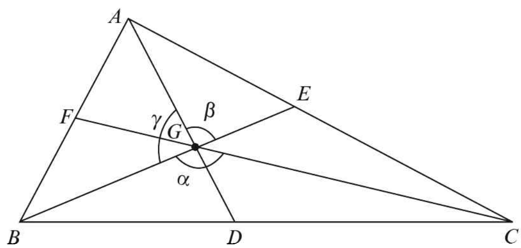
$$\cot\omega = \cot A + \cot B + \cot C$$

Squaring this yields

$$\cot^2\omega = \cot^2 A + \cot^2 B + \cot^2 C + 2$$

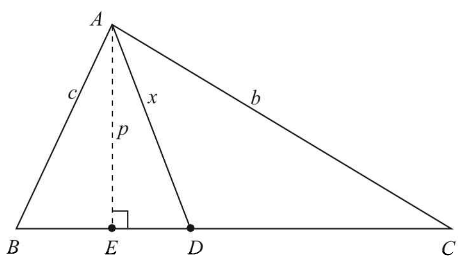
$$\Rightarrow \quad \text{cosec}^2\omega - 1 = (\text{cosec}^2 A - 1) + (\text{cosec}^2 B - 1) + (\text{cosec}^2 C - 1) + 2$$
$$\Rightarrow \quad \text{cosec}^2\omega = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$$

S47.



Let the centroid be G

To proceed, we first evaluate the lengths of the medians of the triangle in terms of the sides. Consider the following diagram:



We wish to evaluate the length of AD, *i.e.*, x.

Note that $BD = DC = \dfrac{a}{2}$

By the Pythagoras theorem,

$$x^2 = p^2 + DE^2 = \frac{1}{2}\{p^2 + p^2 + 2DE^2\}$$

Note that $p^2 = c^2 - BE^2 = b^2 - CE^2$, and $DE = \frac{a}{2} - BE = CE - \frac{a}{2}$. Thus,

$$\begin{aligned}
x^2 &= \frac{1}{2}\left\{(c^2 - BE^2) + (b^2 - CE^2) + 2\left(\frac{a}{2} - BE\right)\left(CE - \frac{a}{2}\right)\right\} \\
&= \frac{1}{2}\left\{b^2 + c^2 - (BE^2 + CE^2) - \frac{a^2}{2} + a(CE + BE) - 2BE \cdot CE\right\} \\
&= \frac{1}{2}\left\{b^2 + c^2 - \frac{a^2}{2} + a^2 - (BE + CE)^2\right\} \\
&= \frac{1}{2}\left\{b^2 + c^2 - \frac{a^2}{2}\right\} = \frac{2b^2 + 2c^2 - a^2}{4} \qquad (1)
\end{aligned}$$

Similar expressions will hold for the lengths of the other medians. Now, we return to the original problem. Applying the cosine rule in ΔBGC, we have

$$BC^2 = a^2 = BG^2 + CG^2 - 2BG \cdot CG \cdot \cos\alpha$$

Now, since G is the centroid, we have $BG = \frac{2}{3}BE,\ CG = \frac{2}{3}CF$, so that

$$a^2 = \frac{4}{9}BE^2 + \frac{4}{9}CF^2 - \frac{8}{9}BE \cdot CF \cdot \cos\alpha \qquad (2)$$

Also, if the area of ΔBGC is Δ_1, then

$$\Delta_1 = \frac{1}{2} \cdot BG \cdot CG \cdot \sin\alpha = \frac{2}{9}BE \cdot CF \cdot \sin\alpha$$

But Δ_1 is one third of the total area Δ of ΔABC (why?), so that

$$BE \cdot CF = \frac{3\Delta}{2\sin\alpha} \qquad (3)$$

Using (3) in (2), we have

$$\begin{aligned}
a^2 &= \frac{4}{9}(BE^2 + CF^2) - \frac{4\Delta}{3}\cot\alpha \\
\Rightarrow \quad \cot\alpha &= \frac{4(BE^2 + CF^2) - 9a^2}{12\Delta}
\end{aligned}$$

Similar expressions hold for $\cot\beta$ and $\cot\gamma$. Summing the expressions for the three 'cot's, and using the result of (1) and simiplfying, we have

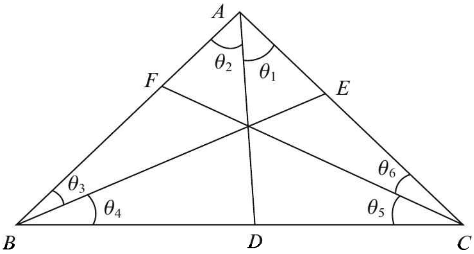
$$\begin{aligned}
\cot\alpha + \cot\beta + \cot\gamma &= \frac{8(AD^2 + BE^2 + CF^2) - 9(a^2 + b^2 + c^2)}{12\Delta} \\
&= \frac{-(a^2 + b^2 + c^2)}{4\Delta} \qquad (4)
\end{aligned}$$

Also, we can easily calculate the sum $\cot A + \cot B + \cot C$:

$$\cot A + \cot B + \cot C = \frac{(a^2 + b^2 + c^2)}{4\Delta} \qquad (5)$$

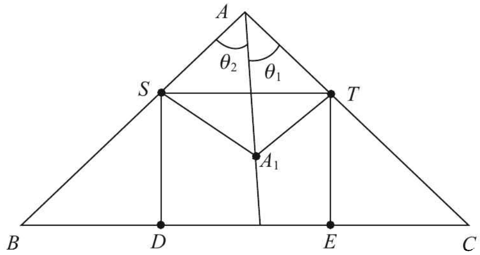
This has been done elsewhere, and is left here to the reader as an exercise. From (4) and (5), the required sum is zero.

S48. To show concurrency, we make use of Ceva's theorem:



$$\frac{\sin\theta_1}{\sin\theta_2}\cdot\frac{\sin\theta_3}{\sin\theta_4}\cdot\frac{\sin\theta_5}{\sin\theta_6}=1$$

This theorem is a simple consequence of the law of sines, and the reader may want to take a minute and prove this theorem herself. Coming back to the present problem:



We note by the law of sines that:

$$\frac{AA_1}{SA_1}=\frac{\sin\angle ASA_1}{\sin\theta_2} \quad (1)$$

$$\frac{TA_1}{AA_1}=\frac{\sin\theta_1}{\sin ATA_1} \quad (2)$$

Also, $SA_1 = TA_1$, $\angle ASA_1 = \angle B + 45°$ and $\angle ATA_1 = \angle C + 45°$. Multiplying (1) and (2), we have

$$1=\frac{\sin\theta_1}{\sin\theta_2}\cdot\frac{\sin(B+45°)}{\sin(C+45°)} \quad\Rightarrow\quad \frac{\sin\theta_1}{\sin\theta_2}=\frac{\sin(C+45°)}{\sin(B+45°)}$$

We will obtain similar expressions for $\frac{\sin\theta_3}{\sin\theta_4}$ and $\frac{\sin\theta_5}{\sin\theta_6}$ (the angles $\theta_3,\theta_4,\theta_5,\theta_6$ are not shown). Thus,

$$\frac{\sin\theta_1}{\sin\theta_2}\cdot\frac{\sin\theta_3}{\sin\theta_4}\cdot\frac{\sin\theta_5}{\sin\theta_6}=1$$

and so the lines AA_1, BB_1 and CC_1 are concurrent by Ceva's theorem.

S49. The value of the given expression is exactly equal to I_n.
We note the following facts:

$$OP = OA_1 = OA_2 = 1$$

$$OB_1 = OB_2 = \sec\frac{\pi}{n}$$

$$PB_1 = PB_2 = \tan\frac{\pi}{n}$$

$$QA_1 = QA_2 = \sin\frac{\pi}{n}$$

$$OQ = \cos\frac{\pi}{n}$$

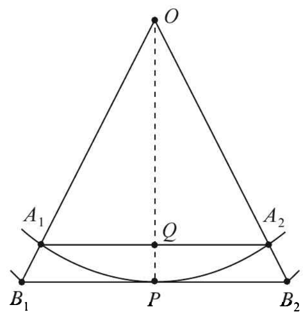


Figure shows one 'sector' of the inscribed and circumscribed polygon.

We thus have

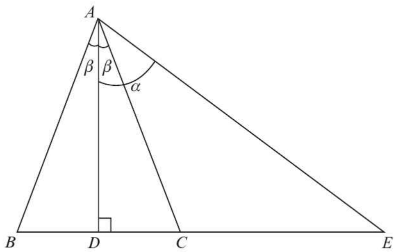
$$I_n = n(\text{area}(\Delta OA_1A_2)) = n\times\frac{1}{2}\times 2\sin\frac{\pi}{n}\times\cos\frac{\pi}{n} = n\sin\tfrac{\pi}{n}\cos\tfrac{\pi}{n}$$

$$O_n = n(\text{area}(\Delta OB_1B_2)) = n\times\frac{1}{2}\times 2\tan\frac{\pi}{n}\times 1 = n\tan\tfrac{\pi}{n}$$

$$\Rightarrow\quad \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right) = \frac{n}{2}\tan\frac{\pi}{n}\left(1+\sqrt{1-\sin^2\frac{2\pi}{n}}\right)$$

$$= \frac{n}{2}\tan\frac{\pi}{n}\left(1+\cos\frac{2\pi}{n}\right) = n\tan\frac{\pi}{n}\cos^2\frac{\pi}{n} = n\sin\frac{\pi}{n}\cos\frac{\pi}{n} = I_n$$

S50. We have assumed angles α and β as shown in the figure below:



Note that

$$\angle EAC = \alpha-\beta,\ \angle EAB = \alpha+\beta$$

It is given that $\tan(\alpha-\beta)$, $\tan\alpha$ and $\tan(\alpha+\beta)$ form a GP. Thus,

$$\tan^2\alpha = \tan(\alpha-\beta)\tan(\alpha+\beta) = \frac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}$$

This gives $\tan\alpha = 1$, *i.e.*, $\alpha = 45°$. Thus,

$AD = DE = 5\sqrt{2}$, so that

$$area(\Delta ABC) = \frac{1}{2}\times BC\times AD = CD\times AD = 50\tan\beta$$

We now need to evaluate β. For that, we use the fact that $\cot\alpha$, $\cot(\alpha-\beta)$ and $\cot\beta$ form an AP. Thus,

$$2\cot(45°-\beta) = 1+\cot\beta \quad\Rightarrow\quad \cot\beta = 3$$

The last step is obtained after some manipulations, doing which is left to the student as an exercise. We thus have

$$\text{area}(\Delta ABC) = \frac{50}{3}$$

S51. Combining $\cos x - \cos 3x$ on the LHS, and writing $1 - \cos 2x = 2\sin^2 x$ on the RHS, we'll have:

$$|2\sin 2x(\sin x - 1)| = 2\sin x(\sin x - 1)$$

Since $\sin x - 1 \le 0$, we have

$$(\sin x - 1)\{\sin x + |\sin 2x|\} = 0 \quad \text{(How?)}$$

Thus, two cases are possible:

Case 1: $\sin x = 1 \Rightarrow x = 2n\pi + \dfrac{\pi}{2},\ n \in \mathbb{Z}$

Case 2: $\sin x + 2|\sin x||\cos x| = 0$. In this case, two sub-cases are possible:

(a) $\sin x \ge 0 \Rightarrow \sin x + 2\sin x|\cos x| = 0 \Rightarrow \sin x = 0$ or $1 + 2|\cos x| = 0$

$$\Rightarrow \sin x = 0 \Rightarrow x = n\pi,\ n \in \mathbb{Z}$$

(b) $\sin x < 0 \Rightarrow \sin x - 2\sin x|\cos x| = 0 \Rightarrow |\cos x| = \dfrac{1}{2} \Rightarrow \cos x = \pm\dfrac{1}{2}$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3},\ 2n\pi \pm \frac{2\pi}{3},\ n \in \mathbb{Z}$$

S52. We note that $\cos^2 x - 3\sin^2 x = 4\cos^2 x - 3 = 2\cos 2x - 1$. The equation is defined if

(i) $2\cos 2x - 1 \ne 0 \Rightarrow x \ne n\pi \pm \frac{\pi}{6},\ n \in \mathbb{Z}$

(ii) $\tan x \ne 0 \Rightarrow x \ne n\pi,\ n \in \mathbb{Z}$

Also, $\tan x$ should itself be defined, which means that $x \ne (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$. Given that these constraints are satisfied, the equation reduces to

$$\lambda \sin x = \lambda + \sin x \quad \Rightarrow \quad \sin x = \frac{\lambda}{\lambda - 1} \quad \Rightarrow \quad -1 \le \frac{\lambda}{\lambda - 1} \le 1$$

$$\Rightarrow \frac{\lambda}{\lambda - 1} \le 1 \quad \text{and} \quad \frac{\lambda}{\lambda - 1} \ge -1$$

$$\Rightarrow \frac{1}{\lambda - 1} \le 0 \quad \text{and} \quad \frac{2\lambda - 1}{\lambda - 1} \ge 0$$

$$\Rightarrow \lambda < 1 \quad \text{and} \quad \lambda \le \frac{1}{2} \text{ or } \lambda \ge 1$$

$$\Rightarrow \lambda \le \frac{1}{2}$$

If $\lambda = \frac{1}{2}$, the equation becomes $\sin x = -1$. Thus,

$$x = \cdots, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \cdots$$

But this is not allowed since at these values, $\tan x$ becomes undefined. Thus, $\lambda < \frac{1}{2}$. Also, since $x \neq n\pi, n\pi \pm \frac{\pi}{6}$, we have

$$\sin x = \frac{\lambda}{\lambda - 1} \neq 0, \frac{1}{2}, -\frac{1}{2} \Rightarrow \lambda \neq 0, -1, \frac{1}{3}$$

$$\Rightarrow \quad \lambda \in \left(-\infty, \frac{1}{2}\right) \backslash \left\{-1, 0, \frac{1}{3}\right\}$$

S53. Since, S and T are permutations of each other, the sum of the elements in the two sets must be equal:

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha = \cos\alpha + \cos 2\alpha + \cos 3\alpha$$

Some manipulation and simplification leads to:

$$(2\cos\alpha + 1)(\sin 2\alpha - \cos 2\alpha) = 0$$

Case 1: $\boxed{2\cos\alpha + 1 = 0}$

$$\Rightarrow \quad \cos\alpha = -\frac{1}{2} \quad \Rightarrow \quad \alpha = 2k\pi \pm \frac{2\pi}{3}$$

For these values of α, it can be easily verified that $S \neq T$, and other than that, both of them may not even be three-element sets.

Case 2: $\boxed{\sin 2\alpha = \cos 2\alpha}$

$$\Rightarrow \quad \tan 2\alpha = 1 \quad \Rightarrow \quad \lambda = \frac{k\pi}{2} + \frac{\pi}{8}$$

For these values of α, we can prove that S and T are three-element sets such that $S = T$. The required values of α are thus $\frac{k\pi}{2} + \frac{\pi}{8}$, for $k \in \mathbb{Z}$.

S54. Applying the AM-GM inequality on the left side (LHS), we have

$$LHS \geq 2(\sin x \cos x)^{2\left(y - \frac{y^2}{4}\right)}$$

Since, $\sin x \cos x < 1$, we must have $y - \frac{y^2}{4} \geq 1$ (why ?)

$$\Rightarrow \quad (1 - \tfrac{y}{2})^2 \leq 0 \quad \Rightarrow \quad y = 2$$

Using $y = 2$ in the given equality yields $x = \frac{\pi}{4}$. The required pair is $(\frac{\pi}{4}, 2)$.

S55. We have $\cos(p\sin x) = \sin(p\cos x) = \cos(\frac{\pi}{2} - p\cos x)$, so that

$$p\sin x = \pm 2k\pi + \frac{\pi}{2} - p\cos x$$

$$\Rightarrow \quad p(\sin x + \cos x) = \pm 2k\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad \sqrt{2}p\sin\left(x + \frac{\pi}{4}\right) = \pm 2k\pi + \frac{\pi}{2} \tag{1}$$

To find the minimum positive value of p possible which satisfies (1), we take the minimum positive value of the RHS, which is $\frac{\pi}{2}$, and the maximum possible value of $\sin(x + \frac{\pi}{4})$, *i.e.*, 1 (at say $x = \frac{\pi}{4}$). Thus,

$$\sqrt{2}p_{\text{min}} = \frac{\pi}{2}$$

$$\Rightarrow \quad p_{\text{min}} = \frac{\pi}{2\sqrt{2}}$$

Straight Lines

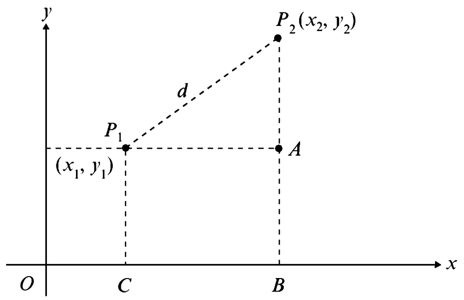
PART-A: Summary of Important Concepts

1. Basic Results

In this section, we summarize some basic results which are based on the concept of straight lines, but are widely used throughout other topics in Coordinate Geometry.

1.1 Distance Formula

This helps us calculate the distance between any two points in the plane whose coordinates are known, and its justification is based on the Pythagoras Theorem, as depicted in the figure below:



Note that:

(i) $OC = x_1$; $OB = x_2$

$\Rightarrow \quad BC = AP_1 = x_2 - x_1$

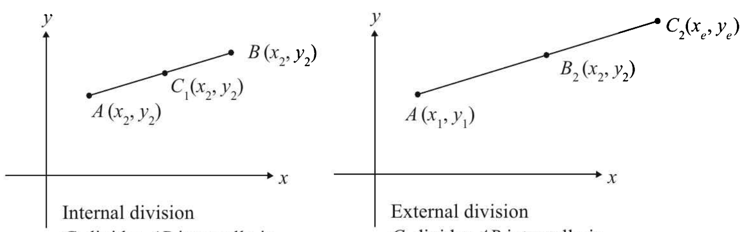
(ii) $CP_1 = y_1$; $BP_2 = y_2$

$\Rightarrow \quad AP_2 = y_2 - y_1$

$$d = \sqrt{AP_1^2 + AP_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.2 Section Formula

Suppose that we are given two fixed points in the co-ordinate plane, say $A(x_1, y_1)$ and $B(x_2, y_2)$. We need to find the co-ordinates of the point C which divides the line segment AB in the ratio $m : n$. Observe that two such points will exist. Name them C_1 and C_2. One of them will divide the line segment AB *internally* in the ratio $m : n$ while the other will divide AB in the same ratio *externally*, as shown in the figure below:



Internal division
C_1 divides AB internally in the ratio $m : n$. Thus,

$$\frac{AC_1}{C_1B} = \frac{m}{n}$$

External division
C_2 divides AB internally in the ratio $m : n$. Thus,

$$\frac{AC_2}{C_2B} = \frac{m}{n}$$

The coordinates of C_1 (internal division) will be

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

while the coordinates of C_2 (external division) will be

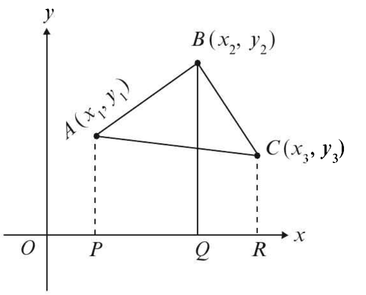
$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

We note that the mid-point of AB will have the coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Using the section formula, we can show that the coordinates of the centroid G of a triangle ABC, in terms of the coordinates of its vertices, are

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

1.3 Area of a Triangle

Suppose we are given three points in the co-ordinate plane: $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. We intend to find the area of ΔABC in terms of the given co-ordinates. How to evaluate this area is described in the figure below:



Note that
area (ΔABC) =
area (trapezium $APQB$)
+
area (trapezium $BQRC$)
−
area (trapezium $APRC$)

From the observations above, we can show that the required area is

$$\text{area}(\Delta ABC) = \left|\frac{1}{2}(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))\right|$$

$$= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

where we have taken the modulus of the determinant to obtain the area as a positive value.

2. Equations Representing a Straight Line

To specify a line uniquely in the plane, you can specify:

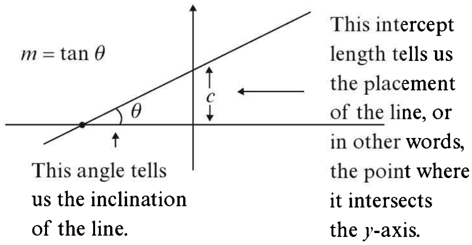
(a) one point on the line, and its slope (b) two points on the line

The different equations representing a straight line are based on one of the two cases above. We summarize these briefly:

2.1 Specifying a point on the line and its slope

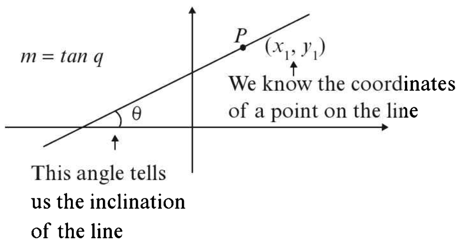
(a) Slope-Intercept Form

$$\boxed{y = mx + c}$$



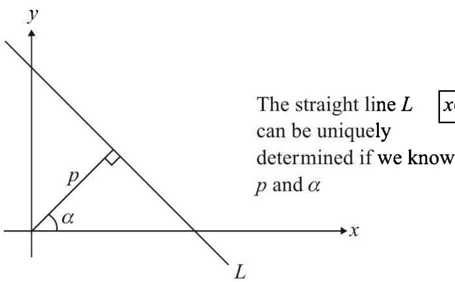
(b) Point-Slope Form

$$\boxed{y - y_1 = m(x - x_1)}$$



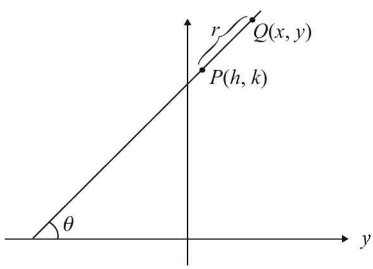
(c) Normal Form

We observe that we can also specify a point on the line and its slope by (instead) specifying the length of the perpendicular dropped from the origin to that line, and the inclination of that perpendicular:



$$\boxed{x\cos\alpha + y\sin\alpha = p}$$

(d) Polar Form



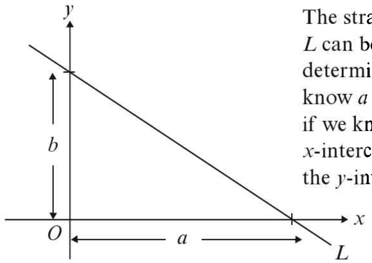
$$\boxed{\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r}$$

θ and $P(h, k)$ are known.
$Q(x, y)$ is an arbitrary point on the line
r is the distance between P and Q

2.2 Specifying two points on the line

(a) Intercept Form

A line can be uniquely determined if we are given the two points where this line intersects the x-axis and the y-axis:

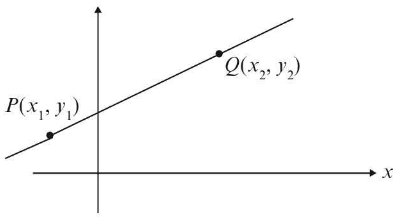


The straight line L can be uniquely determined if we know a and b, *i.e.*, if we know the x-intercept and the y-intercept

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$

(b) Two-point Form

Knowing *any* two points on the line can also enable us to specify the line uniquely:



$$\boxed{\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}}$$

2.3 General Form

The most general form for the equation of a straight line is $Ax+By+C=0$, where $A, B, C \in \mathbb{R}$ and at least one of A, B is non-zero.

We note that the various forms are inter-convertible amongst themselves.

3. Important Facts and Formulae

3.1 Point and Angle of Intersection

Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c=0$ be two arbitrary lines; the slopes of these lines will (respectively) be

$$m_1=-\frac{a_1}{b_1},\ m_2=-\frac{a_2}{b_2}$$

The coordinates of the point of intersection are given by

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

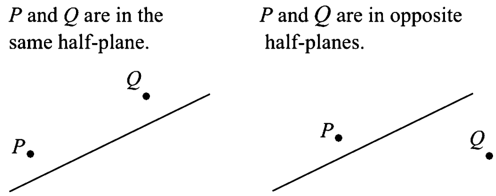
The acute angle of intersection θ will be given by

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$$

We see that the lines are parallel if $m_1=m_2$ and perpendicular if $m_1m_2=-1$.

3.2 Half-Planes

Any line $L \equiv ax + by + c = 0$ divides the plane into two half-planes. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. Two cases arise:

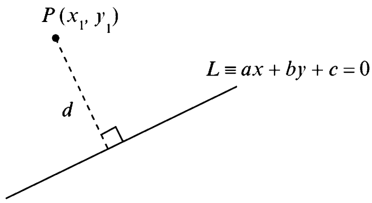


We can show that in the first case, $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ are of the same sign, while in the second case, they are of the opposite signs.

3.3 Length of perpendicular

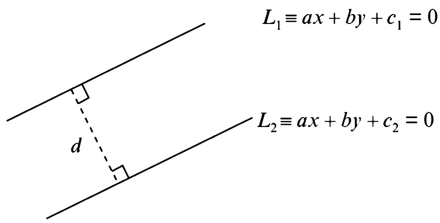
(a) The distance of the point P from the line L, or the length of the perpendicular from P to L, is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



(b) The length of the perpendicular between two parallel lines is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$



3.4 Concurrency

Consider any three non-parallel lines $L_i \equiv a_i x + b_i y + c_i = 0$, where $i = 1, 2, 3$. Any two of these three lines will always intersect, but if we want the third line also to pass through the same point of intersection (that is, we want the lines to be concurrent), a special condition must be satisfied:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

An alternative algebraic condition for proving concurrency is to show that there exist non-zero real constants λ_1 and λ_2 such that

$$L_3 = \lambda_1 L_1 + \lambda_2 L_2$$

Can you prove why this condition is sufficient to demonstrate concurrency?

3.5 Family of lines

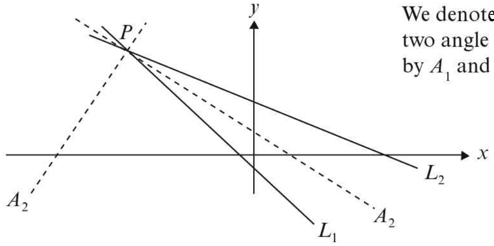
Any line passing through the intersection point P of $L_1 = 0$ and $L_2 = 0$, where $\lambda \in \mathbb{R}$, can be specified as

$$L_1 + \lambda L_2 = 0$$

As λ is varied, different lines are obtained, *all passing through P*. On many occasions, specifying a line through the family of lines approach is extremely powerful, as we'll see in the worked out examples and the problems.

3.6 Angle Bisectors

For any two intersecting lines, two angle bisectors will exist:



We denote the two angle bisectors by A_1 and A_2.

From a straightforward geometrical consideration that any point on A_1 or A_2 must be equidistant from L_1 and L_2, we can show that the equations of A_1 and A_2 will be:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

where one sign (+ or –) will correspond to A_1 and the other to A_2. Often, the problem arises as to which sign corresponds to which bisector. For example, which is the bisector of the angle containing the origin?

To determine that, we have a simple technique: we first arrange the equations of L_1 and L_2 so that c_1 and c_2 are both of the same sign. Subsequently, the bisector given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the angle containing the origin. Try justifying this result.

4. Pair of Straight Lines

It is extremely important to properly understand the concept of algebraically representing two lines simultaneously through a joint equation. Consider the following two lines:

$$L_1 \equiv y - m_1x - c_1 = 0$$

$$L_2 \equiv y - m_2x - c_2 = 0$$

What do you think will $L_1L_2 = 0$ represent ? It is obvious that any point lying on L_1 and L_2 will satisfy $L_1L_2 = 0$, and thus $L_1L_2 = 0$ represents the set of points constituting both the lines, *i.e.*,

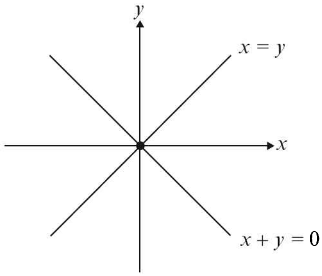
$$\boxed{L_1L_2 = 0 \text{ represents the pair of straight lines given by } L_1 = 0 \text{ and } L_2 = 0}$$

For example, consider the equation $y^2 - x^2 = 0$. What does this represent? We have

$$y^2 - x^2 = 0 \tag{1}$$

$$\Rightarrow \quad (y+x)(y-x) = 0$$

$\Rightarrow$ (1) represents the pair of straight lines $x = y$ and $x + y = 0$.



The pair of lines given by $y^2 - x^2 = 0$

There is nothing special about considering a pair. We can similarly define the joint equation of *n* straight lines $L_i \equiv y - m_i x - c_i = 0 \ (i = 1, 2, \ldots, n)$ as

$$L_1 L_2 \ldots L_n = 0$$

$$\Rightarrow \quad (y - m_1 x - c_1)(y - m_2 x - c_2)\cdots(y - m_n x - c_n) = 0 \tag{2}$$

Any point lying on any of these *n* straight lines will satisfy (2), and thus (2) represents the set of all points constituting the *n* lines, *i.e.*, (2) represents the joint equation of the *n* straight lines.

4.1 Pair of lines passing through the origin

Let $L_1 \equiv y - m_1 x_1 = 0$ and $L_2 \equiv y - m_2 x = 0$ be two lines passing through the origin. Their joint equation is:

$$L_1 L_2 = 0 \quad \Rightarrow \quad y^2 + m_1 m_2 x^2 - (m_1 + m_2)xy = 0 \tag{1}$$

The general equation of a pair of straight lines passing through the origin will be

$$ax^2 + 2hxy + by^2 = 0 \tag{2}$$

We can compare (1) and (2) to see that

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1 m_2 = \frac{a}{b} \tag{3}$$

This is because m_1 and m_2 will be the roots of the following quadratic (obtained from (2) by dividing it by x^2):

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

(3) enables us to calculate the angle between the two lines:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{2\sqrt{h^2 - ab}}{|a+b|}$$

The two lines are:

(i) parallel (in fact coincident since both pass through the origin) if $h^2 = ab$.
(ii) perpendicular if $a + b = 0$.

4.2 General Equation of a Pair of Straight Lines

The general equation representing a pair of straight lines is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

when the following condition is satisfied:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

The conditions for the two lines to be parallel or perpendicular, and the expression giving the angle between the two lines are the same as in the homogeneous case. That these formulae in the homogenous and the general case are the same should not be surprising since the slope of any line is independent of the constant term appearing in its equation.

4.3 Angle Bisectors

The joint equation of the angle bisectors of the lines represented by $ax^2 + 2hyxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

5. Homogenizing

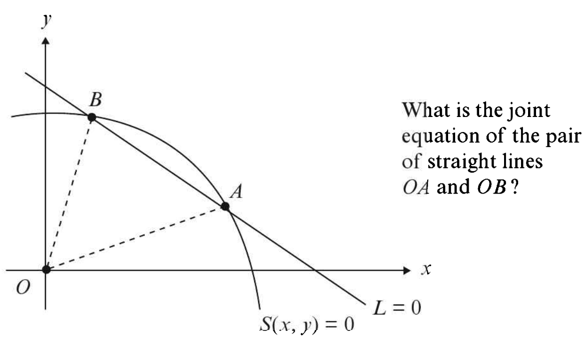
The technique of homogenizing is a very powerful application of the concept of pair of straight lines, and must be addressed in some detail. Consider a second degree curve $S(x, y)$ with the equation

$$S(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a straight line

$$L \equiv px + qy + r = 0$$

intersecting $S = 0$ in A and B. Let O be the origin. What is the joint equation of OA and OB?



The insight that we use here is that since both OA and OB pass through the origin, their joint equation will be homogenous. We now construct a homogenous equation and show that both A and B satisfy it; that equation is then guaranteed to jointly represent OA and OB.

First of all, observe that since A and B satisfy the equation of L, *i.e.*, $px+qy+r=0$, they will also satisfy the relation

$$\frac{px+qy}{-r}=1: \quad \text{Both } A \text{ and } B \text{ will satisfy this relation.}$$

Now, we *homogenize* the equation of the second degree curve $S(x, y)$ using the relation above; consider this equation:

$$ax^2+2hxy+by^2+2gx\left(\frac{px+qy}{-r}\right)+2fy\left(\frac{px+qy}{-r}\right)+c\left(\frac{px+qy}{-r}\right)^2=0 \qquad (1)$$

Can you understand why we've done this? The equation we obtain above is a second degree homogenous equation, and so it must represent two straight lines passing through the origin. Which two straight lines? Since A and B satisfy the equation of the original curve as well as the relation $\frac{px+qy}{-r}=1$, A and B both satisfy the homogenized equation in (1). What does this imply ? That (1) is the joint equation of OA and OB!

Go over the discussion again if you find it confusing. You must fully understand this technique since it will find very wide usage in subsequent chapters.

IMPORTANT IDEAS AND TIPS

1. *Forms of a Line's Equation*. In this chapter, we study many forms of equations which can represent a straight line. But remember that these forms are all based on the concept that to uniquely specify a line, you need

(a) either a point on the line and its slope
(b) or two points on the line

2. *Polar Form*. An extremely powerful way to represent points on a line is to use the polar (or parametric) form. In particular, whenever we are dealing with distances along a line, the polar form may help in a significant simplification of calculations. Refer to some of the examples in this chapter where the polar form has been used, and try to work out alternate solutions to be able to appreciate the power of this form.

3. *Half-Planes*. Any line (say $ax+by+c=0$) divides the plane into two halves. Always remember that any two points (x_1, y_1) and (x_2, y_2) will lie in the same half-plane if ax_1+by_1+c and ax_2+by_2+c are of the same sign, and in opposite half-planes if they are of the opposite signs.

4. *Angle Bisectors*. Some students have trouble remembering the relation representing the angle bisectors of a pair of lines. Keep the following basic facts in mind:

(a) There will be two angle bisectors, both perpendicular to each other.
(b) Any point on any of the two angle bisectors will be equidistant from the two lines. This is the fact used to deduce the formula for the angle bisectors.

These facts along with the fact about half-planes mentioned above should enable you to determine the angle bisector of that angle which contains a specified point, say, the origin.

5. *Pair of Lines*. A useful fact to remember is that the homogeneous (joint) equation $ax^2+2hxy+by^2=0$ representing two lines passing through the origin can equivalently be seen as a quadratic in $m: bm^2+2hm+a=0$, whose two roots will represent the slopes of the two lines. Similarly, a third degree homogeneous equation like $ax^3+bx^2y+cxy^2+dy^3=0$ which represent three lines passing through the

origin, can equivalently be seen as a third degree equation in $m : dm^3 + cm^2 + bm + a = 0,$ the three roots of which will correspond to the slopes of the three lines. This can obviously be extended to *n*th degree homogeneous equations.

6. *Homogenizing*. It is important to understand why the technique of homogenizing works. When you homogenize the equation of a second degree curve S with the relevant expression from a straight line L, the resulting joint equation must represent two lines passing through the origin, and those two lines must also pass through the two intersection points of S and L. This technique is frequently used throughout subsequent chapters of coordinate geometry.

Straight Lines

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

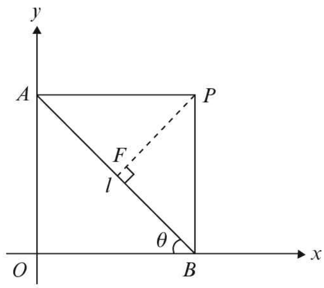
Example 1

A rod AB of length l slides with its ends on the coordinate axes. Let O be the origin. The rectangle $OAPB$ is completed. The locus of the foot of the perpendicular drawn from P onto AB is

(A) $x^2 + y^2 = l^2$ (C) $x^{3/4} + y^{3/4} = l^{3/4}$ (E) None of these

(B) $x^{3/2} + y^{3/2} = l^{3/2}$ (D) $x^{2/3} + y^{2/3} = l^{2/3}$

Solution:



We need to find the locus of F as AB slides between the axes.

In terms of the variable θ, A and B, and hence P, have the coordinates

$$A \equiv (0, l\sin\theta), \quad B \equiv (l\cos\theta, 0), \quad P \equiv (l\cos\theta, l\sin\theta)$$

The slope of PF can be observed to be $\cot\theta$, so that its equation is

$$y - l\sin\theta = \cot\theta(x - l\cos\theta) \qquad (1)$$

The equation of AB is

$$x\sec\theta + y\,\text{cosec}\;\theta = l \qquad (2)$$

The intersection of (1) and (2) gives us the point $F(h, k)$:

$$h = l\cos^3\theta, \quad k = l\sin^3\theta \Rightarrow \quad \cos\theta = \left(\frac{h}{l}\right)^{1/3}, \quad \sin\theta = \left(\frac{k}{l}\right)^{1/3}$$

Eliminating θ, we have

$$h^{2/3} + k^{2/3} = l^{2/3}$$

Thus, the locus of F is

$$x^{2/3}+y^{2/3}=l^{2/3}$$

We see that the correct option is (D). ■

Example 2

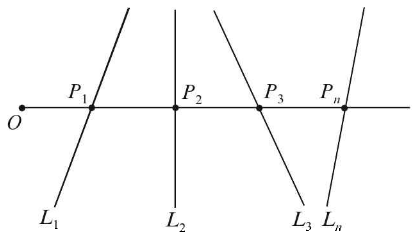
Consider a fixed point O and n fixed straight lines. Through O, a (variable) line is drawn intersecting the fixed lines in $P_1, P_2, \ldots, P_n$. On this variable line, a point P is taken such that

$$\frac{n}{OP}=\frac{1}{OP_1}+\frac{1}{OP_2}+\cdots+\frac{1}{OP_n}$$

The locus of P is

(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola (E) none of these

Solution:



A figure illustrating the situation described.

Assume the equations of the fixed lines to be

$$L_i \equiv a_i x+b_i y+c_i=0, \quad i=1,2,\ldots,n$$

and the coordinates of the fixed point O to be (h,k). Let the slope of the variable line be represented by $\tan\theta$. Thus, the points P_i have the coordinates

$$P_i \equiv (h+OP_i\cos\theta, k+OP_i\sin\theta) \quad i=1,2,\ldots,n$$

Since each P_i satisfies L_i, we have

$$a_i(h+OP_i\cos\theta)+b_i(k+OP_i\sin\theta)+c_i=0$$

$$\Rightarrow \quad OP_i=\frac{-(ha_i+kb_i+c_i)}{a_i\cos\theta+b_i\sin\theta}$$

Assume the coordinates of P (whose locus we wish to determine) to be (x,y). Thus, we have

$$x=h+OP\cos\theta, \quad y=k+OP\sin\theta \tag{1}$$

From the relation given in the equation, we have

$$\frac{n}{OP}=\sum\frac{1}{OP_i} \quad \Rightarrow \quad \frac{n}{OP}=-\sum\frac{a_i\cos\theta+b_i\sin\theta}{ha_i+kb_i+c_i}$$

$$=\left(-\sum\frac{a_i}{ha_i+kb_i+c_i}\right)\cos\theta+\left(-\sum\frac{b_i}{ha_i+kb_i+c_i}\right)\sin\theta$$

$$=\lambda\cos\theta+\mu\sin\theta$$

(These substitutions have been done for convenience)

From (1), we have

$$\frac{n}{OP} = \frac{\lambda(x-h)}{OP} + \frac{\mu(y-k)}{OP}$$

$$\Rightarrow \quad \lambda x + \mu y - (\lambda h + \mu k + n) = 0$$

This is the locus of the point P; it is evidently a straight line. Therefore, none of the first four options is correct. The correct option is (E). ■

Example 3

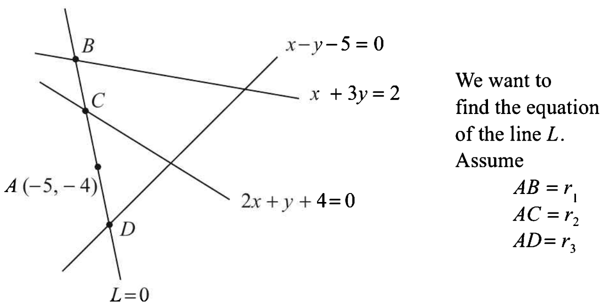
A line through $A(-5,-4)$ meets the lines $x+3y+2=0$, $2x+y+4=0$ and $x-y-5=0$ at the points B, C and D respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2,$$

and the line's equation is written in the form $2x+by+c=0$, the value of c is

(A) 18 (B) 20 (C) 22 (D) 24

Solution: This is a very good example to illustrate the advantages of using the polar form to represent a line. Consider the following figure:



The figure above roughly sketches the situation described in the equation. Let B, C and D be at distances r_1, r_2 and r_3 from A along the line $L=0$, whose equation we wish to determine. Assume the inclination of L to be θ. Thus, B, C and D have the coordinates (respectively):

$$B \equiv (-5 + r_1\cos\theta, \quad -4 + r_1\sin\theta)$$

$$C \equiv (-5 + r_2\cos\theta, \quad -4 + r_2\sin\theta)$$

$$D \equiv (-5 + r_3\cos\theta, \quad -4 + r_3\sin\theta)$$

Since these three points (respectively) satisfy the three given equations, we have:

Point B: $(-5 + r_1\cos\theta) + 3(-4 + r_1\sin\theta) + 2 = 0 \quad \Rightarrow \quad r_1 = \dfrac{15}{\cos\theta + 3\sin\theta}$

Point C: $2(-5 + r_2\cos\theta) + (-4 + r_2\sin\theta) + 4 = 0 \quad \Rightarrow \quad r_2 = \dfrac{10}{2\cos\theta + \sin\theta}$

Point D: $(-5 + r_3\cos\theta) - (-4 + r_3\sin\theta) - 5 = 0 \quad \Rightarrow \quad r_3 = \dfrac{6}{\cos\theta - \sin\theta}$

It is given that

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$$i.e., \left(\frac{15}{r_1}\right)^2 + \left(\frac{10}{r_2}\right)^2 = \left(\frac{6}{r_3}\right)^2$$

$$\Rightarrow \quad (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$$\Rightarrow \quad 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad (2\cos\theta + 3\sin\theta)^2 = 0$$

$$\Rightarrow \quad \tan\theta = \frac{-2}{3} \quad \Rightarrow \quad m = \frac{-2}{3}$$

Thus, we obtain the slope of L as $\frac{-2}{3}$. The equation of L can now be easily written:

$$L: y - (-4) = \frac{-2}{3}(x - (-5))$$

$$\Rightarrow \quad L: 2x + 3y + 22 = 0$$

Comparing it with the expression $2x + by + c = 0$, we see that $c = 22$. The correct option is (C) ■

Example 4

The angle bisector of the angle between the straight lines $L_1: 3x - 4y + 7 = 0$ and $L_2: 12x - 5y - 8 = 0$ which contains the origin is

(A) $99x - 77y + 51 = 0$ (C) $77x + 99y + 51 = 0$

(B) $99x - 77y + 31 = 0$ (D) $77x + 99y + 31 = 0$

Solution: Following the discussion in the theory, we first modify the equations L_1 and L_2 so that the constant terms in both the equations are of the same sign (say both positive):

$$L_1: 3x - 4y + 7 = 0$$

$$L_2: -12x + 5y + 8 = 0$$

The angle bisector of the angle containing the origin is

$$\frac{(3x - 4y + 7)}{\sqrt{3^2 + 4^2}} = +\frac{(-12x + 5y + 8)}{\sqrt{12^2 + 5^2}}$$

$$\Rightarrow \quad 99x - 77y + 51 = 0$$

Therefore, the correct option is (A). ■

Example 5

Consider the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point (1, –3). Let $(a,1)$ and $(b,2)$ be two points lying on this line. The value of $a - b$ is

(A) 0 (B) 1 (C) 2 (D) 3

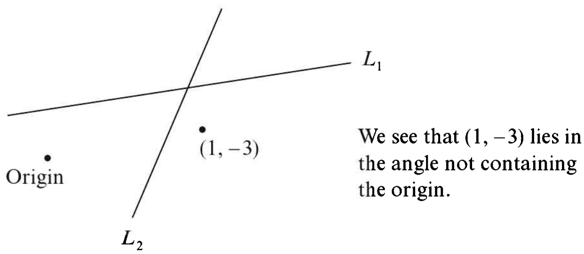
Solution: We arrange the equations of the two lines such that constant terms are positive:

$$L_1: -x-2y+11=0$$
$$L_2: -3x+6y+5=0$$

Note that

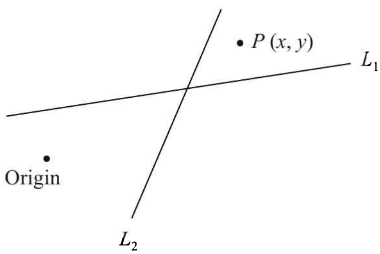
$$L_1(\text{at origin})>0 \text{ and } L_1(\text{at}(1,-3))>0$$
$$\Rightarrow \quad \text{Origin and } (1,-3) \text{ are on the same side of } L_1.$$
$$L_2(\text{at origin})>0 \text{ and } L_2(\text{at}(1,-3))<0$$
$$\Rightarrow \quad \text{Origin and } (1,-3) \text{ are on the opposite sides of } L_2.$$

This means that the point (1, –3) does not lie in the same region as the origin, since (1, –3) must be on the opposite side of the origin with respect to L_2. The example figure below will make this clear:



Thus, it is clear that (1, –3) lies in the angle not containing the origin.

On a side-note, we make an important observation. If suppose a point $P(x, y)$ lies on the opposite side of the origin with respect to both L_1 and L_2, it would lie in the vertically opposite angle to the angle in which the origin lies; in such a case, the angle bisector of the angle containing P is the same as that of the one containing the origin.



Coming back to our problem, to determine the angle bisector of the angle containing (1, –3), we simply determine the angle bisector of the angle not containing the origin, *i.e.*,

$$\frac{-x-2y+11}{\sqrt{5}}=-\frac{-3x+6y+5}{3\sqrt{5}}$$
$$\Rightarrow \quad x=\frac{19}{3}$$

Note that to determine the angle bisector of the angle containing the point P as in the figure above, we would have chosen the angle bisector of the angle *containing* the origin.

Since the line is vertical, the values of a and b will be the same, and hence the required value is 0. The correct option is (A). ■

Example 6

The equation $ax^3+bx^2y+cx^2+dy^3=0$ is a third degree homogenous equation and hence represents three straight lines passing through the origin. The condition so that two of these three lines may be perpendicular will be

(A) $(a+d)^2+ac+bd=0$ (B) $a^2+ac+bd+d^2=0$

(C) $a^2+d^2+ad+bc=0$ (D) None of these

Solution: We divide the given equation by x^3 and substitute $\frac{y}{x}=m$ to obtain:

$$dm^3+cm^2+bm+a=0 \tag{1}$$

This has three roots, say m_1, m_2, m_3, corresponding to the three straight lines. Since we want two of these lines to be perpendicular, we can assume

$$m_1m_2=-1$$

From (1), we have

$$m_1\, m_2\, m_3=\frac{-a}{d} \Rightarrow m_3=\frac{a}{d}$$

Substituting this value of m_3 back in (1) (since m_3 is a root of (1)), we obtain

$$\frac{da^3}{d^3}+\frac{ca^2}{d^2}+\frac{ba}{d}+a=0 \quad \Rightarrow \quad a^2+ac+bd+d^2=0$$

The correct option is (B). ■

Example 7

(a) Show that the four lines given by the equations

$$3x^2+8xy-3y^2=0$$

$$3x^2+8xy-3y^2+2x-4y-1=0$$

form a square.

(b) If A represents the area of this square, the value of $100A$ is

(A) 5 (B) 10 (C) 20 (D) 30

Solution: The first joint equation can be easily factorised to yield

$$(3x-y)(x+3y)=0$$

$$\Rightarrow \quad 3x-y=0, \quad x+3y=0 \tag{1}$$

These are perpendicular lines intersecting at the origin. The second joint equation can now be factorised as

$$(3x-y+\alpha)(x+3y+\beta)=0$$

where α and β can be determined by the comparison of coefficients:

$$\alpha=-1, \quad \beta=1$$

The other two sides are thus

$$3x-y-1=0, \quad x+3y+1=0 \tag{2}$$

From (1) and (2), it should be evident that the four lines form a square. The length l of the sides of this square can be evaluated by determining the perpendicular distance between any pair of opposite sides, say $3x-y=0$ and $3x-y-1=0$:

$$l=\frac{|0-(-1)|}{\sqrt{1^2+3^2}}=\frac{1}{\sqrt{10}}$$

It is evident that $100A=100l^2=10$. The correct option is (B). ■

Example 8

(a) Show that all the chords of the curve $3x^2-y^2-2x+4y=0$ which subtend a right angle at the origin pass through a fixed point.

(b) If that point is represented as (x,y), then the value of $y-x$ is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: Let $y=mx+c$ be a chord of the curve which subtends a right angle at the origin. The joint equation of the lines joining the origin to points of intersection of $y=mx+c$ and the curve is

$$3x^2-y^2+(4y-2x)\left(\frac{y-mx}{c}\right)=0$$

This represents two perpendicular lines if

$$\text{Coeff. of } x^2+\text{Coeff. of } y^2=0$$

$$3+\frac{2m}{c}+\frac{4}{c}-1=0 \quad\Rightarrow\quad c+m+2=0$$

$$\Rightarrow\quad (-2)=m(1)+c \qquad (1)$$

(1) shows that $y=mx+c$ always passes through the fixed point (–2, 1). Therefore, $y-x=3$. The correct option is (D). ■

SUBJECTIVE TYPE EXAMPLES

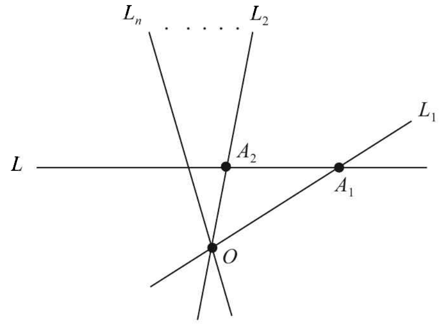
Example 9

Lines are drawn to intersect *n* concurrent lines at the points $A_1, A_2, \ldots, A_n$ such that

$$\sum_{i=1}^{n} \frac{1}{OA_i} = \text{constant}$$

where O is the point of concurrency. Show that the variable lines all pass through a fixed point.

Solution: There's no loss of generality in assuming O to be the origin since we are dealing only with lengths which are invariant with respect to the choice of the coordinate axes.



L_i's are all fixed lines.
L is the variable line.

The inclinations of the fixed lines can be assumed to be θ_i so that the points A_i have the coordinates

$$A_i \equiv (OA_i \cos\theta_i, OA_i \sin\theta_i)$$

Let the variable line have the equation $ax + by + c = 0$. Since all the A_i's lie on this line, we have

$$aOA_i \cos\theta_i + bOA_i \sin\theta_i + c = 0$$

$$\Rightarrow \quad OA_i = \frac{-c}{a\cos\theta_i + b\sin\theta_i} \tag{1}$$

According to the condition specified in the question,

$$\sum_{i=1}^{n} \frac{1}{OA_i} = \text{constant} = K \text{ (say)} \tag{2}$$

Thus, using (1) in (2), we have

$$\sum_{i=1}^{n} \frac{a\cos\theta_i + b\sin\theta_i}{-c} = K$$

$$\Rightarrow \quad a\left(\frac{\sum_{i=1}^{n}\cos\theta_i}{K}\right) + b\left(\frac{\sum_{i=1}^{n}\sin\theta_i}{K}\right) + c = 0 \tag{3}$$

(3) shows that the variable line L always passes through the fixed point $\left(\frac{\sum_{i=1}^{n}\cos\theta_i}{K}, \frac{\sum_{i=1}^{n}\sin\theta_i}{K}\right)$. ■

Example 10

Prove that the three lines L_1, L_2 and L_3 given by $a_i x + b_i y + c_i = 0$, $i = 1,2,3$, are concurrent if we can find three constants λ_1, λ_2 and λ_3 such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

Solution: Assume that L_1 and L_2 intersect at the point P whose co-ordinates are (x_0, y_0). P should satisfy the equations of both L_1 and L_2.

$$L_1(\text{at } P) \equiv a_1 x_0 + b_1 y_0 + c_1 = 0 \tag{1}$$

$$L_2(\text{at } P) \equiv a_2 x_0 + b_2 y_0 + c_2 = 0 \tag{2}$$

Now assume that we can find three non-zero constants λ_1, λ_2 and λ_3 such that $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$. We will prove that due to this condition, L_3 will definitely have to pass through P:

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

$$\Rightarrow \quad L_3 = \left(-\frac{\lambda_1}{\lambda_3}\right)L_1 + \left(-\frac{\lambda_2}{\lambda_3}\right)L_2$$

If we evaluate the value of L_3 at P, we get

$$\begin{aligned} L_3(\text{at } P) &= \left(-\frac{\lambda_1}{\lambda_3}\right) \times L_1(\text{at } P) + \left(-\frac{\lambda_2}{\lambda_3}\right) \times L_2(\text{at } P) \\ &= \left(-\frac{\lambda_1}{\lambda_3}\right) \times 0 + \left(-\frac{\lambda_2}{\lambda_3}\right) \times 0 \qquad \left\{\begin{matrix}\text{By (1)} \\ \text{and (2)}\end{matrix}\right\} \\ &= 0 \end{aligned}$$

Since the value of L_3 is 0 at P, the line L_3 must pass through P. Thus, L_1, L_2 and L_3 are concurrent. Try using this result to show that the medians in a triangle are concurrent. ■

Example 11

The curves

$$C_1 : a_1 x^2 + 2h_1 xy + b_1 y^2 + 2g_1 x = 0$$

$$C_2 : a_2 x^2 + 2h_2 xy + b_2 y^2 + 2g_2 x = 0$$

intersect at two points A and B other than the origin. Find the condition for OA and OB to be perpendicular.

Solution: Assume the equation of AB to be $y = mx + c$. Thus, using the homogenizing technique, we can write the joint equation of OA and OB:

Homogenizing C_1: $a_1 x^2 + 2h_1 xy + b_1 y^2 + 2g_1 x\left(\dfrac{y - mx}{c}\right) = 0$

$$\Rightarrow \quad \left(a_1 - \frac{2mg_1}{c}\right)x^2 + 2h_1 xy + b_1 y^2 + \frac{2g_1 xy}{c} = 0$$

This is the joint equation of OA and OB. OA and OB are perpendicular if

$$a_1 - \frac{2mg_1}{c} + b_1 = 0 \quad \Rightarrow \quad \frac{m}{c} = \frac{a_1 + b_1}{2g_1} \tag{1}$$

Homogenizing C_2: Similarly, we can again evaluate the joint equation of OA and OB by homogenizing the equation of C_2:

$$\left(a_2 - \frac{2mg_2}{c}\right)x^2 + 2h_2xy + b_2y^2 + \frac{2g_2xy}{c} = 0$$

The perpendicularity condition gives

$$a_2 - \frac{2mg_2}{c} + b_2 = 0 \quad \Rightarrow \quad \frac{m}{c} = \frac{a_2 + b_2}{2g_2} \tag{2}$$

From (1) and (2), the necessary required condition is

$$\frac{a_1 + b_1}{g_1} = \frac{a_2 + b_2}{g_2}$$

■

Example 12

Show that the equation $\lambda(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to one another.

Solution: Observe that since the equation is homogenous, it will represent three straight lines passing through the origin. Let the slopes of the three lines be m_1, m_2 and m_3. Thus m_1, m_2 and m_3 are the roots of the equation

$$\lambda(1 - 3m^2) + m^3 - 3m = 0 \qquad \left(\text{where } m = \frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{3m - m^3}{1 - 3m^2} = \lambda$$

Since $m = \frac{y}{x} = \tan\theta$, where θ is the inclination of the line, we have

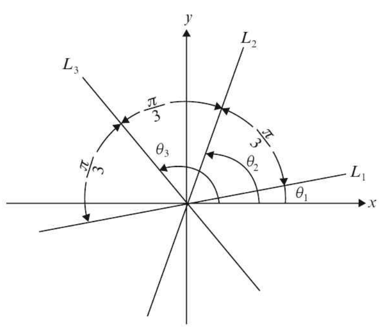
$$\lambda = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta \quad \Rightarrow \quad \tan 3\theta = \lambda$$

$$\Rightarrow \quad 3\theta = n\pi + \tan^{-1}\lambda \quad \Rightarrow \quad \theta = \frac{n\pi + \tan^{-1}\lambda}{3}$$

Since there are three lines corresponding to the joint equation, we'll have three corresponding angles of inclination:

$$\theta_1 = \frac{\tan^{-1}\lambda}{3}, \quad \theta_2 = \frac{\pi}{3} + \tan^{-1}\lambda, \quad \theta_3 = \frac{2\pi}{3} + \tan^{-1}\lambda$$

The angles of inclination show that the three lines are equally inclined to one another.



■

Straight Lines

PART-C: Advanced Problems

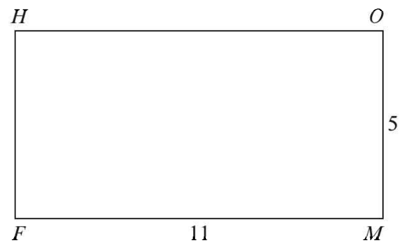
OBJECTIVE TYPE EXAMPLES

P1. (a) Consider a line segment AB where $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$. In what ratio does a line $L \equiv ax + by + c = 0$ divide AB?

(b) A line intersects BC, CA and AB (one of them extended) in ΔABC at P, Q and R respectively. The magnitude of the product $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ is

(A) 1 (B) 2 (C) 3 (D) 4

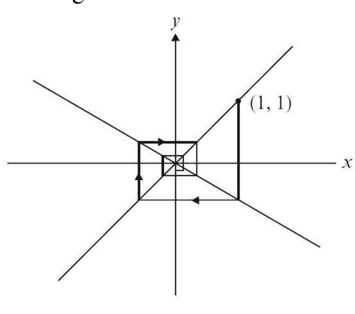
P2. $OHFM$ is a rectangle. ΔABC has H as the intersection of the altitudes, O the center of the circumcircle, M the mid-point of BC and F the foot of altitude from A.



The length of BC is

(A) 24 (B) 26 (C) 28 (D) 30

P3. Consider two fixed lines $y - x = 0$ and $ay + x = 0$, $a > 1$. A particle P starts from (1, 1) to reach the origin in the manner depicted in the figure.



The total distance covered by the particle is

(A) $\dfrac{a+1}{a-1}$ (B) $\dfrac{2(a+1)}{a-1}$ (C) $\dfrac{a-1}{a+1}$ (D) $\dfrac{2(a-1)}{a+1}$ (E) None of these

P4. Let (h, k) be the point which is at unit distance from the lines $L_1 \equiv 3x-4y+1=0$ and $L_2 \equiv 8x+6y+1=0$ and lies below L_1 and above L_2. The magnitude of h/k is

(A) 4 (B) 6 (C) 8 (D) 12

P5. A particle starts from the point $P(0, b)$ to reach $Q(a, 0)$ in the following manner. Initially, it moves towards OQ perpendicularly, and after reaching OQ, it moves towards PQ perpendicularly, and then again alternatively towards OQ and PQ until it reaches Q. The total distance covered by the particle is

(A) $\dfrac{a\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}-a}$ (B) $\dfrac{b\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}-a}$ (C) $\dfrac{2ab}{\sqrt{a^2+b^2}-a}$ (D) None of these

P6. If the points $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$ are collinear for distinct a, b and c, the value of $abc-(ab+bc+ca)+3(a+b+c)$ is

(A) 0 (B) 1 (C) -1 (D) None of these

P7. The difference between the squares of the shortest and longest intercepts made by the family of lines $ax+by=a+2b$ on the circle $x^2+y^2-10x-10y+1=0$ is

(A) 50 (B) 52 (C) 54 (D) 56

P8. The sides of a rhombus are parallel to $y=2x+3$ and $y=7x+2$. The diagonals of the rhombus intersect at $(1, 2)$. One vertex of the rhombus lies on the Y-axis. The possible values of the ordinate of this vertex are

(A) $\dfrac{5}{\sqrt{10}-1}$ (B) $\dfrac{5}{\sqrt{10}+1}$ (C) $\dfrac{-5}{\sqrt{10}-1}$ (D) $\dfrac{-5}{\sqrt{10}+1}$

P9. A triangle has two of its sides along the lines $y=m_1x$ and $y=m_2x$ where m_1, m_2 are the roots of the equation $3x^2+10x+1=0$. It is given that $H(6,2)$ is the orthocentre of the triangle. If the equation of the third side is $ax+by+1=0$, the value of $a+b$ is

(A) 1 (B) 2 (C) 3 (D) 4

P10. Which of the following are possible values of the slope of the line which passes through (1, 1) and intersects the lines $2x^2+5xy+2y^2=0$ at equal distances from the origin?

(A) $m=1$ (B) $m=2$ (C) $m=\dfrac{1}{2}$ (D) $m=\dfrac{1}{3}$ (E) $m=\dfrac{1}{4}$

P11. Let $ax^2+2hxy-ay^2+2gx+2fy+c=0$ be the joint equation of two perpendicular lines intersecting at P. From the origin O, perpendiculars are drawn to the two lines, intersecting them in Q and R. The area of the rectangle $OPQR$ is

(A) $\dfrac{c}{\sqrt{h^2+a^2}}$ (B) $\dfrac{c}{\sqrt{h^2+2a^2}}$ (C) $\dfrac{2c}{\sqrt{h^2+2a^2}}$ (D) $\dfrac{c}{\sqrt{h^2+4a^2}}$ (E) None of these

P12. Which of the following conditions must be satisfied so that two of the lines represented by the equation

$$ay^4+by^3x+cy^2x^2+dyx^3+ax^4=0$$

will bisect the angle between the other two?

(A) $b+d=0$ (B) $bd=1$ (C) $c+4a=0$ (D) $c+6a=0$ (E) $ac+1=0$

SUBJECTIVE TYPE EXAMPLES

P13. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$.

P14. A straight line L through the origin meets the lines $x + y = 1$ and $x + y = 3$ at P and Q respectively. Through P and Q, two straight lines L_1 and L_2 are drawn, parallel to $2x - y = 5$ and $3x + y = 5$ respectively. Lines L_1 and L_2 intersect at R. Find the locus of R as L varies.

P15. In a certain geometry, define the distance $d(P_1, P_2)$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as

$$d(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|$$

Define the μ–distance between a point P and a line L as

$$d(P, L) = \min\{d(P, P_1), P_1 \in L\}$$

Find the μ–distance of the point $(1, 0)$ from the line $y = x\sqrt{3} + 3$.

P16. Consider the straight line $2x + 3y = 6$. When the coordinate axes are rotated by a certain amount, the line cuts intercepts of length α on the new axes. Find α.

P17. (a) Find the area of the parallelogram formed by the lines $ax + by + c = 0$, $ax + by + d = 0$, $a'x + b'y + c' = 0$ and $a'x + b'y + d' = 0$.

(b) Find the condition for this parallelogram to be a rhombus.

P18. The vertices of a triangle are $(-1,3), (-2,2)$ and $(3,-1)$. A triangle is formed by translating the sides of the given triangle by one unit inwards. Find the equation of that side of the new triangle which is nearest to the origin.

P19. Determine all values of α for which the point (α, α^2) lies inside the triangles formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, and $5x - 6y - 1 = 0$.

P20. A variable point P moves in the first quadrant such that the sum of its distances from the coordinate axes is a constant equal to a. The perpendiculars drawn from P onto the coordinate axes intersect the axes at points A and B. Find the locus of the foot of the perpendicular drawn from P onto AB.

P21. Let the sides of a parallelogram be $U = a, U = b, V = a'$ and $V = b'$, where $U \equiv lx + my + n$ and $V \equiv l'x + m'y + n'$. Show that the equation of the diagonal through the intersection points of $U = a, V = a'$ and $U = b, V = b'$ is given by

$$\begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix}$$

P22. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L_3 different from L_2 which passes through P and makes the same angle θ with L_1.

P23. Show that the straight lines $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ and $Ax + By + C = 0$ form an equilateral triangle, and find its area.

P24. What is the condition such that the straight lines joining the origin to the other two points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ will be at right angles.

P25. Find the orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$.

P26. Find the area of the triangle formed by the lines represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and the X-axis.

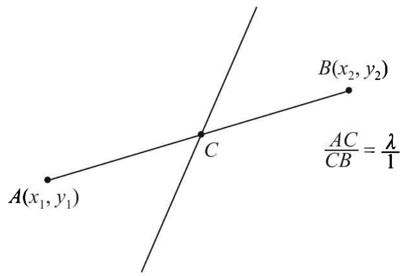
P27. Prove that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two parallel straight lines if $h^2 = ab$ and $bg^2 = af^2$. Also find the distance between them.

Straight Lines

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. (a) Let the required ratio be $\lambda:1$. Consider the following figure:



The coordinates of C are (from the internal division formula)

$$C \equiv \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)$$

Since this lies on L, we have

$$a\left(\frac{\lambda x_2 + x_1}{\lambda + 1} \right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1} \right) + c = 0$$

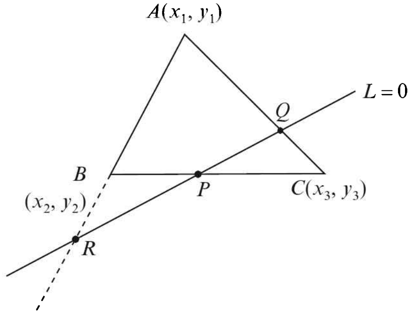
$$\Rightarrow \quad \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0$$

$$\Rightarrow \quad \lambda = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$$

$$\Rightarrow \quad \boxed{\lambda = -\frac{L(x_1, y_1)}{L(x_2, y_2)}}$$

This is a useful result (as you can see from the solution to the next part) and it would be worth memorizing it.

(b) Consider the following figure:



Using the result of part (a), we have

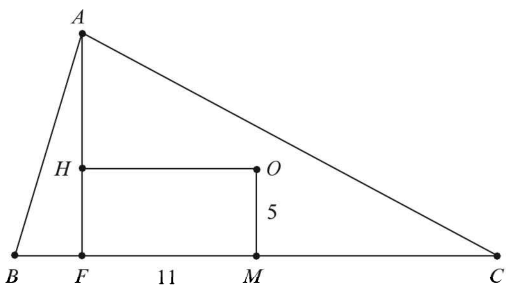
$$\frac{AR}{RB} = -\frac{L(x_1, y_1)}{L(x_2, y_2)} \qquad (1)$$

$$\frac{BP}{PC} = -\frac{L(x_2, y_2)}{L(x_3, y_3)} \qquad (2)$$

$$\frac{CQ}{QA} = -\frac{L(x_3, y_3)}{L(x_1, y_1)} \qquad (3)$$

From (1), (2) and (3), it should be evident that the product in the problem has the value –1, and therefore our answer will be 1.

S2. We assume an appropriate reference frame. Take $O = (0, 0)$, $H = (-11, 0)$ as shown below:



Now that we have fixed our reference axes, we can easily write the coordinates of the following points:

$$F = (-11, -5) \text{ and } M = (0, -5)$$

$$B = (-x, -5),\ C = (x, -5) \text{ and } A = (-11, y) \text{ (} x \text{ and } y \text{ are still unknown)}$$

The slope of the altitude through B, *i.e.*, BH, is $\frac{5}{x-11}$. The slope of AC is $\frac{-(y+5)}{x+11}$. Thus,

$$5(y+5) = (x-11)(x+11).$$

Also,

$$OA = OB \Rightarrow y^2 + 11^2 = x^2 + 5^2$$

We thus obtain $x = 14$ and $y = 10$. Hence, $BC = 2x = 28$. The correct option is (C).

S3. The successive coordinates of the particle's path are

$$(1, 1) \to \left(1, -\frac{1}{a}\right) \to \left(-\frac{1}{a}, -\frac{1}{a}\right) \to \left(-\frac{1}{a}, \frac{1}{a^2}\right) \to \left(\frac{1}{a^2}, \frac{1}{a^2}\right) \to \cdots$$

Thus, the total distance covered d is given by the sum of the following GP:

$$\begin{aligned} d &= 1 + \frac{1}{a} + 1 + \frac{1}{a} + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a} + \frac{1}{a^2} + \cdots \infty \\ &= 2\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{a} + \frac{1}{a^2} + \cdots \infty\right) \\ &= 2\left(1 + \frac{1}{a}\right)\left(\frac{1}{1 - \frac{1}{a}}\right) \quad = \frac{2(a+1)}{a-1} \end{aligned}$$

The correct option is (B).

S4. Recall that a point (h, k) lies on one side or another of a line $ax + by + c = 0$ accordingly as

$$k > -\frac{(ah + c)}{b} \text{ or } k < -\frac{(ah + c)}{b}$$

Let the point we seek be (h, k). Thus,

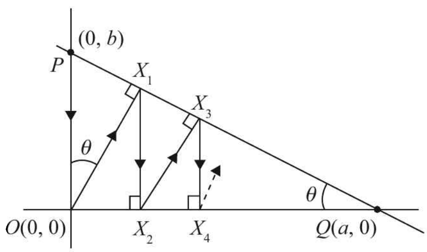
$$\frac{|3h - 4k + 1|}{5} = 1 \text{ and } \frac{|8h + 6k + 1|}{10} = 1 \qquad (1)$$

The condition that (h, k) lies below L_1 and above L_2 forces us to take the positive sign for the mod in both the expressions in (1). Thus,

$$3h - 4k + 1 = 5, \; 8h + 6k + 1 = 10 \quad \Rightarrow \quad (h, k) = \left(\frac{6}{5}, -\frac{1}{10}\right)$$

You can also deduce the sign of the mod expressions to be taken by graphically plotting the lines. We see that the magnitude of h/k is equal to 12. The correct option is (D).

S5. Consider the following diagram, which shows the movement of the particle, starting from P:



$$\begin{aligned} OX_1 &= OP\cos\theta \\ X_1X_2 &= OX_1\cos\theta \\ X_2X_3 &= X_1X_2\cos\theta \\ &\vdots \end{aligned}$$

and so on

Total distance $S = OP(1 + \cos\theta + \cos^2\theta + \cdots \infty) = b \cdot \dfrac{1}{1 - \cos\theta}$

$$= b \cdot \frac{1}{1 - \frac{a}{\sqrt{a^2+b^2}}} = \frac{b\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2} - a}$$

The correct option is (B).

S6. This is essentially a problem of manipulating determinants. Since the three points are given to be collinear,

$$\begin{vmatrix} \frac{a^3}{a-1} & \frac{a^2-3}{a-1} & 1 \\ \frac{b^3}{b-1} & \frac{b^2-3}{b-1} & 1 \\ \frac{c^3}{c-1} & \frac{c^2-3}{c-1} & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a^3 & a^2-3 & a-1 \\ b^3 & b^2-3 & b-1 \\ c^3 & c^2-3 & c-1 \end{vmatrix} = 0$$

The way ahead is to split this determinant into four new determinants, by splitting along C_2 and C_3. Doing that will generate four simpler determinants, each with a value of 0. Doing this is left to the reader as an exercise. Therefore, the required answer is 0. The correct option is (A).

S7. The center of the given circle is (5, 5) and its radius is 7. For the line $ax+by=a+2b$ to form an intercept with the circle, the distance of the line from the center of the circle must be less than the circle's radius. Thus,

$$\frac{|(5a+5b)-(a+2b)|}{\sqrt{a^2+b^2}} < 7 \Rightarrow \frac{|4a+3b|}{\sqrt{a^2+b^2}} < 7$$

However, this is *always* true since

$$\frac{|4a+3b|}{\sqrt{a^2+b^2}} \le 4\left(\frac{|a|}{\sqrt{a^2+b^2}}\right)+3\left(\frac{|b|}{\sqrt{a^2+b^2}}\right)$$

$$= 4\cos\theta + 3\sin\theta, \text{ for some } \theta \quad \text{(Note carefully why this is valid)}$$

$$\le \sqrt{4^2+3^2} = 5$$

The line $ax+by=a+2b$ thus always forms an intercept with the given circle. The minimum and maximum distances $d_{\min}$ and $d_{\max}$ can simply be found from the minimum and maximum (non-negative) values of the expression $(4\cos\theta+3\sin\theta)$:

$$d_{\min} = (4\cos\theta+3\sin\theta)_{\min-\text{non}-\text{negative}} = 0$$

$$d_{\max} = (4\cos\theta+3\sin\theta)_{\max} = 5$$

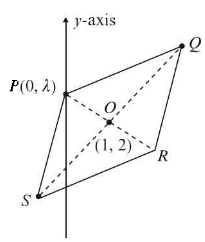
The corresponding intercept lengths are:

$$l_{\min} = 2\sqrt{7^2-d_{\max}^2} = 4\sqrt{6}$$

$$l_{\max} = 2\sqrt{7^2-d_{\min}^2} = 14$$

The difference between their squares is $196-144=52$. The correct option is (B).

S8. We make use of the fact that a rhombus is a parallelogram whose sides are equal. We assume the vertex which lies on the y-axis to be $P(0,\lambda)$. Then, since O is the mid-point of PR, we immediately obtain R as $(2, 4-\lambda)$:



Now, from the figure, *PS* has a slope of 7 while *RS* has a slope of 2. Thus, by the point-slope form, we can write down the equations of both *PS* and *RS*.

$$\left.\begin{array}{l} PS: y = 7x + \lambda \\ RS: y = 2x - \lambda \end{array}\right\} \xrightarrow[\text{Intersection}]{\text{Find}} S \equiv \left(-\frac{2\lambda}{5}, -\frac{9\lambda}{5}\right)$$

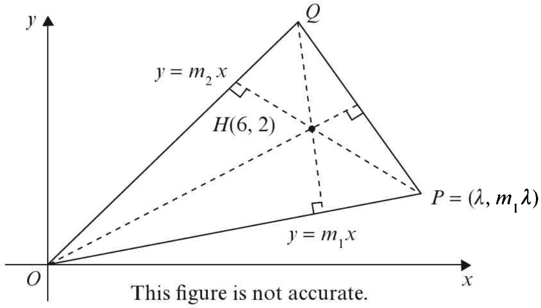
Finally, $PS = RS$ gives us the required value of λ:

$$\left(\frac{2\lambda}{5}\right)^2 + \left(\lambda + \frac{9\lambda}{5}\right)^2 = \left(2 + \frac{2\lambda}{5}\right)^2 + \left(4 - \lambda + \frac{9\lambda}{5}\right)^2$$

$$\Rightarrow \quad \lambda = \frac{5}{\sqrt{10}-1}, \frac{-5}{\sqrt{10}+1}$$

The correct options are (A) and (D). For more clarity, you are urged to draw the rhombus corresponding to these two values of λ.

S9. We have to find the equation of *PQ* as shown in the figure below:



This figure is not accurate.

Note that $m_{OH} = \frac{1}{3}$, so $m_{PQ} = -3$. Also,

$$m_1 + m_2 = -\frac{10}{3}, \; m_1 m_2 = \frac{1}{3}$$

Now, the equation of *PQ* can be written as $y = -3x + c$. If $P \equiv (\lambda, m_1\lambda)$, then

$$m_1\lambda = -3\lambda + c \quad \Rightarrow \quad \lambda = \frac{c}{m_1 + 3} \tag{1}$$

Since *PH* is perpendicular to *OQ*,

$$m_{PH} \times m_{OQ} = -1 \quad \Rightarrow \quad \frac{m_1\lambda - 2}{\lambda - 6} \times m_2 = -1 \tag{2}$$

From (1) and (2),

$$(m_1 m_2 + 1)\left(\frac{c}{m_1 + 3}\right) = 2(m_2 + 3)$$

Using $m_1 m_2 = \frac{1}{3}$, this reduces to

$$3(m_1 + 3)(m_2 + 3) = 2c \quad \Rightarrow \quad 3(m_1 m_2 + 3(m_1 + m_2) + 9) = 2c.$$

Again, using $m_1 + m_2 = \frac{-10}{3}$ and $m_1 m_2 = \frac{1}{3}$, we obtain c:

$$c = -1$$

This gives

$$PQ:\quad 3x+y+1=0$$

Comparing the coefficients with $ax+by+1=0$, the value of $a+b$ is 4. The correct option is (D).

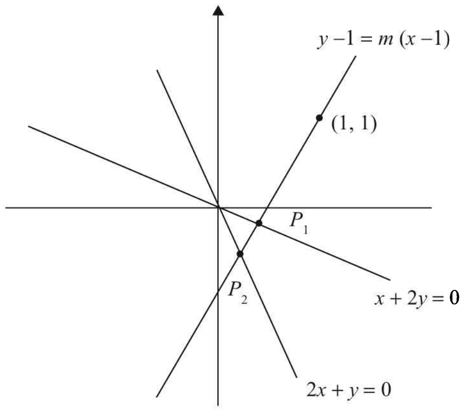
S10. The equation $2x^2+5xy+2y^2=0$ represents the pair of lines

$$(2x+y)(x+2y)=0 \qquad (1)$$

Let the slope we are trying to determine be m. Then, the equation of the to-be-determined line is

$$y-1=m(x-1)$$

Solving this simultaneously with the equations of the two lines in (1) one-by-one, we obtain the two points of intersection P_1 and P_2:



$$P_1:\left(\frac{2(m-1)}{2m+1},-\frac{(m-1)}{2m+1}\right)=(2\lambda,-\lambda)\quad \text{(say)}$$

$$P_2:\left(\frac{m-1}{m+2},-\frac{2(m-1)}{m+2}\right)=(\mu,-2\mu)\quad \text{(say)}$$

Since P_1 and P_2 are equidistant from the origin, we have, by the distance formula:

$$\sqrt{(2\lambda)^2+(-\lambda)^2}=\sqrt{\mu^2+(-2\mu)^2} \quad\Rightarrow\quad \lambda=\pm\,\mu$$

$$\Rightarrow\quad \frac{m-1}{2m+1}=\pm\left(\frac{m-1}{m+2}\right) \quad\Rightarrow\quad m=1 \text{ or } \frac{1}{3}$$

Therefore, the correct options are (A) and (D).

S11. Let the given joint equation be factorized as:

$$(lx+my+n_1)\,(mx-ly+n_2)=0$$

Comparing coefficients, we have

$$lm=a,\ \ n_1 n_2=c,\ \ m^2-l^2=h$$

The required area A is

$$A=OQ\times OR=\frac{n_1n_2}{l^2+m^2}=\frac{c}{\sqrt{h^2+4a^2}}$$

The correct option is (D).

S12. We divide the given equation throughout by ax^4, and express it as a fourth-degree equation in $\frac{y}{x} = m$:

$$m^4 + \frac{b}{a}m^3 + \frac{c}{a}m^2 + \frac{d}{a}m + 1 = 0 \qquad (1)$$

Assuming that the four slopes are m_1, m_2, m_3, m_4, we have $m_1\, m_2\, m_3\, m_4 = 1$. Now, we are given that a pair of these lines bisects the angles between the other pair. Thus, two of these lines must be perpendicular. Assuming $m_1\, m_2 = -1$, we then have $m_3\, m_4 = -1$, which means that the other two lines are also perpendicular. Thus, (1) can be written as

$$(m - m_1)(m - m_2)(m - m_3)(m - m_4) = (m - m_1)\left(m + \frac{1}{m_1}\right)(m - m_3)\left(m + \frac{1}{m_3}\right)$$

$$= \left(m^2 + \left(\frac{1 - m_1^2}{m_1}\right)m - 1\right)\left(m^2 + \left(\frac{1 - m_3^2}{m_3}\right)m - 1\right)$$

$$= (m^2 + \lambda m - 1)(m^2 + \mu m - 1) \quad \text{(say)} \qquad (2)$$

Comparing the coefficients in (1) and (2), we have

$$\lambda + \mu = \frac{b}{a}, \quad \lambda\mu - 2 = \frac{c}{a}, \quad \lambda + \mu = -\frac{d}{a} \qquad (3)$$

Thus, $\qquad \frac{b}{a} = -\frac{d}{a} \quad \Rightarrow \quad b + d = 0$

Additionally, we still have to make use of the angle bisectors fact. Therefore,

$$\left|\frac{m_1 - m_3}{1 + m_1 m_3}\right| = \left|\frac{m_3 + \frac{1}{m_1}}{1 - \frac{m_3}{m_1}}\right| \quad \Rightarrow \quad (m_1 - m_3)^2 = (1 + m_1 m_3)^2 \qquad (4)$$

From (3), $\qquad \lambda\mu - 2 = \frac{c}{a} \quad \Rightarrow \quad \left(\frac{1 - m_1^2}{m_1}\right)\left(\frac{1 - m_3^2}{m_3}\right) - 2 = \frac{c}{a} \qquad (5)$

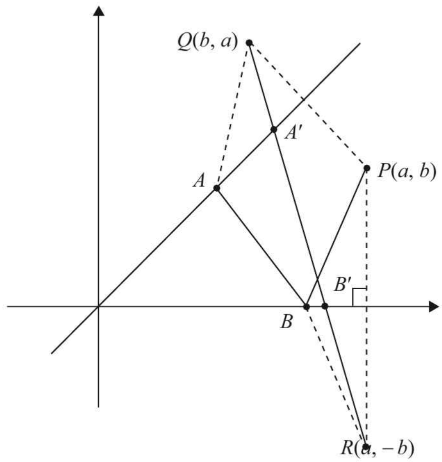
Making use of (4) and (5), we obtain the condition $c + 6a = 0$. Therefore, the required condition is

$$b + d = c + 6a = 0$$

We see that the correct options are (A) and (D).

SUBJECTIVE TYPE EXAMPLES

S13. Let PAB be a triangle satisfying the given constraint. Let Q be the reflection of P in $y = x$, and R be the reflection of P in the x-axis.

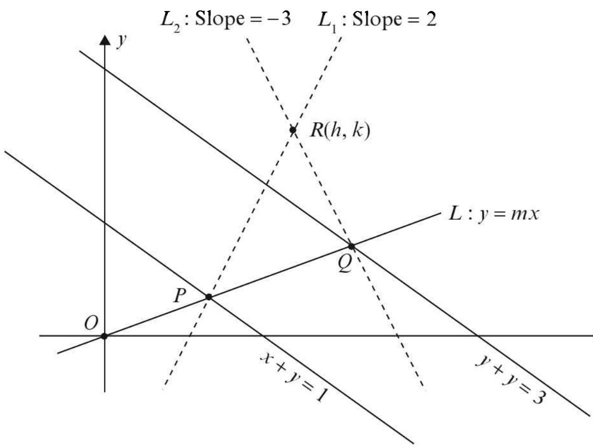


Now,

$$\text{Perimeter } (\Delta PAB) = PA + AB + PB = QA + AB + BR \geq QR,$$

since the shortest distance between Q and R is along the segment joining the two points. Therefore, the minimum perimeter is $QR = \sqrt{2(a^2 + b^2)}$. Note that the corresponding triangle is $PA'B'$.

S14. Observe the following diagram carefully:



We make some simple observations:

$$P \equiv \left(\frac{1}{1+m}, \frac{m}{1+m}\right), \ Q \equiv \left(\frac{3}{1+m}, \frac{3m}{1+m}\right)$$

$$PR : y - \left(\frac{m}{1+m}\right) = 2\left(x - \left(\frac{1}{1+m}\right)\right)$$

$$QR: y-\left(\frac{3m}{1+m}\right)=-3\left(x-\left(\frac{3}{1+m}\right)\right)$$

Solving for $R(h,k)$, we have

$$h=\frac{2m+11}{5(1+m)},\ k=\frac{9m+12}{5(1+m)}$$

Eliminating m from these equation is a trivial matter, and we obtain

$$m=\frac{5h-11}{2-5h}=\frac{5k-12}{9-5k}$$

Cross-multiplying, simplifying and using $(h \to x, k \to y)$, we obtain the required locus of R as the following straight line:

$$x-3y+5=0$$

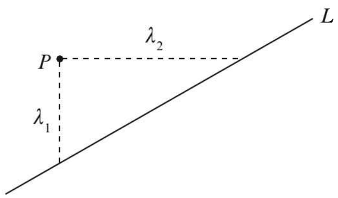
S15. Assume any point on the given line as $(a, a\sqrt{3}+3)$. Now, the μ-distance of $(1, 0)$ from this point is

$$d=|a-1|+|a\sqrt{3}+3|$$

We now need to consider different intervals of a:

$$d=\begin{cases} 1-a-3-a\sqrt{3}, & a \le -\sqrt{3} \\ 1-a+3+a\sqrt{3}, & -\sqrt{3} < a \le 1 \\ a-1+3+a\sqrt{3}, & a > 1 \end{cases}$$

The minimum value of d is $\sqrt{3}+1$, when $a=-\sqrt{3}$. Graphically, the situation may be understood better through the following figure.

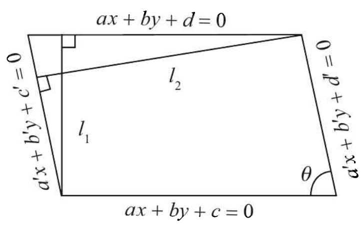


$$d(P, L)=\min(\lambda_1, \lambda_2)$$

S16. This is an interesting problem which might look complicated at first glance. The trick is to visualize that even after the rotation, the distance of the line from the origin must remain the same. Therefore, we have

$$\frac{6}{\sqrt{2^2+3^2}}=\frac{\alpha}{\sqrt{2}} \Rightarrow \alpha=\frac{6\sqrt{2}}{\sqrt{13}}$$

S17. **(a)** Consider the following figure:



Note that the area A of the parallelogram can be written as

$$A = \frac{l_1}{\sin\theta} \times l_2 = \frac{l_1 l_2}{\sin\theta}$$

l_1 and l_2 are straightforward to evaluate since they are distances between parallel lines:

$$l_1 = \frac{|c-d|}{\sqrt{a^2+b^2}}, \quad l_2 = \frac{|c'-d'|}{\sqrt{a'^2+b'^2}}$$

Now, we need sin θ. Since θ is the angle between the lines $ax+by+c=0$ and $a'x+b'y+c'=0$, we have

$$\tan\theta = \left|\frac{ab'-a'b}{aa'+bb'}\right| \quad \Rightarrow \quad \sin\theta = \frac{|ab'-a'b|}{\sqrt{(a^2+b^2)(a'^2+b'^2)}}$$

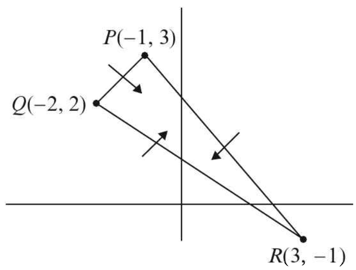
Thus,

$$A = \left|\frac{(c-d)(c'-d')}{ab'-a'b}\right|$$

(b) For a rhombus, l_1 and l_2 must be equal, *i.e.*,

$$(c-d)^2 (a'^2+b'^2) = (c'-d')^2 (a^2+b^2)$$

S18. It would be instructive to draw a figure of the triangle specified, to scale:



We note the following:

(i) $PQ: x-y+4=0$, Dist. from origin $= 2\sqrt{2}$

(ii) $QR: 3x+5y-4=0$, Dist. from origin $= \frac{4}{\sqrt{34}}$

(iii) $PR: x+y-2=0$, Dist. from origin $= \sqrt{2}$

The new distances from the origin will respectively be:

$$2\sqrt{2}-1, \quad 1+\frac{4}{\sqrt{34}}, \quad \sqrt{2}-1 \quad \text{(how?)}$$

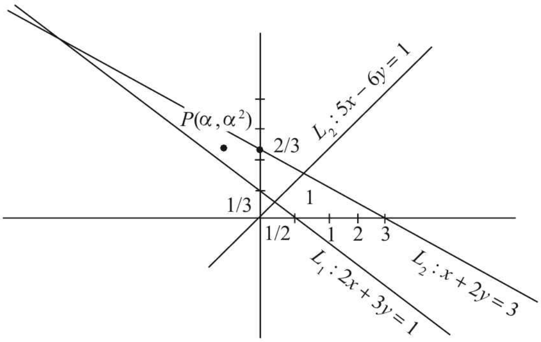
The side nearest to the origin is (the shifted version of) PR. If we assume its equation as $x+y+c=0$ $(c<0)$, we have

$$\frac{|c|}{\sqrt{2}} = \sqrt{2}-1 \quad \Rightarrow \quad c = -(2-\sqrt{2})$$

Thus, the required equation is

$$x+y = 2-\sqrt{2}$$

S19. We start by drawing an accurate figure corresponding to the problem:



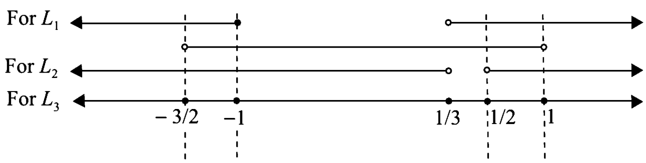
We make use of the fact that for L_1, P lies on the side *opposite* to the side containing the origin, while for L_2 and L_3, P lies on the *same* side as the origin. Thus, we will obtain three inequalities:

$$\text{For } L_1\text{:}\quad 2\alpha + 3\alpha^2 - 1 > 0 \quad \Rightarrow \quad \alpha < -1 \text{ or } \alpha > \frac{1}{3}$$

$$\text{For } L_2\text{:}\quad \alpha + 2\alpha^2 - 3 < 0 \quad \Rightarrow \quad -\frac{3}{2} < \alpha < 1$$

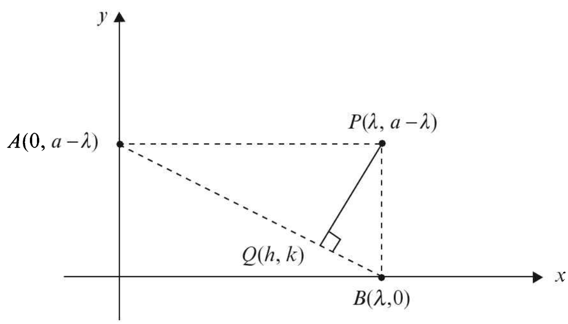
$$\text{For } L_3\text{:}\quad 5\alpha - 6\alpha^2 - 1 < 0 \quad \Rightarrow \quad \alpha < \frac{1}{3} \text{ or } \alpha > \frac{1}{2}$$

We take the values of α common to the three solution sets:



$$\Rightarrow \quad \alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$$

S20. We note that the coordinates of P can be assumed as $(\lambda, a - \lambda)$, which immediately gives us the coordinates of A and B, as shown.



We make use of two simple facts:

Fact-1: PQ is perpendicular to AB

$$\left(\frac{a-\lambda-k}{\lambda-h}\right)\left(\frac{a-\lambda}{-\lambda}\right)=-1$$

$$\Rightarrow \quad \lambda h-(a-\lambda)k=a(2\lambda-a) \tag{1}$$

Fact-2: Q lies on AB, so $m_{QB}=m_{AB}$:

$$\frac{k}{h-\lambda}=\frac{a-\lambda}{-\lambda}$$

$$\Rightarrow \quad (a-\lambda)h+\lambda k=\lambda(a-\lambda) \tag{2}$$

From (1) and (2), we can eliminate λ as follows:

Step-1: Solve for h and k:

$$h=\frac{\lambda^3}{(a-\lambda)^2+\lambda^2},\quad k=\frac{(a-\lambda)^3}{(a-\lambda)^2+\lambda^2} \tag{3}$$

Step-2: Use componendo and dividendo as follows:

$$\frac{h}{\lambda^3}=\frac{k}{(a-\lambda)^3}=\left(\frac{h^{1/3}+k^{1/3}}{a}\right)^3$$

$$\Rightarrow \quad \lambda=\frac{ah^{1/3}}{(h^{1/3}+k^{1/3})},\quad a-\lambda=\frac{ak^{1/3}}{(h^{1/3}+k^{1/3})} \tag{4}$$

Step-3: Substitute the values in (4) into the expression for h in (3), and simplify

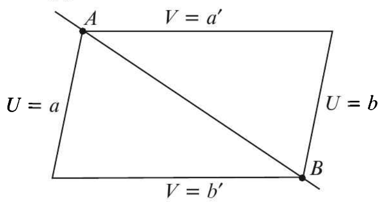
This gives

$$(h^{1/3}+k^{1/3})(h^{2/3}+k^{2/3})=a$$

The required locus for Q is thus

$$(x^{1/3}+y^{1/3})(x^{2/3}+y^{2/3})=a$$

S21. We make use of a family of lines approach. Consider the following diagram:



The idea is to write AB in two different ways, once as a member belonging to the family of lines passing through $U=a$, $V=a'$, and once corresponding to $U=b$, $V=b'$. Thus, for some λ, μ,

$$(U-a)+\lambda(V-a')=(U-b)+\mu(V-b') \tag{1}$$

The next step is to use $U\equiv lx+my+n$ and $V\equiv l'x+m'y+n'$ and compare the coefficients on both sides of (1). This leads us to the actual values of λ and μ.

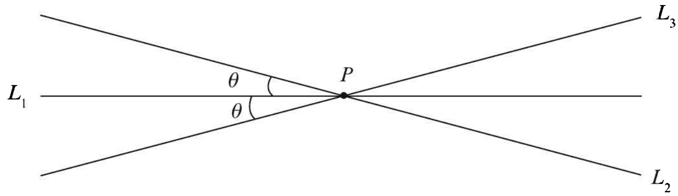
$$\lambda=\mu=-\left(\frac{a-b}{a'-b'}\right)$$

Using this in one of the expressions in (1), we obtain the required equation of the diagonal as the determinant in the problem.

S22. The equation of L_3 can be written as

$$L_3 \equiv L_1 + \lambda L_2 = 0$$

$$\Rightarrow \quad (a+\lambda l)x + (b+\lambda m)y + c + \lambda n = 0 \qquad (1)$$



We note that L_1 is the angle bisector of the angle between L_2 and L_3. The slopes of L_1, L_2 and L_3 are respectively:

$$m_1 = -\frac{a}{b}, \quad m_2 = -\frac{l}{m}, \quad m_3 = -\frac{a+\lambda l}{b+\lambda m}$$

Now,

$$\frac{m_2 - m_1}{1+m_1m_2} = \frac{m_1 - m_3}{1+m_1m_3}$$

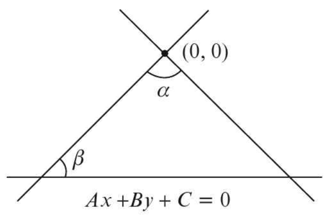
$$\Rightarrow \quad \frac{\frac{a}{b} - \frac{l}{m}}{1+\frac{al}{bm}} = \frac{\frac{a+\lambda l}{b+\lambda m} - \frac{a}{b}}{1+\frac{a(a+\lambda l)}{b(b+\lambda m)}}$$

$$\Rightarrow \quad \lambda = -\left(\frac{a^2+b^2}{al+bm}\right)$$

Substituting this value of λ in (1), we obtain the required equation as

$$(abm + b^2l)x + (abl - a^2m)y + acl + bcm - n(a^2+b^2) = 0$$

S23. Note that the two lines represented by the joint equation pass through the origin.



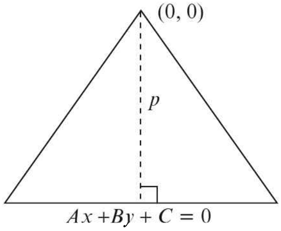
The angle α between these two lines can be evaluated using the relevant formula:

$$\tan\alpha = \frac{2\sqrt{h^2 - ab}}{a+b}$$

Using $h = 4AB$, $a = A^2 - 3B^2$ and $b = B^2 - 3A^2$, we obtain $\alpha = 60°$. To obtain β, we need the slope of one of the two lines represented by the joint equation. Using this equation as a quadratic in $\frac{y}{x}$, the two slope values are

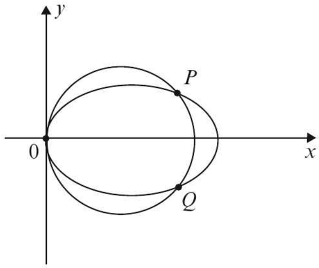
$$m_{1,2} = -\frac{4AB \pm \sqrt{3}(A^2+B^2)}{B^2 - 3A^2}$$

while the slope of the third line is $m = -\frac{A}{B}$. Either of the two values of the slope can now be used with m, and β comes out the be 60°. Thus, the triangle is equilateral.
Now to the area problem. We are lucky that the triangle is equilateral, since its area Δ can simply be expressed in terms of the perpendicular distance of one vertex from the opposite side.



$$\Delta = \frac{\sqrt{3}}{4} \times \left(\frac{p}{\sin 60^\circ}\right)^2$$
$$= \frac{C^2}{\sqrt{3}(A^2 + B^2)}$$

S24. Consider the following situation as an example:



Consider the line joining the points P and Q to have the equation $lx + my + n = 0$, or (since we will use homogenizing) more conveniently,

$$px + qy = 1$$

Thus, the joint equation of OP and OQ can be written in two different ways.

$$ax^2 + 2hxy + by^2 + 2gx(px + qy) = 0$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x(px + qy) = 0$$

Since we want OP and OQ to be perpendicular,

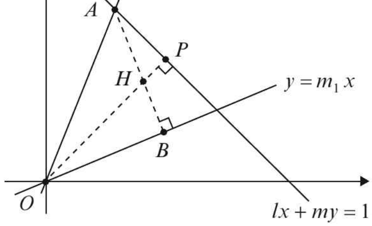
$$a + 2pg + b = a' + 2pg' + b' = 0$$

Eliminating p, we have

$$g(a' + b') = g'(a + b)$$

This is the required condition.

S25. The equation for the pair of lines is homogenous, which means that the lines pass through the origin. Let the slopes of these two lines be m_1 and m_2. Consider the following diagram:



We need to find the coordinates (h, k) of the orthocenter H.

The obvious way forward is to write down the equations for OP and AB, and solve them to find H.

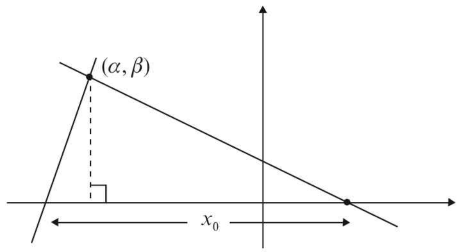
$$OP: \quad mx - ly = 0; \quad AB: \left(y - \frac{m}{l + mm_2}\right) = -\frac{1}{m_1}\left(x - \frac{1}{l + mm_2}\right)$$

In writing the equation of AB, we have first found out the point A, and have used the fact that the slope of AB is $-\frac{1}{m_1}$. Solving for the coordinates of H gives us the required answer:

$$\frac{h}{l} = \frac{k}{m} = \frac{a+b}{am^2 + 2hlm + bl^2}$$

$$\Rightarrow \quad (h, k) = \left(\frac{l(a+b)}{am^2 + 2hlm + bl^2}, \frac{m(a+b)}{am^2 + 2hlm + bl^2}\right)$$

S26. Consider the following diagram:



If we put $y = 0$ in the given two degree equation, we have

$$ax^2 + 2gx + c = 0$$

If the two roots of this equation be x_1, x_2, then

$$x_0 = |x_1 - x_2| = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = \frac{2\sqrt{g^2 - ac}}{a}$$

The area of the triangle A can be determined as $\frac{1}{2}x_0\,|\beta|$, so that we now need to evaluate β, the ordinate of the point of intersection of the two lines. There are more than one ways to do that, but the easiest is by 'completing the squares' and rearranging the second degree equation to

$$ax + hy + g = \pm\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}$$

We now impose the constraint that the expression inside the square root must be a perfect square (of a linear expression, say $py + q$) so that

$$ax + hy + g = \pm(py + q)$$

$$\Rightarrow \quad ax + (h - p)y + g - q = 0 \quad \text{and} \quad ax + (h + p)y + g + q = 0$$

From these two equations, (α, β) becomes evident.

$$|\beta| = \left|-\frac{q}{p}\right| = \sqrt{\frac{g^2 - ac}{h^2 - ab}} \quad \text{(why?)}$$

$$\Rightarrow \quad A = \frac{g^2 - ac}{a\sqrt{h^2 - ab}}$$

S27. The obvious way is to consider the given two-degree expression as a product of two linear factors with the 'same slope', say m, since it represents parallel lines. Thus,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \lambda(y - mx + k_1)(y - mx + k_2)$$
$$= \lambda(m^2x^2 - 2mxy + y^2 - m(k_1 + k_2)x + (k_1 + k_2)y + k_1k_2)$$

Comparing coefficients, we have

$$\frac{m^2}{a} = -\frac{m}{h} = \frac{1}{b} = -\frac{m(k_1 + k_2)}{2g} = \frac{k_1 + k_2}{2f} = \frac{k_1\, k_2}{c}$$

The conditions $h^2 = ab$ and $bg^2 = af^2$ are immediately evident. The distance d between them is

$$d = \frac{k_1 - k_2}{\sqrt{1+m^2}} = \frac{\sqrt{(k_1 + k_2)^2 - 4k_1\, k_2}}{\sqrt{1+m^2}}$$

Using $k_1\, k_2 = \frac{c}{b}$ and $(k_1 + k_2)^2 = \frac{4g^2}{ab}$, and $m^2 = \frac{a}{b}$, we have

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

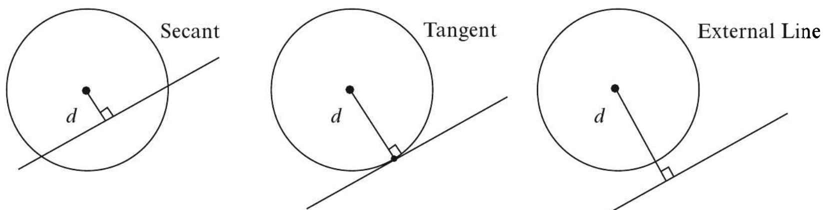
Circles

PART-A: Summary of Important Concepts

1. Basics of Circles

1.1 Fundamental Geometric Properties of a Circle

1. A circle is a geometrical figure described by a moving point in the Euclidean plane such that its distance from a fixed point is always constant; the fixed point is the center, while the fixed distance is the radius of the circle.
2. The ratio of the circumference to the diameter in any circle is fixed, and equal to π, an irrational number. The area of the circle is πr^2, where r is the radius.
3. Three non-collinear points are sufficient to uniquely determine a circle in the plane (passing through those points). In other words, one and only one circle will pass through three non-collinear points.
4. A given line may be a secant to a given circle, a tangent to the circle or may not intersect the circle at all. In terms of d, the distance of the center of the circle from the line, and the radius r of the circle, we have the following:

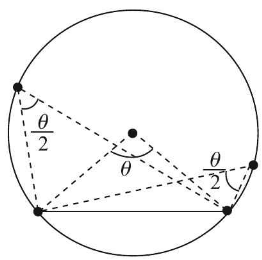


- Two points of intersection
- $d < r$

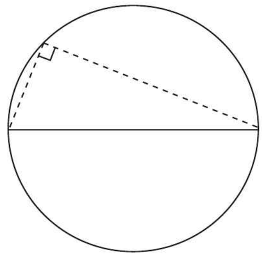
- One points of intersection
- $d = r$

- No points of intersection
- $d > r$

5. If a chord subtends angle θ at the centre of the circle, it subtends angle $\frac{\theta}{2}$ at any point on the circumference:



This implies that any diameter of a circle subtends a right angle at any point on its circumference:



1.2 Equations Describing Circles

If we know the center $C(x_0, y_0)$ of a circle and its radius r, we can use the distance formula to write its equation as

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

From this, we can deduce that the general equation representing a circle is of the form

$$S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$

and comparing the two forms, we can conclude that the center of the circle is $(-g, -f)$ while its radius is given by $\sqrt{g^2 + f^2 - c}$.

Another form of a circle's equation exists in terms of the coordinates of the end-points of any diameter of the circle. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the end-points of a diameter; the equation of the circle can be written as

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Let $P(x_1, y_1)$ be a point in the plane, and $S(x, y) = 0$ represent a circle. The position of P with respect to the circle is governed by the following conditions:

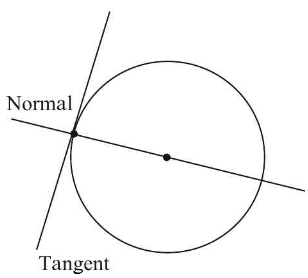
P lies inside the circle $\Rightarrow$ $S(x_1, y_1) < 0$
P lies on the circle $\Rightarrow$ $S(x_1, y_1) = 0$
P lies outside the circle $\Rightarrow$ $S(x_1, y_1) > 0$

We note that any point P lying on the circle $x^2 + y^2 = a^2$ can be represented in the polar form $P \equiv (a\cos\theta, a\sin\theta)$.

2. Tangents and Normals

2.1 Basics

As we know, a tangent is a line which touches the circle, or in other words, intersects the circle in just one point. The line perpendicular to a tangent at the point of contact is a normal to the circle:



- The distance of a tangent from the centre is equal to the radius.
- A normal passes through the centre.

2.2 Important Results

We now summarize some of the important results of tangents and normals:

1. A line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if

$$c^2 = a^2(1+m^2)$$

This means that any line of the form

$$y = mx \pm a\sqrt{1+m^2}$$

will always be a tangent to the circle $x^2 + y^2 = a^2$, whatever the value of m may be. The corresponding normal will have a slope of $-\frac{1}{m}$ and will pass through the origin.

2. For a point $P(x_1, y_1)$ lying on the circle $S(x, y) \equiv x^2 + y^2 = a^2$, the tangent to the circle at P has the equation $T(x, y) \equiv xx_1 + yy_1 = a^2$.

If (x_1, y_1) has been specified in polar form, that is, in the form $(a\cos\theta, a\sin\theta)$, the equation of the tangent becomes

$$T(x, y) \equiv x\cos\theta + y\sin\theta = a$$

3. If the equation of the circle is in the general form

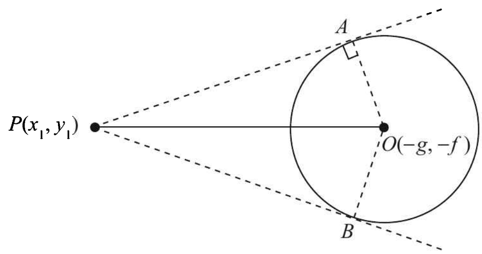
$$S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$

the equation of the tangent to the circle at a point $P(x_1, y_1)$ lying on the circle is

$$T(x, y) \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c =$$

4. The length l of the tangent to the circle $S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ drawn from the external point $P(x_1, y_1)$ is given by

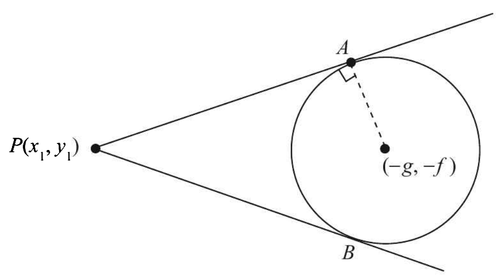
$$\ell = \sqrt{S(x_1, y_1)} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$



O $(-g, -f)$ is the centre of S.
$P(x_1, y_1)$ is an external point.
Note that two tangents, PA and PB can be drawn from P to S.
Also, $PA = PB$.

In other words, to calculate the required length, you substitute the coordinates for P into the circle's equation, and take the square root of the resulting value. Note that P must not be internal to the circle, otherwise $S(x_1, y_1)$ will be negative.

5. From an external point $P(x_1, y_1)$, (two) tangents are drawn to the circle $S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. These tangents touch the circle at A and B.



The joint equation of PA and PB is given by

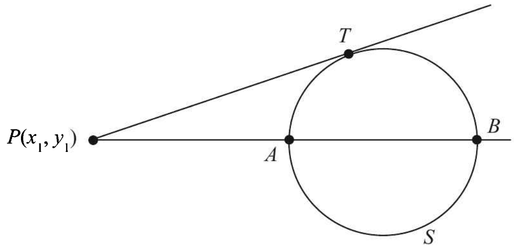
$$T^2(x_1, y_1) = S(x, y)S(x_1, y_1) \quad \text{or} \quad T^2 = SS_1 \quad \text{(in brief)}$$

Here, $T(x_1, y_1) = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ is the *expression* we encounter in the general equation of a circle's tangent.

6. In the figure above, suppose that A and B are joined. AB is referred to as the chord of contact from the point P to the circle $S = 0$. The equation of the chord of contact is

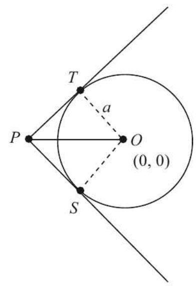
$$T(x_1, y_1) = 0$$

7. From an external point P, a line is drawn intersecting a circle S in two distinct points A and B. A tangent is also drawn from P touching the circle S at T.



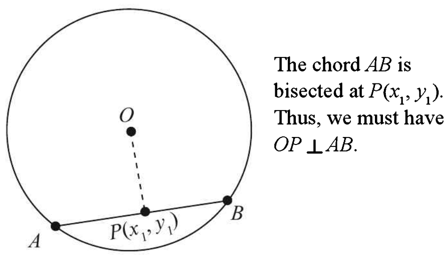
We note the very important result that $PA \cdot PB = PT^2$.

8. A point P moves such that the tangents drawn from it to the circle $x^2 + y^2 = a^2$ are perpendicular:



The locus of P is $x^2 + y^2 = 2a^2$, a circle which is known as the *director circle* of the given circle. In general, for any circle, such a director circle will exist which is concentric with the given circle and has a radius equal to $\sqrt{2}$ times the radius of the given circle.

9. This is a very important result related to chords. Consider a circle $S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. A chord of this circle is bisected at the point $P(x_1, y_1)$.



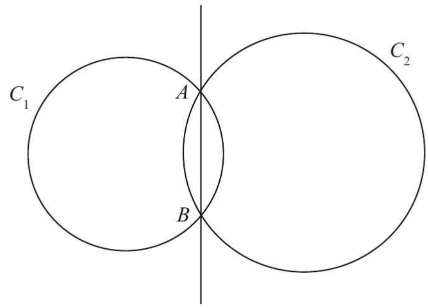
The equation of this chord is given by

$$T(x_1, y_1) = S(x_1, y_1) \quad \text{or} \quad T = S_1 \quad \text{(in brief)}$$

3. Common Chords and Radical Axis

3.1 Common Chords

Consider two intersecting circles as shown below:



AB is the common chord of the circles C_1 and C_2. The equations of C_1 and C_2 are given to be $S_1 = 0$ and $S_2 = 0$ respectively.

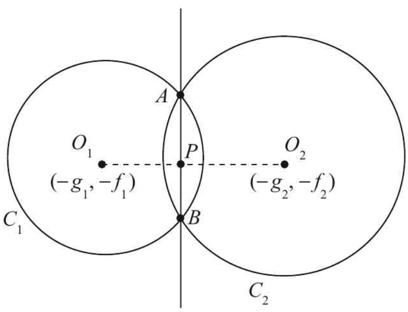
The equation of AB, the common chord of the two circles, is given by

$$S \equiv S_1 - S_2 = 0$$

or

$$S \equiv (2g_1 - 2g_2)x + (2f_1 - 2f_2)y + c_1 - c_2 = 0$$

The common chord is perpendicular to the line joining the centres of C_1 and C_2, which is expected:



Slope of $O_1 O_2$ is

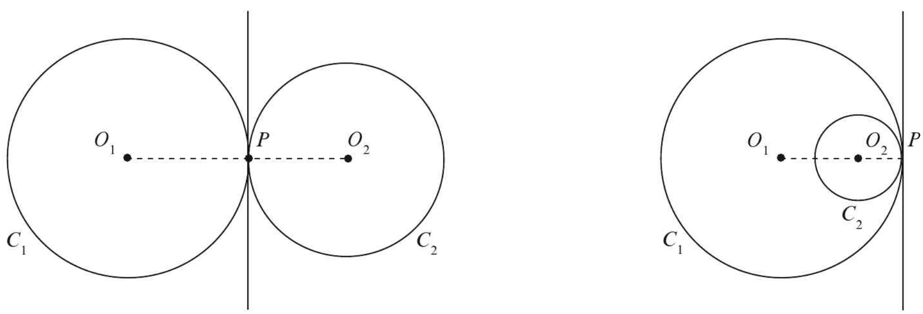
$$m_1 = \frac{f_2 - f_1}{g_2 - g_1}$$

Slope of AB is from (1)

$$m_2 = \frac{g_2 - g_1}{f_2 - f_1}$$

Thus, $m_1 m_2 = -1$

If the length of the common chord is 0, it is actually the common tangent to the two circles C_1 and C_2, which will touch each other externally or internally:

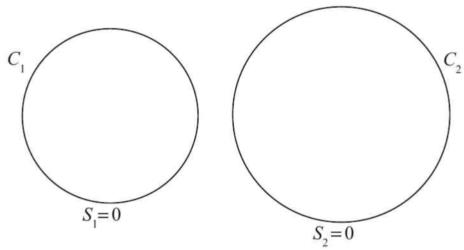


If the two circles touch each other externally or internally (at P), then the common chord $S_1 - S_2 = 0$ actually represents the common tangents to the two circles at P.

If C_1 and C_2 lie completely external to each other, or one of the two circles lies completely inside the other, no common chord will exist.

3.2 Radical Axis

Suppose that C_1 and C_2 lie external to each other and do not intersect. What does $S_1 - S_2 = 0$ represent in this case?

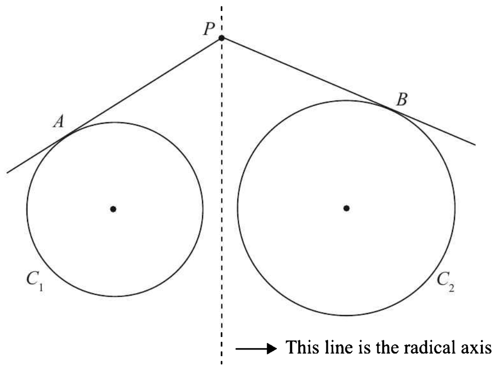


What does $S_1 - S_2 = 0$ represent in this case? Note that no common chord or common tangent can exist.

Let a point $P(x_1, y_1)$ be such that it satisfies $S_1 - S_2 = 0$. Thus,

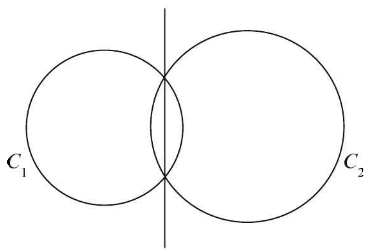
$$S_1(x_1, y_1) = S_2(x_1, y_1) \Rightarrow \sqrt{S_1(x_1, y_1)} = \sqrt{S_2(x_1, y_1)}$$

This says that P is a point such that the lengths of the tangents drawn from it to the two circles are equal. Thus, any point lying on the straight line $S_1 - S_2 = 0$ will possess the property that the tangents drawn from it to the two circles are equal. This line is termed the radical axis of the two circles.

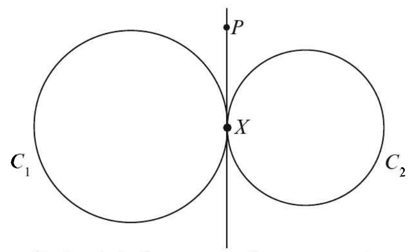


For any point P on the radical axis, tangents drawn from it to C_1 and C_2 are of equal length, *i.e.*, $PA = PB$

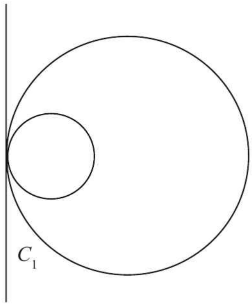
It should be obvious that in case of intersecting (or touching) circles, the common chord (or the common tangent) is itself the radical axis. For a situation as in the figure above, the radical axis exists but no common chord exists. For a circle lying inside another circle, neither the radical axis nor the common chord exist:



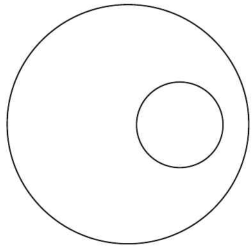
The radical axis is the same as the common chord.



The radical axis is the same as the common tangent.This should otherwise be obvious also since for any point on this line, the length of tangents to both C_1 and C_2 is PX.

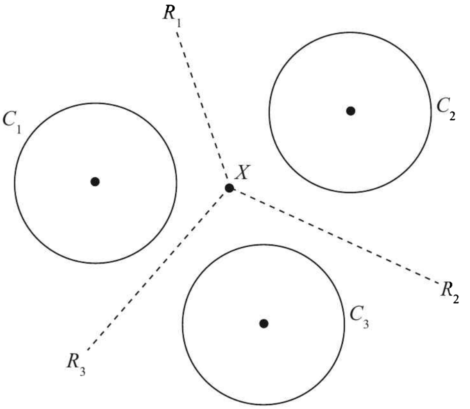


The radical axis is the same as the common tangent which is again obvious.



No common chord or radical axis exist.

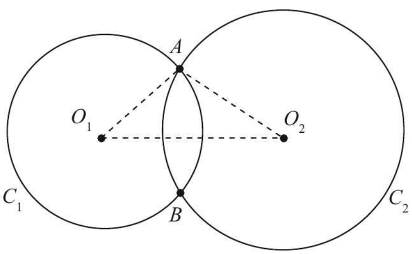
We also note the important fact that the radical axes of three circles (whose centers are non-collinear) taken two at a time, are concurrent:



R_1, R_2 and R_3 are the three radical axes, while X, the point of concurrency is termed the radical centre. Tangents drawn from X to the three circles will be of equal lengths.

4. Angle of Intersection of Circles

Consider two intersecting circles C_1 and C_2 with radii r_1 and r_2 respectively and centres at O_1 and O_2 respectively:



C_1 and C_2 intersect at A and B.

The angle of intersection of the two circles can be defined as the angle between the tangents to the two circles at their point(s) of intersection, which will be the same as the angle between the two radii at the point(s) of intersection. In particular, for example, C_1 and C_2 in the figure above intersect at an angle $\angle O_1AO_2$.

The most important case we need to consider pertaining to intersecting circles is orthogonal circles, meaning that the angle of intersection of the two circles is a right angle. In that case, $\angle O_1AO_2$ above will become a right angle, so that

$$O_1A^2 + O_2A^2 = O_1O_2^2 \tag{1}$$

If the two circles C_1 and C_2 have the equations

$$S_1: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0; \quad r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$$

$$S_2: x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0; \quad r_2 = \sqrt{g_2^2 + f_2^2 - c_2},$$

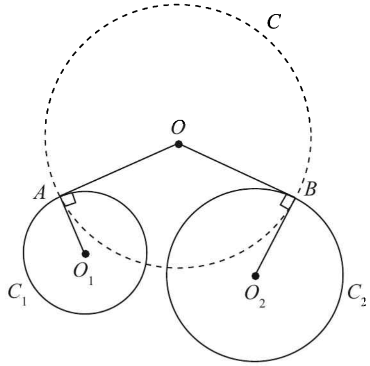
then the condition (1) becomes

$$r_1^2 + r_2^2 = O_1O_2^2$$

$$\Rightarrow \quad g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$$

$$\Rightarrow \quad 2(g_1g_2 + f_1f_2) = c_1 + c_2$$

We note that the locus of the centers of the circles cutting two fixed circles orthogonally is their radical axis:



C intersects C_1 and C_2 orthogonally.
Thus,

$OA \perp O_1A$ and

$OB \perp O_2B$.

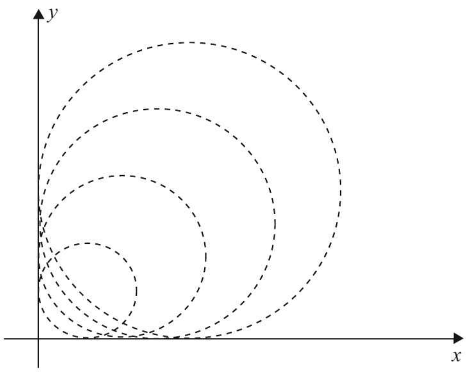
5. Family of circles

By a family of circles, we will mean a set of circles satisfying some given property (or properties). For example, the family of circles with each circle having its centre lying in the first quadrant and touching both the co-ordinate axes can be represented by the equation

$$(x-a)^2 + (y-a)^2 = a^2 \tag{1}$$

where a is a positive real number.

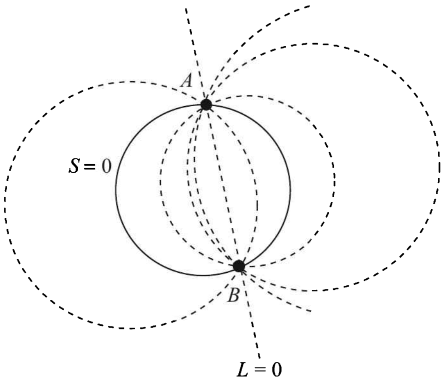
The important point to observe is that a is a variable here. As we vary a, we get different circles belonging to this family, but due to the constraint imposed by (1), all circles of this family satisfy the specified property.



Some members of the family of circles given by (1).
The centre of a circle is (a, a) and its radius is a, where a is a real positive variable. As a is varied, we obtain different members in this family.

We intend to discuss in this section certain families that are of significant importance. In all cases, the family will be represented by an equation containing a real variable, which when varied will give rise to different members of this family.

TYPE 1: Family of circles passing through the intersection points of a given circle and a given line



The circles $S = 0$ and $L = 0$ are fixed. The dotted circles represent some of the members of the family of circles passing through the intersection points A and B of $S = 0$ and $L = 0$.

Any circle F belonging to this family can be written as

$$F \equiv S + \lambda L = 0, \text{ where } \lambda \in \mathbb{R}$$

This general type gives rise to the following two special cases:

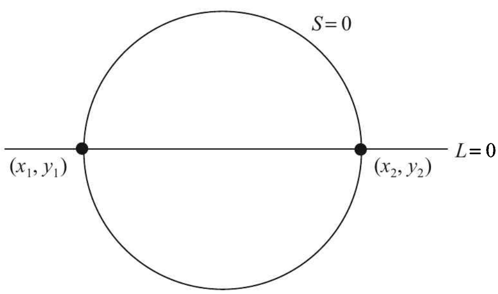
(a) Family of circles passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$

We first write the equation of the (fixed) circle with P and Q as the end-points of a diameter:

$$S \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

We then write the equation of the (fixed) line through P and Q:

$$L : \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



Given two fixed points (x_1, y_1) and (x_2, y_2), we define the fixed circle as the one with these points being the end-points of a diameter. The fixed line is simply the line passing through these two points.

The required family can now be written as

$$F \equiv S + \lambda L = 0$$

(b) Family of circles touching a line $L = 0$ at $P(x_1, y_1)$

For this case, we can use the result in the previous part and let $x_2 \to x_1$ and $y_2 \to y_1$, because by applyng these limits, $L = 0$ will simply become the tangent to any member of the family. Thus, the required equation will become

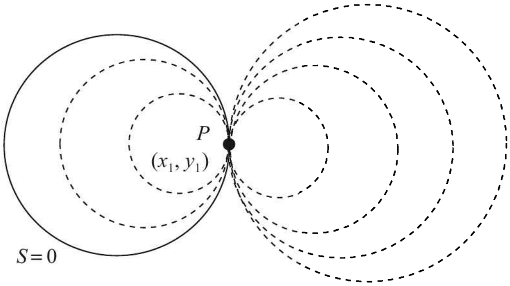
$$F \equiv (x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$$

TYPE 2: Family of circles touching a given circle at a given point.

Let the equation of the fixed circle be

$$S : x^2 + y^2 + 2gx + 2fy + c = 0$$

and let there be a point $P(x_1, y_1)$ lying on this circle. We wish to determine the equation of the family of circles touching S at P.



The dotted circles are some of the members of the family of circles in which each circle touches S at P.

We can write the equation of the tangent to $S = 0$ at P as

$$T : xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Once we have a circle ($S = 0$) and a line ($T = 0$) intersecting or touching the circle, we can write the equation of the family of circles passing through the point (s) of intersection of the circle and the line. Thus, the required family can be represented as

$$\boxed{F : S + \lambda T = 0}$$

$$\Rightarrow \quad F : x^2 + y^2 + 2gx + 2fy + c + \lambda(xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c) = 0$$

$$\Rightarrow \quad F : x^2 + y^2 + (2g + \lambda x_1 + \lambda g)x + (2f + \lambda y_1 + \lambda f)y + c + \lambda gx_1 + \lambda fy_1 + \lambda c = 0$$

As we vary λ, we will obtain different members belonging to this family.

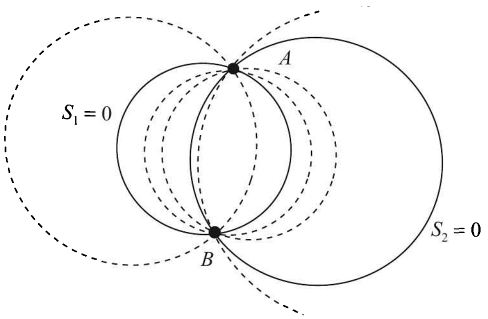
TYPE 3: Family of circles passing through the intersection point(s) of two given circles

Let the two fixed circles be

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

and their points of intersection be $A(x_1, y_1)$ and $B(x_2, y_2)$. In case the two circles touch each other, A and B will be the same. We wish to determine the family of circles passing through A and B.



The dotted circles represent some members of the family of circles in which each member passes through A and B.

You might be able to extrapolate from the last few cases that the equation representing this family will be

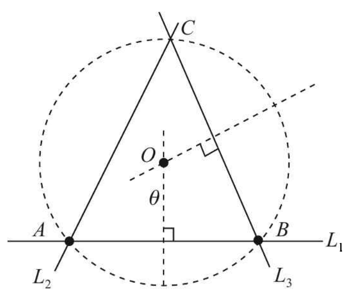
$$\boxed{F : S_1 + \lambda S_2 = 0}$$

IMPORTANT IDEAS AND TIPS

1. *A Powerful Technique in Circles.* The most prominent feature of a circle's equation is that it is a second degree equation in which (i) the coefficient of x^2 is equal to the coefficient of y^2 and (ii) the coefficient of $xy = 0$. Sometimes, this fact can be put to a very creative use, as we discuss in the following special example.

Problem: Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.

Solution: Let us first consider the general case wherein we've been given three lines L_1, L_2 and L_3 and we need to find the circle circumscribing the triangle that these three lines form:



One way to do it would be as follows:

- Find the intersection points A, B and C of the three lines.
- Use these intersection points to write any two perpendicular bisectors.
- Find the intersection of these two perpendicular bisectors which gives us the centre O.
- Finally, find the radius (which will equal OA, OB and OC).

This procedure will definitely become quite lengthy. We look instead for a a more elegant method. We first try to write the equation of an arbitrary second-degree curve S passing through the intersection points of L_1, L_2 and L_3. Think carefully and you'll realise that such a curve can be written in terms of two arbitrary constants λ and μ as follows:

$$S \equiv L_1L_2 + \lambda\, L_2L_3 + \mu\, L_3L_1 = 0$$

That such a curve S will pass through all the three intersection points can be verified by observing that the substitution of the co-ordinates of any of the three points in the equation above will make both sides identically 0. Once we have such a curve, we can impose the necessary constraints to make it a circle. Coming back to the current example, the equation of an arbitrary curve passing through the intersection points of the three lines can be written as:

$$S \equiv (x+y-6)(2x+y-4)+\lambda(2x+y-4)(x+2y-5)+\mu(x+2y-5)(x+y-6)=0$$

To make S the equation of a circle, we simply impose the following constraints:

$$\textbf{Coeff. of } \boldsymbol{x^2} = \textbf{Coeff. of } \boldsymbol{y^2} \;\Rightarrow\; 2+2\lambda+\mu = 1+2\lambda+2\mu \quad (1)$$

$$\Rightarrow\; \mu = 1$$

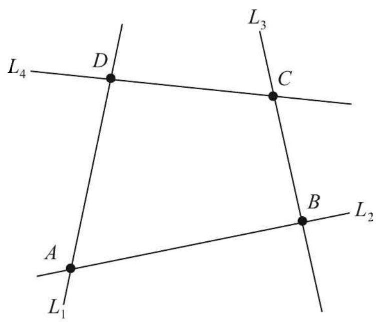
$$\textbf{Coeff. of } \boldsymbol{xy} = \textbf{0} \;\Rightarrow\; 3+5\lambda+3\mu = 0$$

$$\Rightarrow\; \lambda = -\frac{6}{5} \quad (2)$$

We substitute λ and μ back into S to obtain the required equation as:

$$S \equiv x^2 + y^2 - 17x - 19y + 50 = 0$$

As another example of following such an approach, suppose that we are given four straight lines and are told that they intersect at four concyclic points, as shown below:



A, B, C, D are four concyclic points.

What approach will you follow if you're told to find the equation of the circle circumscribing this quadrilateral? Obviously, one can always proceed by explicitly determining the centre and the radius of the said circle, but as in the previous question, a much more elegant method exists. Convince yourself that any second degree curve S passing through A, B, C, D can we written as

$$S \equiv L_1L_3 + \lambda\, L_2L_4 = 0$$

Observe carefully that the substitution of the co-ordinates of any of the four points A, B, C, D will make both sides identically 0, implying that these four points lie on S. We now simply impose the necessary constraint (on λ) to make S represent a circle, thus obtaining S!

2. *Pitfalls in Writing Expressions.* Let the equation of a circle be $S(x, y) = 0$,where $S(x, y)$ is the expression $x^2 + y^2 + 2gx + 2fy + c$. Note that $S(x, y)$ is an *expression*, not an *equation*. The equation of the tangent

to this circle at some point, say (x_1, y_1), is $T(x_1, y_1) = 0$, where $T(x_1, y_1)$ is again the *expression* (and not the equation) $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$. We have seen three important results in the theory:

Problem	Answer	Points to be Noted
Joint equation of tangents from an *external* point $P(x_1, y_1)$	$T^2 = SS_1$	(a) Here, T is short for $T(x_1, y_1)$, and is the *expression* obtained by using the values of x_1 and y_1 in $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$. Therefore, note that $T(x_1, y_1)$ is a *linear expression* in x and y. (b) S is short for the *expression* $S(x, y)$ or $x^2 + y^2 + 2gx + 2fy + c$, which is a *second degree expression in* x and y. (c) $S(x_1, y_1)$ is obtained by substituting the values of x_1 and y_1 in place of x and y in the expression $S(x, y)$. Note that S_1 is a *constant*–it has no *variable* term. Dimensional Correctness: $(Linear)^2 = (Second\ Degree) \times (Constant)$ On simplifying this, we get a second-degree expression which corresponds to the pair of tangents.
Chord of contact from an external point $P(x_1, y_1)$	$T = 0$	Once again, T is short for $T(x_1, y_1)$ and is the *expression* obtained by using the values of x_1 and y_1 in $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$, and is linear, which is how it should be, since the chord of contact has to be represented by a linear equation. Dimensional correctness: $(Linear) = 0$
Chord bisected at an internal point $P(x_1, y_1)$	$T = S_1$	T and S_1 have the same meanings as above; the important thing is the dimensional correctness: $(Linear) = (Constant)$ Note that this is a linear equation, which is how it should be if it has to represent a straight line.

Many students are unfamiliar with the justifications behind these importants results. That in itself is a pitfall–we strongly urge the reader to take out some time and understand the proofs of these results.

3. *Family of Circles*. One of the most powerful techniques you have at your disposal is the family of circles approach. In the theory, we have summarized the various ways of writing the equations of families of circles and we hope you have observed them carefully. Many advanced problems on circles can be solved using the family of circles approach. We urge as much practice as possible, and also an effort on your part in trying and comparing a family of circles approach with alternate solutions to the same problems. That will enable you to better appreciate the power of this technique.

Circles

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

A point P moves in the Euclidean plane in such a way that $PA = \lambda PB$, where A and B are fixed points and $\lambda > 0$. Which of the following can be the possible loci (plural of locus) for P?

(A) Straight Line (B) Circle (C) Parabola (D) Ellipse (E) Hyperbola

Solution: The easiest case is when $\lambda = 1$; then $PA = PB$ and P will hence lie on the perpendicular bisector of AB. We consider the case when $\lambda \neq 1$. Let A and B be assigned the co-ordinates $(a, 0)$ and $(-a, 0)$ (for convenience). This can always be done by an appropriate choice of the co-ordinate axes. Now, let P have the co-ordinates (x, y). We have,

$$PA^2 = \lambda^2 PB^2$$

$$\Rightarrow \quad (x-a)^2 + y^2 = \lambda^2\{(x+a)^2 + y^2\}$$

$$\Rightarrow \quad (1-\lambda^2)x^2 + (1-\lambda^2)y^2 - 2ax(1+\lambda^2)x + a^2(1-\lambda^2) = 0$$

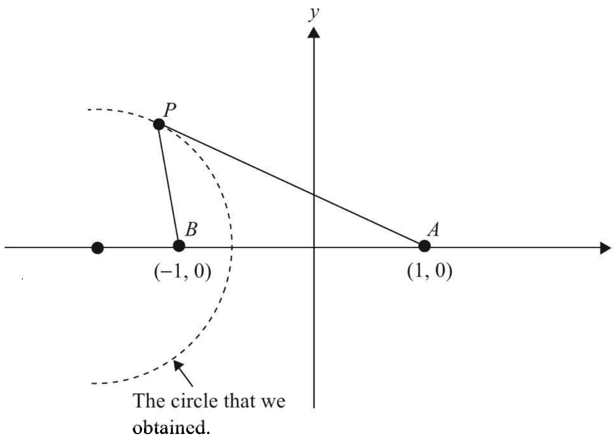
$$\Rightarrow \quad x^2 + y^2 - 2a\frac{(1+\lambda^2)}{(1-\lambda^2)}x + a^2 = 0 \qquad (1)$$

This is obviously the equation of a circle centred at $\left(\frac{a(1+\lambda^2)}{(1-\lambda^2)}, 0\right)$. Note that this circle does not pass through either A or B. Let us consider an example of this. Let AB be 2 units, so that we can assign $(1, 0)$ and $(-1, 0)$ as the co-ordinates A and B. Let P move in such a way that $PA = 2PB$, *i.e.*, $\lambda = 2$. From (1), the locus of P is the following circle:

$$x^2 + y^2 - \frac{2(1+4)}{1-4}x + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 + \frac{10}{3}x + 1 = 0$$

The centre of this circle is $(-\frac{5}{3}, 0)$ and its radius is $\sqrt{(-\frac{5}{3})^2 + (0)^2 - 1} = \frac{4}{3}$.



For any point P taken on the circumference of this circle, we will have $PA = 2PB$.

Coming back to our options, we see that (A) and (B) are both correct. ■

Example 2

(a) A fixed line L_1 intersects the co-ordinate axes at $P(a, 0)$ and $Q(0, b)$. A variable line L_2, perpendicular to L_1, intersects the axes at R and S. Show that the locus of the points of intersection of PS and QR is a circle.

(b) The center of that circle is

(A) $\left(\frac{a}{4}, \frac{b}{4}\right)$ (B) $\left(\frac{a}{3}, \frac{b}{3}\right)$ (C) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (D) None of these

Solution: The equation of L_1, using intercept form, can be written as

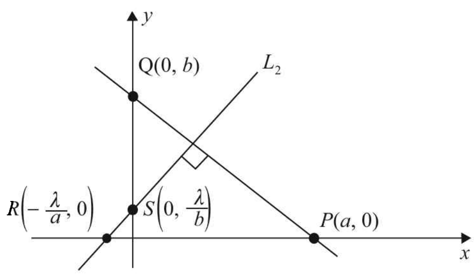
$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad bx + ay = ab$$

Since L_2 is perpendicular to L_1, its equation can be written as

$$L_2 \equiv ax - by + \lambda = 0$$

where λ is a real parameter. Using the equation of L_2, we can determine R and S to be $(-\frac{\lambda}{a}, 0)$ and $(0, \frac{\lambda}{b})$ respectively.



Points P and Q are fixed whereas R and S will vary as λ varies.

We now write the equations to PS and QR using the two-point form:

$$PS: \frac{y}{x-a} = \frac{-\lambda}{ab} \quad \Rightarrow \quad \lambda x + aby = a\lambda \tag{1}$$

$$QR: \frac{y-b}{x} = \frac{ab}{\lambda} \quad \Rightarrow \quad -abx + \lambda y = b\lambda \tag{2}$$

The relation that the intersection point of PS and QR will satisfy can be evaluated by eliminating λ from (1) and (2). We thus obtain

$$\lambda = \frac{aby}{a-x} = \frac{abx}{y-b}$$

$$\Rightarrow \quad aby^2 - ab^2y = a^2bx - abx^2$$

$$\Rightarrow \quad x^2 + y^2 - ax - by = 0$$

This represents a circle centered at $(\frac{a}{2}, \frac{b}{2})$ and passing through the origin. Thus, in part-(b), option (C) is correct. ■

Example 3

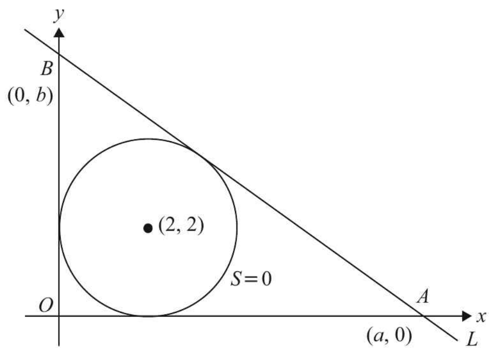
The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. If the locus of the circumcentre of the triangle is

$$x + y - xy + k\sqrt{x^2 + y^2} = 0,$$

the value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Solution : The situation is described clearly in the figure below:



The circle $S=0$ is fixed. The line is variable, intersecting the axes in $A(a, 0)$ and $B(0, b)$ respectively. We are concerned with the locus of the circumcentre of ΔOAB.

The equation of L is, using the intercept form,

$$L: \frac{x}{a} + \frac{y}{b} = 1$$

The distance of the centre of S, *i.e.*, (2, 2) from L must equal the radius of S, which is 2. Thus,

$$\frac{\left|\frac{2}{a} + \frac{2}{b} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = 2$$

We now use the fact that $L(2, 2)$ is negative since (2, 2) and the origin lie on the same side of L and $L(0, 0)$ is negative.

$$\Rightarrow \quad 2a + 2b - ab + 2\sqrt{a^2 + b^2} = 0 \tag{1}$$

From pure geometric considerations, the circumcentre C of ΔOAB lies on AB and is in fact, the mid-point of AB. Thus,

$$C \equiv \left(\frac{a}{2}, \frac{b}{2}\right)$$

Slightly manipulating (1), we obtain

$$\frac{a}{2}+\frac{b}{2}-\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)+\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=0 \tag{2}$$

The locus of $(\frac{a}{2}, \frac{b}{2})$ is given by (2). Using (x, y) instead of $(\frac{a}{2}, \frac{b}{2})$, we obtain

$$x+y-xy+\sqrt{x^2+y^2}=0 \tag{3}$$

Upon comparing (3) with the locus specified in the question, we obtain $k=1$. The correct option is (A). ■

Example 4

A circle of radius r passes through the origin O and cuts the axes at A and B. If the locus of the foot of the perpendicular from O to AB is

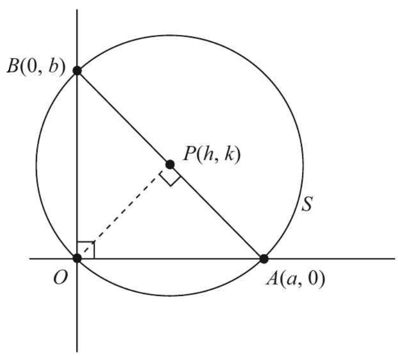
$$(x^2+y^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)=\lambda r^2 a,$$

the value of λ is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: Let the co-ordinates of A and B be $(a, 0)$ and $(0, b)$ respectively, so that the equation to the variable circle becomes

$$x^2+y^2-ax-by=0$$



The equation for S is
$x^2+y^2-ax-by=0$
Note that since $\angle AOB = \frac{\pi}{2}$,
AB is a diameter of
the circle.

We have,

$$a^2+b^2=4r^2 \tag{1}$$

Let the foot of perpendicular P have the co-ordinates (h, k). Since $OP \perp AB$, we obtain

$$\frac{k}{h}\times\frac{b}{-a}=-1$$

$$\Rightarrow \quad \frac{k}{a}=\frac{h}{b}=\frac{\sqrt{h^2+k^2}}{\sqrt{a^2+b^2}}=\frac{\sqrt{h^2+k^2}}{2r}$$

$$\Rightarrow \quad a=\frac{2rk}{\sqrt{h^2+k^2}},\ b=\frac{2rh}{\sqrt{h^2+k^2}} \tag{2}$$

Using (2) in (1) and (x,y) instead of (h,k), we obtain the required locus as

$$(x^2+y^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)=4r^2$$

We see that $\lambda=4$. The correct option is (C). ■

Example 5

Let $(m_i,\frac{1}{m_i}),\ i=1,2,3,4$ be four distinct points lying on a circle. The value of $m_1m_2m_3m_4$ is

(A) −1 (B) 4 (C) 1 (D) None of these

Solution: We first assume an equation for this circle C, in its general form:

$$C: x^2+y^2+2gx+2fy+c=0$$

Since $(m_i,\frac{1}{m_i})$ satisfies the equation of C for i = 1, 2, 3, 4, we have

$$m_i^2+\frac{1}{m_i^2}+2gm_i+\frac{2f}{m_i}+c=0 \quad i=1,2,3,4$$

$$\Rightarrow \quad m_i^4+2gm_i^3+cm_i^2+2fm_i+1=0 \quad i=1,\ 2,\ 3,\ 4$$

This last equation tells us that m_i's are the roots of the following equation in m:

$m^4+2gm^3+cm^2+2fm+1=0$: Roots of this equation are m_i, i = 1, 2, 3, 4

The product of the roots, which is $m_1m_2m_3m_4$, can easily be seen to be 1 from this equation. The correct option is (C). ■

Example 6

In which of the follwing intervals can a lie so that the point $(a-1, a+1)$ lies inside the circle $x^2+y^2-12x+12y-62=0$ but outside the circle $x^2+y^2=8$?

(A) $(-3\sqrt{2},-\sqrt{3})$ (B) $(-\sqrt{3},-1)$ (C) $(-1,1)$ (D) $(1,\sqrt{3})$ (E) $(\sqrt{3},3\sqrt{2})$

Solution: We know that if $S=0$ is the equation of circle and $P\ (x_1\ y_1)$ be any point, then

$$S(x_1,y_1)<0 \quad \Rightarrow \quad P \text{ lies inside } S$$

$$S(x_1,y_1)>0 \quad \Rightarrow \quad P \text{ lies outside } S$$

Using these relations for the current case, we obtain

$$(a-1)^2+(a+1)^2-12(a-1)+12(a+1)-62<0 \tag{1}$$

$$\Rightarrow \quad 2a^2-36<0$$

$$\Rightarrow \quad -3\sqrt{2}<a<3\sqrt{2} \qquad \text{(i)}$$

and

$$(a-1)^2+(a+1)^2-8>0 \tag{2}$$

$$\Rightarrow \quad 2a^2-6>0$$

$$\Rightarrow \quad a>\sqrt{3} \text{ or } a<-\sqrt{3}a \qquad \text{(ii)}$$

The intersection of (i) and (ii) gives us the required values of a as

$$a \in (-3\sqrt{2}, -\sqrt{3}) \cup (\sqrt{3}, 3\sqrt{2})$$

The correct options are (A) and (E). ■

Example 7

Consider two circles with the following equations:

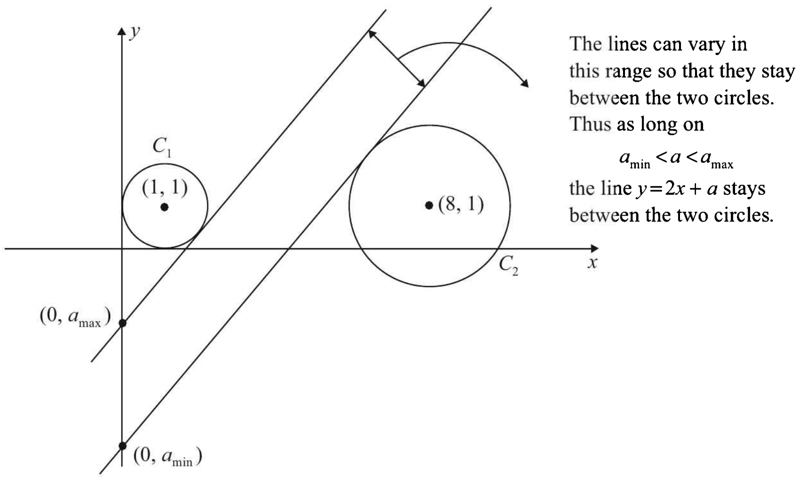
$$C_1 : x^2 + y^2 - 2x - 2y + 1 = 0$$
$$C_2 : x^2 + y^2 - 16x - 2y + 61 = 0$$

What are the values that a can take so that the variable line $y = 2x + a$ lies *between* these two circles without touching or intersecting either of them?

(A) $(\sqrt{5} - 10, -\sqrt{5})$ (B) $(\sqrt{5} - 15, -\sqrt{5} + 1)$ (C) $(\sqrt{5} - 15, -\sqrt{5} - 1)$

(D) $(2\sqrt{5} - 15, -\sqrt{5} - 1)$ (E) None of these

Solution: Observe carefully that what is variable about the variable line $y = 2x + a$ is not its slope but its y-intercept a. Thus, we can always adjust a so that this line stays between the two circles. The following diagram makes this clear:



Evaluate a_{max}: The line $y = 2x + a_{max}$ is a tangent to C_1 if the perpendicular distance of the centre (1, 1) of C_1 from this line is equal to C_1's radius, which is 1. Thus:

$$\frac{|2 - 1 + a_{max}|}{\sqrt{5}} = 1$$

$$\Rightarrow \quad 1 + a_{max} = \pm\sqrt{5}$$

$$\Rightarrow \quad a_{max} = -\sqrt{5} - 1$$

$\left\{ \begin{array}{l} \text{since from the figure we} \\ \text{can see that } a_{max} \text{ is definitely} \\ \text{negative} \end{array} \right\}$

Evaluate $a_{\min}$: The distance of C_2's center (8, 1) from $y = 2x + a_{\min}$ must be equal to its radius, which is equal to 2. Thus:

$$\frac{|16 - 1 + a_{\min}|}{\sqrt{5}} = 2$$

$$\Rightarrow \quad 15 + a_{\min} = \pm 2\sqrt{5}$$

$$\Rightarrow \quad a_{\min} = 2\sqrt{5} - 15$$

$\left\{\begin{array}{l}\text{we have selected the larger of the two} \\ \text{values possible since that is what} \\ \text{corresponds to } a_{\min}, \textit{ i.e.}, \text{ because the} \\ \text{line } y = 2x + a_{\min} \text{ lies } \textit{above } C_2.\end{array}\right\}$

Thus, we obtain the possible values of a as

$$2\sqrt{5} - 15 < a < -\sqrt{5} - 1$$

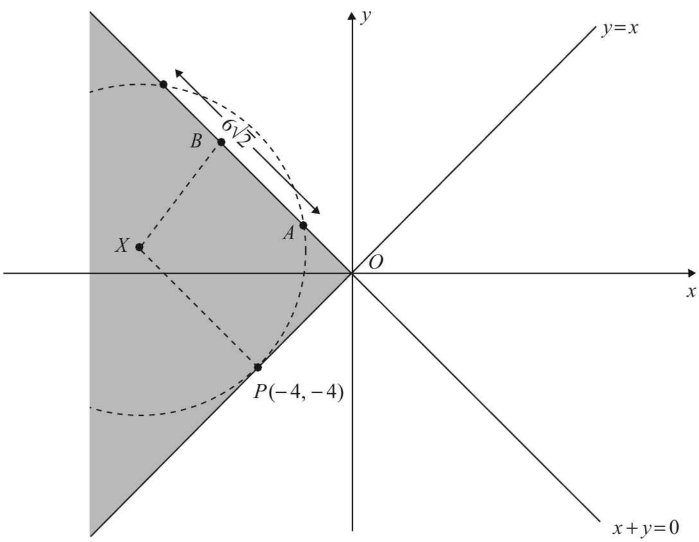
The correct option is (D). ■

Example 8

A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$ where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord-intercept on the line $x + y = 0$ is $6\sqrt{2}$. If the circle's equation is written in the standard form $x^2 + y^2 + 2gx + 2fy + c = 0$, the value of c is

(A) 8 (B) 16 (C) 24 (D) 32

Solution: As always, before starting with the solution, it is a good practice to draw a diagram of the situation described to get as much insight into it as possible. Also, as far as possible, we should try to use pure-geometric considerations to cut down on the (complicated) algebraic manipulations that would result otherwise.



It should be more or less apparent that to be able to contain the point (– 10, 2) inside it, the centre of the circle must lie somewhere in the shaded region.
The point P is then (– 4, – 4) since $OP = 4\sqrt{2}$.
Assume the centre to be at $X(h, k)$. The radius of this circle is then given by $r = XP$ where

$$r^2 = (h + 4)^2 + (k + 4)^2.$$

We have,

$$PX \perp (y = x)$$

$$\Rightarrow \quad \frac{k+4}{h+4} = -1 \tag{1}$$

$$\Rightarrow \quad h+k+8=0 \tag{2}$$

Also, the perpendicular distance of X from $x+y=0$ is given by

$$BX = \sqrt{AX^2 - AB^2} = \sqrt{r^2 - AB^2}$$

$$\Rightarrow \quad \frac{|h+k|}{\sqrt{2}} = \sqrt{(h+4)^2 + (k+4)^2 - (3\sqrt{2})^2} \tag{3}$$

Using (1) and (2) in (3), we obtain

$$(4\sqrt{2})^2 = 2(h+4)^2 - 18$$

$$\Rightarrow \quad h+4 = \pm 5$$

$$\Rightarrow \quad h = -9, 1$$

Given the region in which X lies, h must be –9. Thus, from (2), k is 1 and the radius r is $5\sqrt{2}$. The required equation is therefore

$$(x+9)^2 + (y-1)^2 = (5\sqrt{2})^2$$

$$\Rightarrow \quad x^2 + y^2 + 18x - 2y + 32 = 0 \tag{4}$$

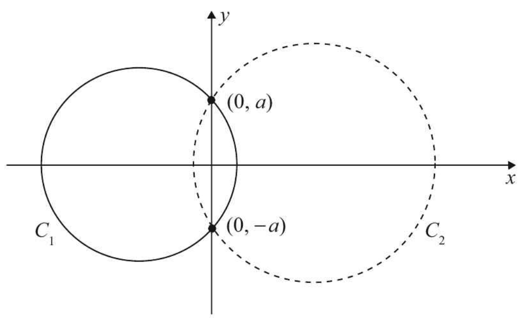
The answer is $c = 32$. The correct option is (D). For students used to rigor, it can finally be verified that the circle given by (4) does indeed contain the point $(-10, 2)$ and thus our initial assumption of the region in which the centre X lies, was correct. ■

Example 9

It is given that two circles, both of which pass through the point $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$, will intersect orthogonally if $c^2 = a^2(\lambda + m^2)$. The value of λ is

(A) 1 (B) 2 (C) 3 (D) 4 (E) None of these

Solution: It should be clear that the y-axis is the common chord of the two circles C_1 and C_2:



y-axis is the common chord of C_1 and C_2
Also, the centres of both C_1 and C_2 will lie on the x-axis.

Let the centres of C_1 and C_2 be $(-g_1, 0)$ and $(-g_2, 0)$, so that their radii become $\sqrt{g_1^2 + a^2}$ and $\sqrt{g_2^2 + a^2}$ respectively. Their equations then become:

$$C_1 : x^2 + y^2 + 2g_1x - a^2 = 0$$

$$C_2 : x^2 + y^2 + 2g_2x - a^2 = 0$$

The line $y = mx + c$ touches both C_1 and C_2 so that the perpendicular distance of the centres of C_1 and C_2 from this line must be respectively equal to their radii. Thus, we obtain

$$\frac{|mg_1 - c|}{\sqrt{1+m^2}} = \sqrt{g_1^2 + a^2} \quad \text{and} \quad \frac{|mg_2 - c|}{1+m^2} = \sqrt{g_2^2 + a^2}$$

$$\Rightarrow \quad g_1^2 + 2mcg_1 + a^2(1+m^2) - c^2 = 0 \quad \text{and} \quad g_2^2 + 2mcg_2 + a^2(1+m^2) - c^2 = 0$$

Thus, g_1 and g_2 are the roots of the equation

$$g^2 + 2mcg + a^2(1+m^2) - c^2 = 0$$

so that

$$g_1 g_2 = a^2(1+m^2) - c^2$$

Finally, C_1 and C_2 are orthogonal if the condition

$$2(g_1 g_2 + f_1 f_2) = c_1 + c_2$$

is satisfied, *i.e.*,

$$2\{(a^2(1+m^2) - c^2) + (0)(0)\} = -2a^2$$

$$\Rightarrow \quad c^2 = a^2(2+m^2)$$

This is the required relation for C_1 and C_2 to be orthogonal. The correct option is (B). ■

Example 10

Suppose we are given two curves C_1 and C_2 whose equation are as follows:

$$C_1 : a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$C_2 : a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$$

It is also given that these curves intersect in four concyclic points. Which of the following options is correct?

(A) $\dfrac{a_1+b_1}{h_1} = \dfrac{a_2+b_2}{h_2}$ (B) $\dfrac{a_1-b_1}{h_1} = \dfrac{a_2-b_2}{h_2}$

(C) $h_1(a_1+b_1) = h_2(a_2+b_2)$ (D) $h_1(a_1-b_1) = h_2(a_2-b_2)$

Solution: We know that any curve C passing through the point(s) of intersection of two given curves $C_1 = 0$ and $C_2 = 0$ can be written as

$$C \equiv C_1 + \lambda C_2 = 0 \text{ where } \lambda \in \mathbb{R}$$

We can do the same in the current example to obtain the equation of the curve passing through the four (concyclic) points of intersection as:

$$(a_1 + \lambda a_2)x^2 + 2(h_1 + \lambda h_2)xy + (b_1 + \lambda b_2)y^2 + 2(g_1 + \lambda g_2)x + 2(f_1 + \lambda f_2 y) + c_1 + \lambda c_2 = 0$$

From the general form of the equation of the circle, we know that this equation (above) will represent the equation of a circle only if:

Coeff. of x^2 = Coeff. of y^2 $\quad \Rightarrow \quad a_1 + \lambda a_2 = b_1 + \lambda b_2$

$$\Rightarrow \quad \lambda = -\frac{b_1 - a_1}{b_2 - a_2} \tag{1}$$

$$\textbf{Coeff. of } xy = 0 \quad \Rightarrow \quad h_1 + \lambda h_2 = 0$$

$$\Rightarrow \quad \lambda = -\frac{h_1}{h_2} \qquad (2)$$

From (1) and (2), we have

$$\frac{b_1 - a_1}{h_1} = \frac{b_2 - a_2}{h_2}$$

Thus, the correct option is (B). ■

Example 11

What is the equation of the circle which passes through the points of intersection of the circles

$$S_1 : x^2 + y^2 - 6x + 2y + 4 = 0$$
$$S_2 : x^2 + y^2 + 2x - 4y - 6 = 0$$

and whose centre lies on the line $y = x$?

(A) $x^2 + y^2 - 6x - 6y + 10 = 0$ (B) $x^2 + y^2 - 8x - 8y - 12 = 0$

(C) $x^2 + y^2 - 10x - 10y - 12 = 0$ (D) $x^2 + y^2 - 12x - 12y - 14 = 0$

(E) None of these

Solution: Let the required equation be $S = 0$. Then, we can find some $\lambda \in \mathbb{R}$ and $\lambda \neq -1$ such that

$$S \equiv S_1 + \lambda S_2 = 0$$

$$\Rightarrow \quad S \equiv (1+\lambda)x^2 + (1+\lambda)y^2 + (2\lambda - 6)x + (2 - 4\lambda)y + 4 - 6\lambda = 0 \qquad (1)$$

The centre of S from this equation comes out to be

$$\text{centre} \equiv \left\{ -\frac{(\lambda - 3)}{1+\lambda}, -\frac{(1 - 2\lambda)}{1+\lambda} \right\}$$

Since the centre lies on the line $y = x$, we have

$$-\frac{(\lambda - 3)}{1+\lambda} = -\frac{(1 - 2\lambda)}{1+\lambda}$$

$$\Rightarrow \quad \lambda = \frac{4}{3}$$

We now substitute this value back in (1) to obtain the equation for S:

$$S : \frac{7}{3}x^2 + \frac{7}{3}y^2 - \frac{10}{3}x - \frac{10}{3}y - 4 = 0$$

$$\Rightarrow \quad S : x^2 + y^2 - 10x - 10y - 12 = 0$$

The correct option is (C). ■

Example 12

A family of circles passing through the points A (3, 7) and B (6, 5) cuts the circle $x^2 + y^2 - 4x - 6y - 3 = 0$.

(a) Show that the common chord of the fixed circle and the variable circle (belonging to the family) will always pass through a fixed point.

(b) The x-coordinate of that point will be

(A) 2 (B) 4 (C) 5 (D) 7

Solution: We can write the equation to the specified family of circles by first writing the equation L of the line AB:

$$L: \frac{y-7}{x-3} = \frac{2}{-3}$$

$$\Rightarrow \quad L: 2x+3y=27$$

The required family can now be written as

$$F: (x-3)(x-6)+(y-7)(y-5)+\lambda L = 0 \text{ where } \lambda \in \mathbb{R}$$

$$\Rightarrow \quad F: x^2+y^2+(2\lambda-9)x+(3\lambda-12)y+(53-27\lambda)=0$$

The common chord of F and the given circle S is:

$$S-F=0$$

$$\Rightarrow \quad (5-2\lambda)x+(6-3\lambda)y+(27\lambda-56)=0$$

$$\Rightarrow \quad (5x+6y-56)-\lambda(2x+3y-27)=0$$

$$\Rightarrow \quad L_1+\mu L_2=0$$

That the common chord can be written like $L_1+\mu L_2=0$ implies that it will always pass through the intersection point of L_1 and L_2, whatever the value of μ may be. This intersection point can be obtained (by simultaneously solving L_1 and L_2) to be $(2, \frac{23}{3})$. The answer to part (b) is 2, and so the correct option is (A). ■

Example 13

What is the equation of the circle which touches the line $x-y=0$ at the origin and bisects the circumference of the circle $x^2+y^2+2y-3=0$?

(A) $x^2+y^2-4x+4y=0$ (B) $x^2+y^2-5x+5y=0$

(C) $x^2+y^2-6x+6y=0$ (D) $x^2+y^2-8x+8y=0$

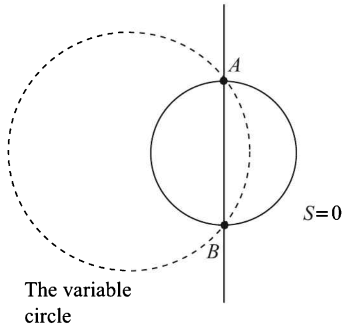
(E) None of these

Solution: We know how to evaluate the equation of the family of circles all touching a given line at a given point. Here, the given line is $x-y=0$ and the given point is (0, 0). Thus, the equation of the family is

$$F: (x-0)^2+(y-0)^2+\lambda(x-y)=0$$

$$\Rightarrow \quad F: x^2+y^2+\lambda x-\lambda y=0 \tag{1}$$

We need to find the value of λ for which the circle in (1) *bisects* the circumference of the given circle $S: x^2+y^2+2y-3=0$, which means that the common chord of the required circle and S will be a diameter of S.



If the variable circle bisects the circumference of S, this mean that the common chord AB must be the diameter of S.

The common chord AB is

$$F - S = 0$$

$$\Rightarrow \quad \lambda x - (\lambda + 2)y + 3 = 0$$

Since this is a diameter of S, the centre of S, *i.e.*, $(0, -1)$ must lie on it (satisfy its equation). Thus, we obtain λ as

$$\lambda(0) - (\lambda + 2)(-1) + 3 = 0$$

$$\Rightarrow \quad \lambda = -5$$

Finally, we substitute this value of λ back in (1) to get the required equation of the circle as

$$x^2 + y^2 - 5x + 5y = 0$$

The correct option is (B). ■

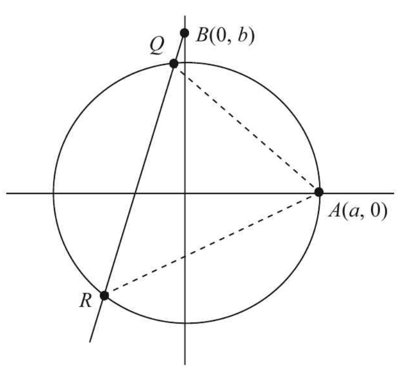
Example 14

Let $A(a, 0)$ be a point on the circle $x^2 + y^2 = a^2$. Through another point $B(0, b)$, chords are drawn to meet the circle at points Q and R. The locus of the centroid of ΔAQR is

(A) $x^2 + y^2 - \frac{a}{3}x - \frac{b}{3}y - \frac{a^2}{9} = 0$ (B) $x^2 + y^2 - \frac{2a}{3}x - \frac{2b}{3}y + \frac{a^2}{9} = 0$

(C) $x^2 + y^2 - ax - by - \frac{a^2}{6} = 0$ (D) None of these

Solution: The following diagram shows an example of the situation described:



The points A and B are fixed. The line BQR is of variable slope. We need to determine the locus of the centroid of ΔAQR.

The equation of BQR can be written as $y = mx + b$, where m is variable. The intersection points of BQR with the circle (Q and R) are given by simultaneously solving the following system:

$$x^2 + y^2 = a^2$$
$$y = mx + b$$

Thus,

$$x^2 + (mx+b)^2 = a^2$$
$$\Rightarrow \quad (1+m^2)x^2 + 2mbx + b^2 - a^2 = 0 \tag{1}$$

If we assume the co-ordinates of Q and R to (x_1, y_1) and (x_2, y_2), we have from (1)

$$x_1 + x_2 = -\frac{2mb}{1+m^2}$$

Thus,

$$y_1 + y_2 = m(x_1 + x_2) + 2b = -\frac{2m^2b}{1+m^2} + 2b = \frac{2b}{1+m^2}$$

Let the centroid of ΔAQR be (h, k). We have

$$\frac{a + x_1 + x_2}{3} = h, \quad \frac{0 + y_1 + y_2}{3} = k$$
$$\Rightarrow \quad x_1 + x_2 = 3h - a, \quad y_1 + y_2 = 3k$$
$$\Rightarrow \quad -\frac{2mb}{1+m^2} = 3h - a, \quad \frac{2b}{1+m^2} = 3k$$

These two relations can be used to eliminate m and obtain a relation in (h, k). To specify the locus conventionally, we use (x, y) instead of (h, k). The equation obtained is (verify):

$$x^2 + y^2 - \frac{2}{3}ax - \frac{2}{3}by + \frac{a^2}{9} = 0$$

The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

Example 15

Let C be any circle with centre $(0, \sqrt{2})$. Prove that at the most two rational points can lie on C. By a rational point, we mean a point which has both its co-ordinates rational.

Solution: Let the equation of C be $x^2 + y^2 + 2gx + 2fy + c = 0$. We can arrive at the result easily be contradiction. Suppose that we have three rational points on the circle with the co-ordinates (x_i, y_i), $i = 1, 2, 3$. These three points must satisfy the equation of the circle. Thus, we obtain a system of linear equations in g and f:

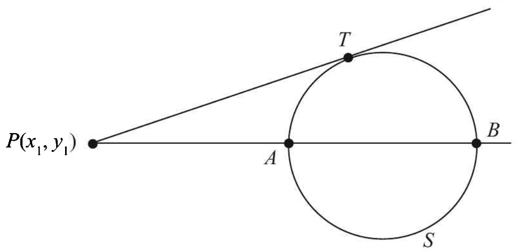
$$\begin{aligned} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 &\Rightarrow \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 &\Rightarrow \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 &\Rightarrow \end{aligned} \left\{\begin{aligned} (2x_1)g + (2y_1)f + c = -(x_1^2 + y_1^2) \\ (2x_2)g + (2y_2)f + c = (-x_2^2 + y_2^2) \\ (2x_3)g + (2y_3)f + c = (-x_3^2 + y_3^2) \end{aligned}\right\}$$

The coefficients in this system of linear equations are all rational by assumption. Thus, when we solve this system, we must obtain g, f, and c to be all rational. But since the centre is $(0, \sqrt{2})$, we have $f = -\sqrt{2}$ which gives us a contradiction. This means that our assumption of taking three rational points on the circle is wrong. Thus, at the most two rational points can lie on this circle. ■

Example 16

In this problem, we prove a widely known result on circles, using coordinate geometry. From an external point P, a line is drawn intersecting a circle S in two distinct points A and B. A tangent is also drawn from P touching the circle S at T. Prove that $PA \cdot PB$ is always constant, and equal to PT^2.

Solution: Let the equation of the circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and the point P be (x_1, y_1):



We need to prove that $PA \cdot PB = PT^2$

Notice that the line PAB passes through the fixed point P. What is variable about it is its slope, which we assume to be $\tan\theta$. Thus, using the polar form for the lines, we obtain the equation of PAB as

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \tag{1}$$

In particular, the co-ordinates of A and B can be obtained in terms of θ using the value of r as PA and PB respectively in (1). Using (1), we can write any point on the line PAB as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If this point lies on the circle, it must satisfy the circle's equation:

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0$$

$$\Rightarrow \quad r^2 + (2g\cos\theta + 2f\sin\theta)r + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \tag{2}$$

The equation (2) in r will have two roots r_1 and r_2 corresponding to PA and PB since A and B lie on the circle. Thus,

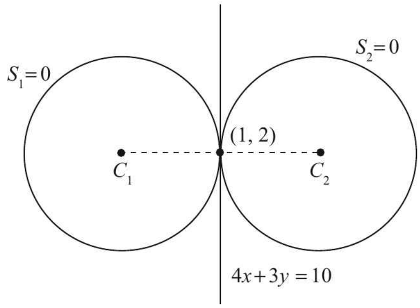
$$PA \cdot PB = r_1 \cdot r_2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S(x_1, y_1) = PT^2$$

You are urged to prove this result using 'pure' geometry. ■

Example 17

Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equations of the circles.

Solution: The following diagram explains the situation better:



The two circles touch each other at (1, 2).

We describe here two alternatives that can be used to solve this problem.

Alternative 1: Find the centres of the two circles.

The slope of the common tangent is $-\frac{4}{3}$. Therefore, the slope of C_1C_2 is $\frac{3}{4}$, *i.e.*,

$$m = \tan\theta = \frac{3}{4}$$

$$\Rightarrow \quad \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

Using the polar form, we can write the co-ordinates of any point on the line C_1C_2:

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$$

$$\Rightarrow \quad x = 1 + r\cos\theta, \quad y = 2 + r\sin\theta$$

Substituting $r = \pm 5$ gives the two centres, as should be apparent from the figure. Thus, the two centres are

$$C_1 \equiv (5, 5) \text{ and } C_2 \equiv (-3, -1)$$

The two equations therefore are

$$S_1 : x^2 + y^2 - 10x - 10y + 25 = 0$$

$$S_2 : x^2 + y^2 + 6x + 2y - 15 = 0$$

Alternative 2: Use a family of circles approach.

The family of circles touching $L = 0$ at (x_1, y_1) can be written, as described earlier, as

$$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$$

In the current case, this becomes

$$(x-1)^2+(y-2)^2+\lambda(4x+3y-10)=0$$

$$\Rightarrow \quad x^2+y^2+(4\lambda-2)x+(3\lambda-4)y+(5-10\lambda)=0 \qquad (1)$$

The radius of the required circle is 5. Thus,

$$(2\lambda-1)^2+\left(\frac{3\lambda-4}{2}\right)^2-(5-10\lambda)=5^2$$

This is a quadratic in λ which gives two values of λ:

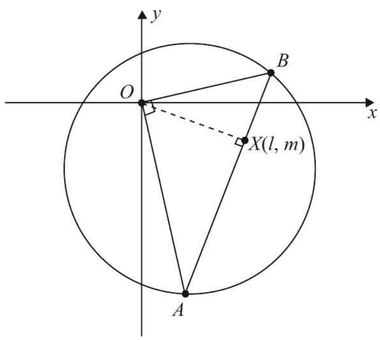
$$\lambda=\pm 2$$

Using these values in (1), we obtained the two required circles. ■

Example 18

Find the locus of the foot of the perpendicular drawn from the origin upon any chord of a circle $S\equiv x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.

Solution: The situation is depicted graphically in the figure below:



O is the origin. AB is a particular chord of the circle S which subtends a right angle at the origin. We need to find the locus of the foot of the perpendicular, *i.e.*, $X(l, m)$.

Observe that the equation of the chord AB can be written as

$$\frac{y-m}{x-l}=\frac{-l}{m} \quad (\because AB\perp OX)$$

$$\Rightarrow \quad lx+my=l^2+m^2$$

Now, if we homogenize the equation of S using the equation of the chord AB, what we'll get is the equation of the pair of straight lines OA and OB. This is what we proceed to do:

$$x^2+y^2+2gx\left(\frac{lx+my}{l^2+m^2}\right)+2fy\left(\frac{lx+my}{l^2+m^2}\right)+c\left(\frac{lx+my}{l^2+m^2}\right)^2=0 \qquad (1)$$

This is the joint equation of OA and OB, and since OA and OB need to be at right angles, we impose the appropriate constraint for perpendicularity on (1):

$$\text{Coeff. of } x^2+\text{Coeff. of } y^2=0$$

$$\left\{1+\frac{2gl}{l^2+m^2}+\frac{cl^2}{(l^2+m^2)^2}\right\}+\left\{1+\frac{2fm}{l^2+m^2}+\frac{cm^2}{(l^2+m^2)^2}\right\}=0$$

$$\Rightarrow \quad 2+\frac{2gl}{l^2+m^2}+\frac{2fm}{l^2+m^2}+\frac{c(l^2+m^2)}{(l^2+m^2)^2}=0$$

$$\Rightarrow \quad l^2+m^2+gl+fm+\frac{c}{2}=0$$

This is the equation of a circle. To be more conventional, we should use (x, y) instead of the variables l and m. Thus, the required locus is

$$x^2 + y^2 + gx + fy + \frac{c}{2} = 0 \qquad \blacksquare$$

Example 19

A circle C passes through (1, 1) and cuts the following two circles orthogonally:

$$S_1 : x^2 + y^2 - 8x - 2y + 16 = 0$$
$$S_2 : x^2 + y^2 - 4x - 4y - 1 = 0$$

Find the equation of C.

Solution: We assume the equation of C to be

$$S : x^2 + y^2 + 2gx + 2fy + c = 0$$

Applying the condition of orthogonality of S with S_1 and S_2, we obtain:

With S_1: $2(-4g - f) = c + 16$

$$\Rightarrow \quad 8g + 2f + c = -16 \qquad (1)$$

With S_2: $2(-2g - 2f) = c - 1$

$$\Rightarrow \quad 4g + 4f + c = 1 \qquad (2)$$

The third condition can be obtained using the fact that C passes through (1, 1):

$$1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$$
$$\Rightarrow \quad 2g + 2f + c = -2 \qquad (3)$$

Solving (1), (2) and (3), we obtain

$$g = -\frac{7}{3}, \quad f = \frac{23}{6}, \quad c = -5$$

Thus, the equation of the circle C is

$$S : 3x^2 + 3y^2 - 14x + 23y - 15 = 0 \qquad \blacksquare$$

Example 20

Prove that the locus of the centres of the circles cutting two given circles orthogonally is their radical axis.

Solution: This assertion means that any circle cutting two given circles orthogonally will have its centre lying on the radical axis of the two given circles. Assume the two fixed given circles to have the following equations:

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

Let the variable circle be $S: x^2+y^2+2gx+2fy+c=0$ so that its centre is $(-g,-f)$ (whose locus we wish to determine). Applying the condition for orthogonality, we obtain

$$2(gg_1+ff_1)=c+c_1 \tag{1}$$

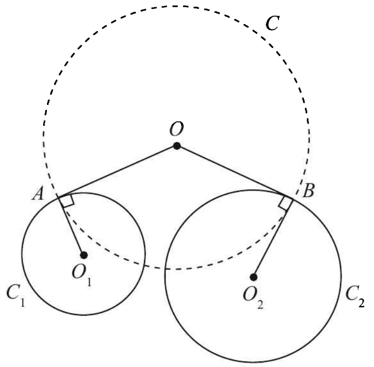
$$2(gg_2+ff_2)=c+c_2 \tag{2}$$

By (1)–(2), we obtain

$$2g(g_1-g_2)+2f(f_1-f_2)=c_1-c_2$$

Using (x,y) instead of $(-g,-f)$, we obtain the locus as $2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0$ which is the same as the equation of the radical axis, *i.e.*, $S_1-S_2=0$.

We can also prove the assertion of this question in a very straightforward manner using a pure geometric approach. Let C be a circle intersecting the two given circles C_1 and C_2 orthogonally as shown in the figure below:



C intersects C_1 and C_2 orthogonally. Thus,

$OA \perp O_1A$ and

$OB \perp O_2B$.

$OA \perp O_1A$ and $OB \perp O_2B$ implies that OA and OB are simply the tangents drawn from O to C_1 and C_2. Since OA and OB are the radii of the same circle C, we have $OA = OB$. Thus, O is a point such that tangents drawn from it to the two circles C_1 and C_2 are equal in length, which implies that O lies on the radical axis of C_1 and C_2.

From this property, a straightforward corollary follows. For three fixed circles, a circle with centre at the radical centre and radius equal to the length of the tangent from it to any of the circles will intersect all the three circles orthogonally. As an exercise, find the equation to the circle C cutting the following three circles orthogonally:

$$C_1: x^2+y^2-2x+3y-7=0$$

$$C_2: x^2+y^2+5x-5y+9=0$$

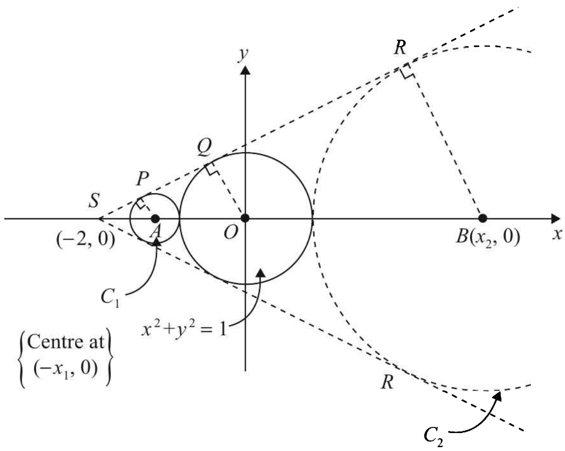
$$C_3: x^2+y^2+7x-9y+29=0$$

■

Example 21

Let T_1 and T_2 be two tangents drawn from (–2, 0) to the circle $x^2+y^2=1$. Determine the circles touching C and having T_1 and T_2 as their pair of tangents. Also find all the possible pair wise common tangents to these circles.

Solution: Note that two such circles can be drawn, as shown in the figure below:



C_1 and C_2 are the two possible circles, with centres $(-x_1, 0)$ and $(x_2, 0)$.

Let the radii of C_1 and C_2 be r_1 and r_2 respectively. We can evaluate the coordinates of both A and B very easily from geometrical considerations. We have

$$\Delta ASP \sim \Delta OSQ \sim \Delta BSR$$

$$\Rightarrow \quad \frac{AS}{AP} = \frac{OS}{OQ} = \frac{BS}{BR}$$

Now, $\quad AS = 2 - x_1,\ AP = r_1,\ OS = 2,\ OQ = 1,\ BS = 2 + x_2,\ BR = r_2$

$$\Rightarrow \quad \frac{2 - x_1}{r_1} = \frac{2}{1} = \frac{2 + x_2}{r_2}$$

Also, note that $x_1 = 1 + r_1$ and $x_2 = 1 + r_2$. Thus,

$$\Rightarrow \quad \frac{2 - x_1}{x_1 - 1} = 2 = \frac{2 + x_2}{x_2 - 1}$$

$$\Rightarrow \quad x_1 = \frac{4}{3} \text{ and } x_2 = 4$$

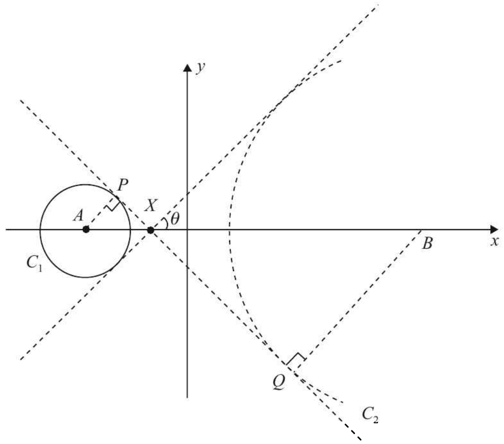
$$\Rightarrow \quad r_1 = \frac{1}{3} \text{ and } r_2 = 3$$

Thus, the equations representing C_1 and C_2 are

$$C_1 : \left(x + \frac{4}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2$$

$$C_2 : (x - 4)^2 + y^2 = 3^2$$

Now we determine all the possible common tangents to C_1 and C_2. Two such possible tangents are simply SR and SR' which both touch C_1 and C_2 on the same side. There will be two other possible common tangents, each having A and B on the opposite side of it as shown in the following figure:



These tangents are termed the transverse tangents to C_1 and C_2.

Let X have the co-ordinates $(x, 0)$. Note that $\Delta APX \sim \Delta BQX$. Thus,

$$\frac{AX}{AP} = \frac{BX}{BQ}$$

$$\Rightarrow \quad \frac{x+\frac{4}{3}}{\frac{1}{3}} = \frac{4-x}{3}$$

$$\Rightarrow \quad x = -\frac{4}{5}$$

Also,

$$\sin\theta = \frac{AP}{AX} = \frac{\frac{1}{3}}{\frac{-4}{5}+\frac{-4}{3}} = \frac{5}{8}$$

$$\Rightarrow \quad \tan\theta = \frac{5}{\sqrt{8^2-5^2}} = \frac{5}{\sqrt{39}}$$

Thus, the two transverse tangents pass through $(\frac{-4}{5}, 0)$ and have slopes $\pm\frac{5}{\sqrt{39}}$. Their equations are therefore

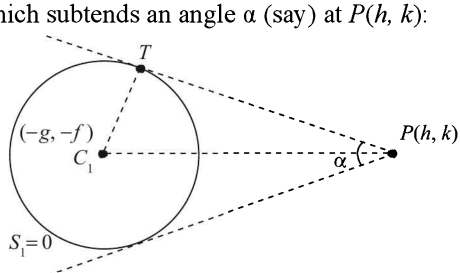
$$y = \pm\frac{5}{\sqrt{39}}\left(x+\frac{4}{5}\right)$$

■

Example 22

Let $S_1 = 0$ and $S_2 = 0$ be two circles with radii r_1 and r_2 respectively. Find the locus of the point P at which the two circles subtend equal angles.

Solution: Consider $S_1 = 0$ which subtends an angle α (say) at $P(h, k)$:



We have,

$$C_1T = r_1 \text{ and } PT = \sqrt{S_1(h,k)}$$

Thus, in ΔC_1PT, we have

$$\tan\frac{\alpha}{2} = \frac{r_1}{\sqrt{S_1(h,k)}} \tag{1}$$

We can write an exactly analogous equation for S_2 which subtends the same angle α at P:

$$\tan\frac{\alpha}{2} = \frac{r_2}{\sqrt{S_2(h,k)}} \tag{2}$$

From (1) and (2), we have

$$\frac{r_1}{\sqrt{S_1(h,k)}} = \frac{r_2}{\sqrt{S_2(h,k)}}$$

Using (x, y) instead of (h,k), we obtain the required locus as

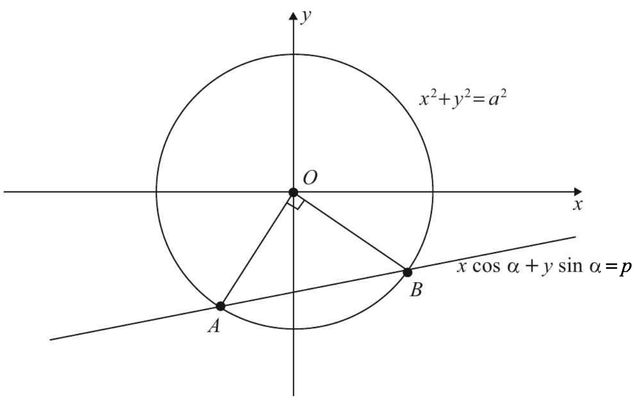
$$\frac{S_1(x,y)}{r_1^2} = \frac{S_2(x,y)}{r_2^2}$$

■

Example 23

Find the condition so that the chord $x\cos\alpha + y\sin\alpha = p$ subtends a right angle at the centre of the circle $x^2 + y^2 = a^2$.

Solution: Consider the following diagram:



We can obtain the joint equation J to the pair of lines OA and OB by homogenizing the equation of the circle using the equation of the chord AB.

$$J: x^2 + y^2 = a^2\left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 \tag{1}$$

For J to represent two perpendicular lines, we must have in (1),

$$\text{Coeff. of } x^2 + \text{Coeff. of } y^2 = 0$$

which upon simplification yields the required condition as

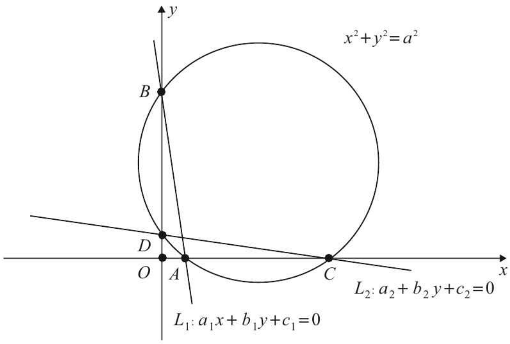
$$a^2 = 2p^2$$

■

Example 24

Suppose that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect the co-ordinate axes at points A, B and C, D respectively. Find the condition that must be satisfied if these four points are to be concyclic.

Solution: The co-ordinates of A, B and C, D can be evaluated to be $(-\frac{c_1}{a_1}, 0)$, $(0, -\frac{c_1}{b_1})$, $(-\frac{c_2}{a_2}, 0)$ and $(0, -\frac{c_2}{b_2})$.



The four points A, B and C, D are given to be concyclic.

Instead of resorting to a detailed calculation, we simply use the result on tangents and secants that we've already derived earlier:

$$OA \cdot OC = OD \cdot OB = l^2$$

where l is the length of the tangent drawn from O to the circle. This gives

$$\left(-\frac{c_1}{a_1}\right) \cdot \left(-\frac{c_2}{a_2}\right) = \left(-\frac{c_2}{b_2}\right) \cdot \left(-\frac{c_1}{b_1}\right)$$

$$\Rightarrow \quad a_1a_2 - b_1b_2 = 0$$

This is the required condition! ■

Example 25

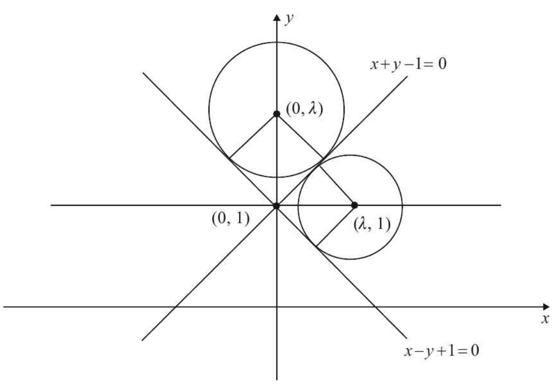
Find the equation of the family of circles touching the lines given by

$$x^2 - y^2 + 2y - 1 = 0$$

Solution: The equation to the pair of lines can be factorised to yield the individual lines as

$$x - y + 1 = 0 \quad \text{and} \quad x + y - 1 = 0$$

From the following diagram, observe carefully that for a circle to touch both the lines above, its centre must be on one of their angle bisectors because only then will the distance of the centre from the two lines be equal.



For a circle to be able to touch both the lines, its centre must lie on one of their angle bisectors.

The angle bisectors can be seen from inspection to be $x = 0$ and $y = 1$. Thus, the centre of the variable circle can be assumed to be:

(i) $\begin{matrix}(0,\lambda)\\ \lambda\in\mathbb{R}\end{matrix}$: The distance of $(0,\lambda)$ from the two lines is $\frac{|\lambda-1|}{\sqrt{2}}$.

Thus, the equation of the variable circle in this case is

$$x^2 + (y-\lambda)^2 = \frac{(\lambda-1)^2}{2} \tag{1}$$

(ii) $\begin{matrix}(\lambda,1)\\ \lambda\in\mathbb{R}\end{matrix}$: The distance of $(\lambda,1)$ from the two lines in $\frac{|\lambda|}{\sqrt{2}}$.

Thus, the equation of the variable circle in this case is

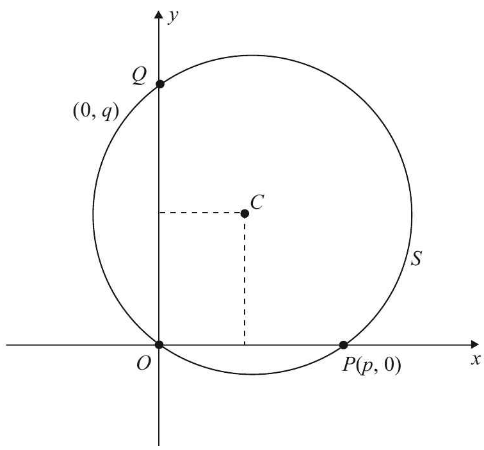
$$(x-\lambda)^2 + (y-1)^2 = \frac{\lambda^2}{2} \tag{2}$$

The equations (1) and (2) represent the required family of circles. ■

Example 26

Circles are drawn passing through the origin O to intersect the co-ordinate axes at points P and Q such that $m\cdot OP + n\cdot OQ$ is a constant. Show that the circles pass through a fixed point other than the origin.

Solution: Consider one such circle as shown in the figure below:



Given that
$mOP + nOQ = k$ (a constant)
we need to show that
S will always pass through
a fixed point.

We have $OP = p$ and $OQ = q$. Thus,

$$mp + nq = k \tag{1}$$

Now, observe that the centre C of the circle will be $(\frac{p}{2},\frac{q}{2})$ and its radius will be OC. Thus, the equation of the circle is

$$\left(x-\frac{p}{2}\right)^2 + \left(y-\frac{q}{2}\right)^2 = \frac{p^2}{4} + \frac{q^2}{4}$$

$$\Rightarrow \quad x^2 + y^2 - px - qy = 0 \tag{2}$$

From (1), we can find q in terms of p and substitute in (2) so that (2) becomes

$$x^2 + y^2 - px + \left(\frac{mp-k}{n}\right)y = 0$$

$$\Rightarrow \quad \{n(x^2 + y^2) - ky\} + p\{-nx + my\} = 0$$

We see that the variable circle can be written as

$$S_0 + pL_0 = 0, p \in \mathbb{R}$$

where $S_0 \equiv n(x^2 + y^2) - ky = 0$ and $L_0 \equiv -nx + my = 0$. Thus, the variable circle will always pass through the two intersection points of S_0 and L_0, one of them obviously being the origin. The other fixed point can be obtained by simultaneously solving the equations of S_0 and L_0 to be

$$\left(\frac{mk}{m^2 + n^2}, \frac{nk}{m^2 + n^2}\right)$$

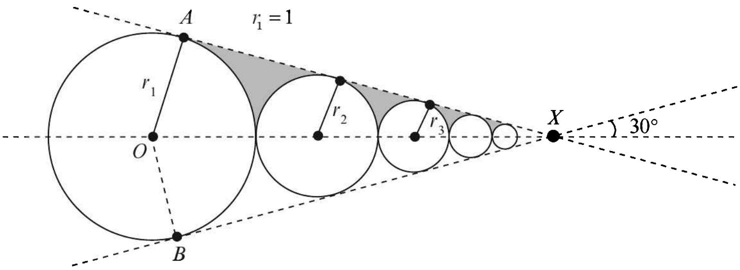
■

Circles

PART-C: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

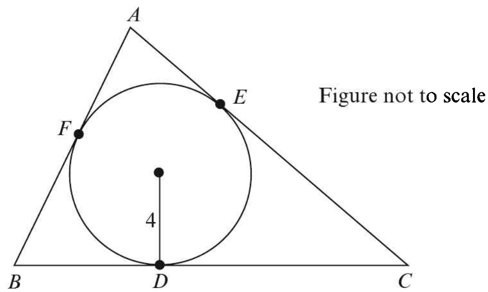
P1. Let S be the sum of this (decreasing) infinite sequence of shaded areas:



The value of $[S]$ (the greatest integer part of S) is

(A) 0 (B) 1 (C) 2 (D) 3

P2. In a triangle ABC, the incircle touches the sides BC, CA and AB at D, E, F respectively, as shown in the figure.



If the radius of the incircle is 4 units and BD, CE and AF are consecutive natural numbers, the mean of the lengths of the sides is

(A) 13 (B) 14 (C) 15 (D) 16

P3. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 4. The ratio of the product of the radii to the sum of the radii of the circles is

(A) 4 (B) 8 (C) 16 (D) 32

P4. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then which of the following are true?

(A) $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$ (B) $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

(C) $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$ (D) $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

P5. If x, y and z be the lengths of the perpendiculars dropped from the circumcentre to the sides BC, CA and AB respectively of a triangle ABC, the reciprocal of the value of the expression $\frac{xyz}{abc}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$ is

(A) 2 (B) 4 (C) 8 (D) 16

P6. Three circles with radii r_1, r_2, r_3 (where $r_1<r_2<r_3$) touch each other externally. If they have a common tangent, the value of $\sqrt{\frac{r_1}{r_2}}+\sqrt{\frac{r_1}{r_3}}$ is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

P7. Consider the two circles

$$C_1: x^2+y^2+4x+4y-1=0$$
$$C_2: x^2+y^2+6x+2y-7=0.$$

and the two lines

$$L_1: x+2y+3=0$$
$$L_2: 2x+3y+\lambda=0$$

If L_1 intersects C_1 at A and B and L_2 intersects C_2 at C and D, the value of λ so that A, B, C, D are concyclic is

(A) 2 (B) 4 (C) 6 (D) 8

P8. Let $L_1: 2x+3y+p-3=0$ and $L_2: 2x+3y+p+3=0$ be two lines and $p\in\mathbb{Z}$. Let $C: x^2+y^2+6x+10y+30=0$. If it is given that at least one of the lines L_1, L_2 is a chord of C, the probability that both are chords of C is

(A) $\frac{2}{7}$ (B) $\frac{3}{7}$ (C) $\frac{4}{11}$ (D) $\frac{5}{11}$ (E) None of these

P9. Curves

$$ax^2+2hxy+by^2-2gx-2fy+c=0$$
$$\text{and}\quad a'x^2-2hxy+(a'+a-b)y^2-2g'x-2f'y+c=$$

intersect at four concylic points A, B, C and D. If P is the point $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$, the value of $\frac{PA^2+PB^2+PC^2}{PD^2}$ is

(A) 1 (B) 2 (C) 3 (D) 4

P10. The set(s) of values of a for which the line $y+x=0$ bisects two chords drawn from the point $P(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2})$ to the circle $2x^2+2y^2-(1+\sqrt{2}a)x-(1-\sqrt{2}a)y=0$ is

(A) $(-\infty, -2)$ (B) $(-2, -1)$ (C) $(-1, 1)$ (D) $(1, 2)$ (E) $(2, \infty)$

P11. Consider a family of circles passing through the intersection point of the lines $\sqrt{3}(y-1)=x-1$ and $y-1=\sqrt{3}(x-1)$ and having its centre on the acute angle bisector of the given lines.

(a) Show that the common chords of each member of the family and the circle $x^2+y^2+4x-6y+5=0$ are concurrent.

(b) If the point of concurrency is (a, b), the value of $a+b$ is

(A) 0 (B) 1 (C) 2 (D) 3

P12. Consider two fixed circles $x^2+y^2+4|x|+3=0$. A triangle ABC is initially located so that its vertices have the following positions:

$$\left\{A \equiv (0, 2), B \equiv (2, 2\sqrt{3}+2), C \equiv (-2, 2\sqrt{3}+2)\right\}$$

It starts translating downwards perpendicular to the x-axis, and stops when its edges hit the circles (AB at the point P_1, and AC at P_2). The ratio in which P_1 divides AB is

(A) $\frac{3-\sqrt{3}}{\sqrt{3}}$ (B) $\frac{4-\sqrt{3}}{\sqrt{3}}$ (C) $\frac{1+\sqrt{3}}{\sqrt{3}}$ (D) $\frac{2+\sqrt{3}}{\sqrt{3}}$ (E) None of these

SUBJECTIVE TYPE EXAMPLES

P13. Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2, r_3 are the radii of the circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, then prove that

$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$$

P14. **(a)** Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Determine the locus of the centre of C.

(b) If C_1 and C_2 lie external to each other, and C touches both C_1 and C_2 externally, what will the locus of the center of C in this case?

P15. Let C_1 and C_2 be circles of radii 1 and 3, and let the distance between their centers be 10. Find the locus of a point M for which there exists points X and Y on C_1 and C_2 respectively such that M is the mid-point of XY.

P16. Consider a chord L subtending a right angle at the center of a given circle C of radius r. A circle C_1 is drawn with L as diameter. Find the locus of the point of intersection of the common tangents of C and C_1.

P17. Consider a curve $ax^2+2hxy+by^2=1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R. If the product $PQ\cdot PR$ is independent of the slope of the line, then show that the curve is a circle.

P18. Show that two circles can be drawn passing through the points $A(a,4a), B(4a,a)$ and touching the X-axis. Also find the angle of intersection of these circles.

P19. From a point P, tangents are drawn to concentric circles C_1 and C_2 such that they are perpendicular.

(a) Show that the lengths of the tangents are inversely proportional to the radii of C_1 and C_2, *i.e.*, the length of the tangent to C_2 is proportional to the radius of C_1, and vice-versa.

(b) Find the locus of P.

P20. Two circles C_1 and C_2 are given. Find the locus of the center of a circle C whose common chords with C_1 and C_2 intersect at right angles.

P21. Find the radius of the circle whose diameter is the common chord of the circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2ay-c=0$.

P22. Show that the circumcircle of the triangle formed by the lines

$$a_ix+b_iy+c_i=0,\ i=1,2,3$$

is
$$\begin{vmatrix} \dfrac{a_1^2+b_1^2}{a_1x+b_1y+c_1} & \dfrac{a_2^2+b_2^2}{a_2x+b_2y+c_2} & \dfrac{a_3^2+b_3^2}{a_3x+b_3y+c_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

P23. Prove that the circles

$$\frac{S_1}{r_1}+\frac{S_2}{r_2}=0 \text{ and } \frac{S_1}{r_1}-\frac{S_2}{r_2}=0$$

will cut orthogonally, where S_1 and S_2 are circles having radii r_1 and r_2 respectively.

P24. Circles are drawn which are orthogonal to both the circles

$$S_1 \equiv x^2+y^2-16=0$$
$$\text{and} \quad S_2 \equiv x^2+y^2-8x-12y+16=0$$

If tangents are drawn from the centre of the variable circles to S_1, find the locus of the mid-point of the chord of contact of the tangents.

P25. Let a circle be given by

$$2x(x-a)+y(2y-b)=0, (a \neq 0, b \neq 0)$$

Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn for the circle from the point $(a, b/2)$.

P26. The line $Ax+By+C=0$ cuts the circle $x^2+y^2+gx+fy+c=0$ at P and Q. The line $A'x+B'y+C'=0$ cuts the circle $x^2+y^2+g'x+f'y+c'=0$ at R and S. If P, Q, R and S are concyclic, show that

$$\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

P27. A is one of the points of intersection of two given circles. A variable line through A meets the two circles again at points P and Q. Show that the locus of the mid-point of P and Q is also a circle passing through A.

P28. Find the equation of the circle, two of whose normals are given by the equation $x^2 + 3x + 6y + 2xy = 0$ and which is large enough to just contain the circle $x(x-4) + y(y-3) = 0$.

P29. A tangent to the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ meets the coordinate axes at points A and B. Find the locus of the circumcentre of ΔOAB, O being the origin.

P30. S is a circle having its centre at (0, a) and radius $b (b < a)$. A variable circle centered at $(\alpha, 0)$ and touching the circle S, meets the x-axis at M and N. Find a point P on the y-axis, such that $\angle MPN$ is a constant for any choice of α.

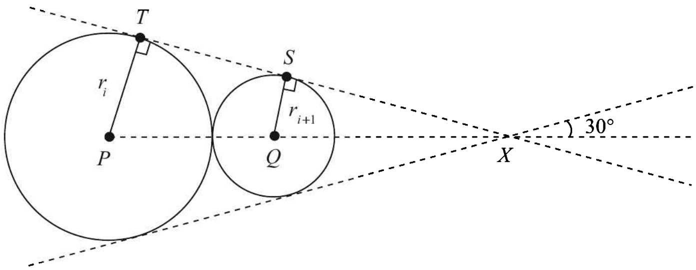
P31. Consider the straight line $x + y = 0$ and the circle $x^2 + y^2 - 2x + \frac{1}{2} = 0$. At $t = 0$, the line starts rotating anti-clockwise about the origin with angular speed ω. The circle too starts moving (anti-clockwise) such that the locus of its center is a circle of radius 1 centered at the origin. The angular speed of the circle's center is 2ω. Find the equation of the line and the circle at the instant when they first meet after $t = 0$.

Circles

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. Since $OA = r_1 = 1$, we have $AX = \sqrt{3}$. Thus, the area of the quadrilateral $OAXB$ is $2\times(\frac{1}{2}\times 1\times\sqrt{3}) = \sqrt{3}$.
Now, let us find the relation between the radii of two successive circles:



We have $PX = r_i \text{ cosec } 30° = PQ + QX = (r_i + r_{i+1}) + r_{i+1} \text{ cosec } 30°$:

$$\Rightarrow \quad \frac{r_{i+1}}{r_i} = \frac{\text{cosec } 30° - 1}{\text{cosec } 30° + 1} = \frac{1}{3}$$

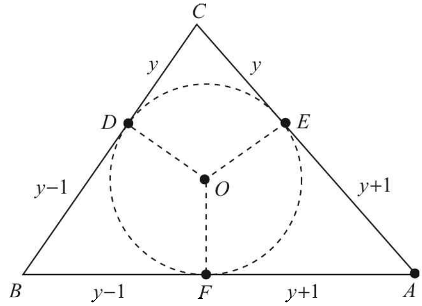
The area in $OAXB$ contained *within* the circles is

$$S' = \frac{1}{3}\pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \cdots\infty = \frac{\pi}{3} + \frac{\pi}{9}\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\infty\right) = \frac{11\pi}{24}$$

Thus, $$S = \sqrt{3} - \frac{11\pi}{24} \quad \Rightarrow \quad [S] = 0$$

The correct option is (A).

S2. We assume $BD = y - 1$, so that $CE = y$ and $AF = y + 1$.
We note that the semi-perimeter s is $3y$. By Heron's formula,



$$\begin{aligned}\Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{3y(y)(y+1)(y-1)} \\ &= \sqrt{3y^2(y^2-1)}\end{aligned}$$

Also,

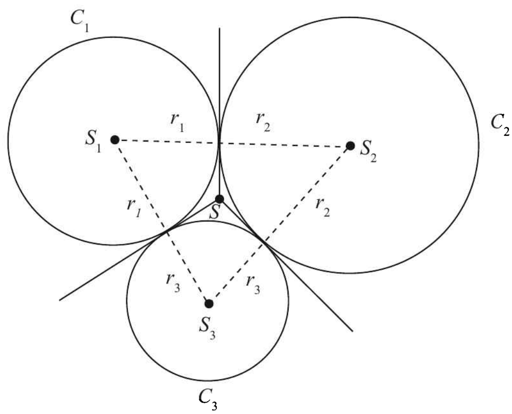
$$r = 4 = \frac{\Delta}{s} = \frac{\sqrt{3y^2(y^2-1)}}{3y}$$

Squaring, we have

$$3y^2(y^2-1) = 144y^2$$
$$\Rightarrow \quad y^2 = 49$$
$$\Rightarrow \quad y = 7$$

The sides are thus 13, 14 and 15 units. The mean of these three is 14. The correct option is (B).

S3. A slight consideration of the figure shown alongside immediately tells us that S is the incenter of $\Delta S_1S_2S_3$, and the distance of S from any point of contact, *i.e.*, 4, is the inradius r.



For $\Delta S_1S_2S_3$, we have $r = \frac{\Delta}{s}$, where

$$s = r_1 + r_2 + r_3, \quad \Delta^2 = s(s-(r_1+r_2))(s-(r_2+r_3))(s-(r_3+r_1))$$
$$= r_1r_2r_3(r_1+r_2+r_3)$$
$$\Rightarrow \quad r = 4 = \frac{\Delta}{s} = \sqrt{\frac{r_1r_2r_3}{r_1+r_2+r_3}}$$
$$\Rightarrow \quad \frac{r_1r_2r_3}{r_1+r_2+r_3} = 16$$

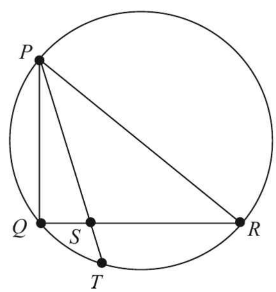
The correct option is (C). Note how no use of coordinates was required for solving this problem.

S4. We have

$$PS \cdot ST = QS \cdot SR$$

Now,

$$\frac{1}{PS} + \frac{1}{ST} = \frac{PS+ST}{PS \cdot ST} = \frac{PS+ST}{QS \cdot SR}$$
$$> \frac{2\sqrt{PS \cdot ST}}{QS \cdot SR} \quad \text{(Using AM-GM inequality)}$$



$$= \frac{2\sqrt{QS \cdot SR}}{QS \cdot SR} = \frac{2}{\sqrt{QS \cdot SR}}$$

Also,

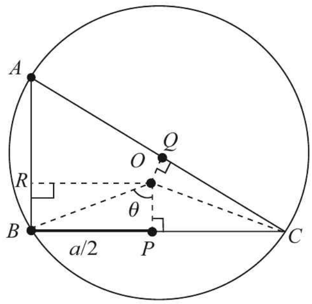
$$\sqrt{QS \cdot SR} < \frac{QS + SR}{2} = \frac{QR}{2} \qquad \text{(Again, by the AM-GM inequality)}$$

Thus,

$$\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \cdot SR}} > \frac{4}{QR}$$

$\Rightarrow$ Options (B) and (D) are true.

S5.



We note the following facts:

$OP = x, OQ = y, OR = z$

$\theta = \angle A$ (angle subtended at center by an arc is twice the angle subtended at the circumference)

$\tan\theta = \tan A = \frac{PB}{PO} = \frac{a}{2x}$

Similarly, $\tan B = \frac{b}{2y}$ and $\tan C = \frac{c}{2z}$

Now, since $\angle A$, $\angle B$ and $\angle C$ are the angles of a triangle, we simply use the fact that

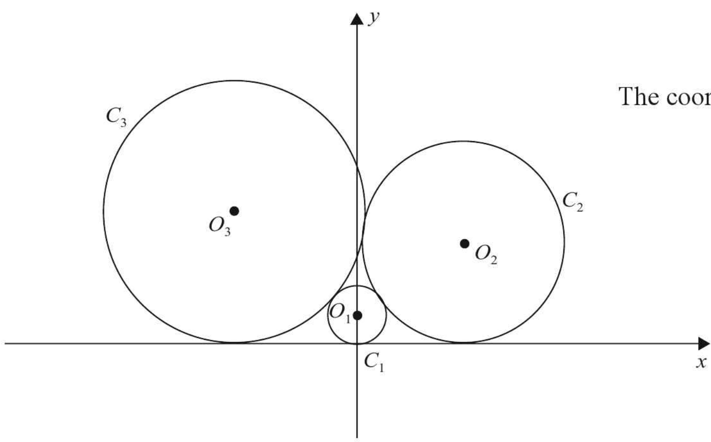
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\Rightarrow \quad \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{abc}{8xyz}$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

We see that the required answer is 4. The correct option is (B).

S6. We choose our reference axes such that the common tangent coincides with the x-axis, and the y-axis passes through the center of the circle with radius r_1, as follows:



The coordinates of the centers will be

$$C_1 : O_1 \equiv (0, r_1)$$
$$C_2 : O_2 \equiv (\alpha, r_2)$$
$$C_3 : O_3 \equiv (\beta, r_3)$$

Now,

$$O_1O_2 = r_1 + r_2 = \sqrt{\alpha^2 + (r_1 - r_2)^2} \Rightarrow \alpha = 2\sqrt{r_1 r_2}$$

$$O_1O_3 = r_1 + r_3 = \sqrt{\beta^2 + (r_1 - r_3)^2} \Rightarrow \beta = -2\sqrt{r_1 r_3} \quad (C_3 \text{ is in the second quadrant})$$

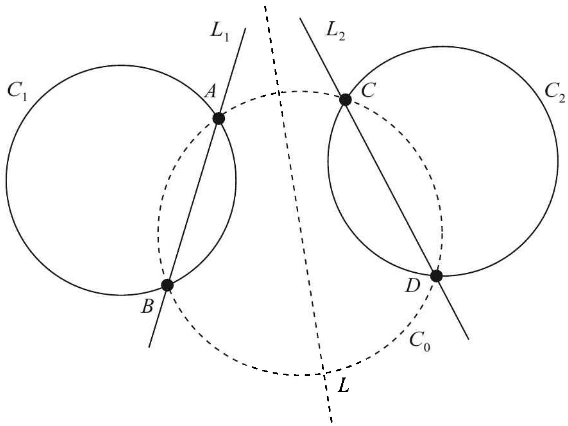
$$O_2O_3 = r_2 + r_3 = \sqrt{(\alpha - \beta)^2 + (r_2 - r_3)^2} \Rightarrow (\alpha - \beta)^2 = 4r_2 r_3$$

$$\Rightarrow 4(\sqrt{r_1 r_2} + \sqrt{r_1 r_3})^2 = 4r_2 r_3$$

$$\Rightarrow \sqrt{\frac{r_1}{r_2}} + \sqrt{\frac{r_1}{r_3}} = 1$$

The correct option is (A).

S7. We represent the given situation through the following diagram (which is not very accurate):



• C_0 is the circle passing through A, B, C and D.

• L is the radical axis of C_1 and C_2, given by.

$$S_1 - S_2 = 0$$
$$\Rightarrow 2x - 2y - 6 = 0$$

We use the fact that the radical axes of three circles taken two at a time are concurrent. Thus, L, L_1 and L_2 are concurrent:

$$\begin{vmatrix} 2 & -2 & -6 \\ 1 & 2 & 3 \\ 2 & 3 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 4$$

The correct option is (B).

S8. The center and radius of the given circle are (–3, –5) and 2 respectively. For L_1 to be a chord of the circle, its distance from the center must be less than the radius:

$$\frac{|p - 24|}{\sqrt{13}} < 2 \Rightarrow p \in (24 - 2\sqrt{13}, 24 + 2\sqrt{13})$$

Since p is an integer, p can take values from the set $S_1 = \{17, 18, \dots, 31\}$. Similarly, for L_2 to be a chord of the circle, p can take values from the set $S_2 = \{11, 12, \dots, 25\}$. Note that $S_1 \cup S_2$ has 21 elements, while $S_1 \cup S_2$ has 9 elements. The required probability is $\frac{9}{21}$ or $\frac{3}{7}$. Therefore, the correct option is (B).

S9. This problem is straightforward, so we simply outline the solution briefly. Denoting the two curves by S_1 and S_2, the equation of any curve S passing through the points of intersection of S_1 and S_2 can be represented as

$$S \equiv S_1 + \lambda S_2 = 0, \quad \text{for some } \lambda$$

Writing out the equation for S in full, and imposing the constraint that S is a circle (coeff. of x^2= coeff. of y^2, coeff. of $xy = 0$), we immediately obtain $\lambda = 1$, and hence:

$$S \equiv (a' + a)(x^2 + y^2) - 2(g' + g)x - 2(f' + f)y + 2c = 0$$

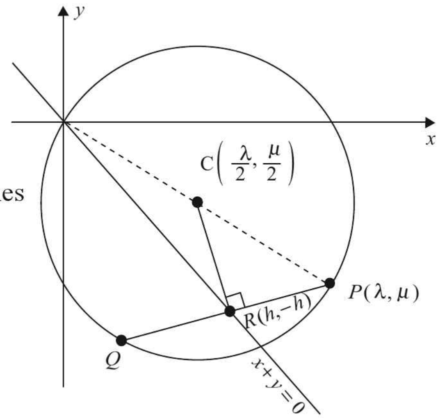
The center of this circle is $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$, which is the same as the point P, *i.e.*, P is the center of this circle. Thus, PA, PB, PC, PD are all radii of this circle, so that

$$PA^2 + PB^2 + PC^2 = 3PD^2$$

The required value is 3. The correct option is (C).

S10. We let

$$\lambda = \frac{1+\sqrt{2}a}{2}, \quad \mu = \frac{1-\sqrt{2}a}{2}$$



for the sake of brevity. The equation of the circle thus becomes

$$x^2 + y^2 - \lambda x - \mu y = 0$$

We assume R as $(h, -h)$, and note that P lies on the circle. We use the following facts to arrive at the solution:

Fact-1: *CR* is perpendicular to *PQ*.

$$\left(\frac{\frac{\mu}{2}+h}{\frac{\lambda}{2}-h}\right) \times \left(\frac{\mu+h}{\lambda-h}\right) = -1$$

$$\Rightarrow \quad 4h^2 + 3(\mu - \lambda)h + (\lambda^2 + \mu^2) = 0 \qquad (1)$$

Fact-2: The discriminant of (1) must be non-negative.

This leads to $7(\lambda^2 + \mu^2) + 18\lambda\mu < 0$ (2)

Now,

$$\lambda^2 + \mu^2 = \frac{1+2a^2}{2}, \lambda\mu = \frac{1-2a^2}{4}.$$

Using this in (2) gives

$$7(1+2a^2) + 9(1-2a^2) < 0 \quad \Rightarrow \quad a^2 > 4$$

Thus, the required values of a are

$$a \in (-\infty, -2) \cup (2, \infty)$$

The correct options are (A) and (E).

S11. It is easy to see that P is the point (1, 1), and the acute angle bisector of the two lines is $y = x$, so that the centre O of the variable C can be assumed to be (λ, λ).

Also, $OP^2 = (\lambda - 1)^2 + (\lambda - 1)^2$

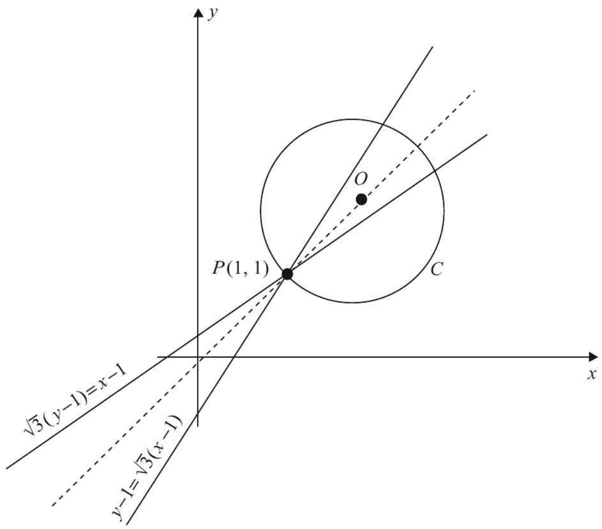
Thus, the equation of C is

$$(x - \lambda)^2 + (x - \lambda)^2 = (\lambda - 1)^2 + (\lambda - 1)^2$$

$$\Rightarrow \quad x^2 + y^2 - 2\lambda x - 2\lambda y + 4\lambda - 2 = 0$$

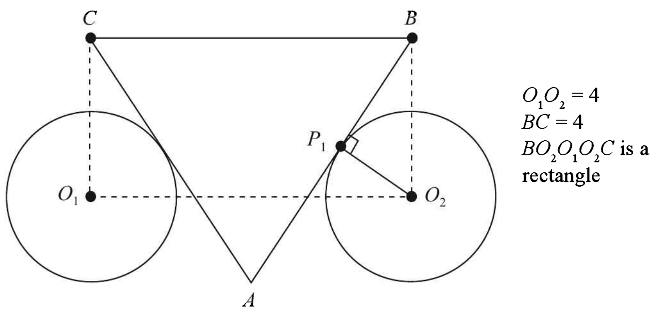
The common chord of C with the circle

$$x^2 + y^2 + 4x - 6y + 5 = 0 \text{ is}$$
$$(4+2\lambda)x + (2\lambda - 6)y + (7 - 4\lambda) = 0$$
$$\Rightarrow \quad (4x - 6y + 7) + 2\lambda(x + y - 2) = 0 \qquad (1)$$



By varying λ, (1) gives us a family of lines passing through the point of intersection of $4x - 6y + 7 = 0$ and $x + y - 2 = 0$, *i.e.*, $(\frac{1}{2}, \frac{3}{2})$. The point of concurrency of the common chords is thus $(\frac{1}{2}, \frac{3}{2})$. The required answer to part-(b) is 2. The correct option is (C).

S12. The two fixed circles have centres $(\pm 2, 0)$ and both have their radii as 1. It is easy to see that the triangle ABC is equilateral, with each side being 4. We now draw the diagram corresponding to the instant when the triangle's edges hit the circles. The reader may observe that no further use of coordinate geometry is required:



In ΔBO_2P_1, $\angle P_1BO_2 = 30°$ and $O_2P_1 = 1$. Thus,

$$\tan 30° = \frac{O_2P_1}{BP_1} \quad \Rightarrow \quad BP_1 = \sqrt{3}$$

Also, $AB = 4$. This implies that the required ratio is

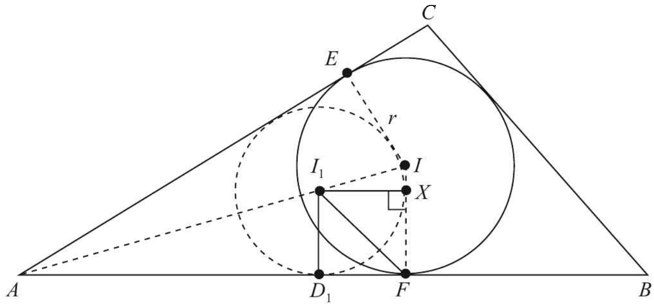
$$\frac{AP_1}{P_1B} = \frac{4 - \sqrt{3}}{\sqrt{3}}$$

The correct option is (B).

SUBJECTIVE TYPE EXAMPLES

S13. In this figure, we are just considering the circle inscribed in $AFIE$. If I_1 is the center of this circle, we note that I_1 must lie on the angle bisector of all the following four angles:

(i) $\angle EAF$ (ii) $\angle EIF$ (iii) $\angle AFI$ (iv) $\angle AEI$



Since AI is the angle bisector of both $\angle EAF$ and $\angle EIF$, I_1 must lie on AI. Also, since $\angle AFI = \angle AEI = 90°$, we draw FI_1 such that $\angle IFI_1 = 45°$. Thus, I_1 is uniquely determined. Note that $I_1D_1 = I_1X = r_1$, whereas $IF = r$. Now, observe carefully that

$$r = IF = IX + XF = I_1X\cot(\angle I_1IF) + I_1X\cot(45°) = r_1\cot\left(90° - \frac{A}{2}\right) + r_1 = r_1\tan\frac{A}{2} + r_1$$

$$\Rightarrow \quad \frac{r_1}{r - r_1} = \cot\frac{A}{2}$$

Similarly, we have

$$\frac{r_2}{r - r_2} = \cot\frac{B}{2}, \quad \frac{r_3}{r - r_3} = \cot\frac{C}{2}$$

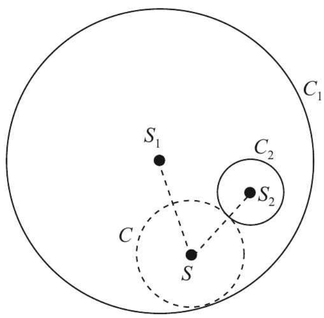
All that remains to be shown is that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ is the same as $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$, when $A + B + C = 180°$. This is straightforward trigonometry and is left to the reader as an exercise.

S14. **(a)** Let the centres of C_1, C_2 and C respectively be S_1, S_2 and S, and their radii respectively be r_1, r_2 and r. Thus,

$$SS_1 = r_1 - r$$

$$SS_2 = r_2 + r$$

$$\Rightarrow \quad SS_1 + SS_2 = r_1 + r_2$$



What does this tell us? Even though S is a moving (variable) point, it moves in a way such that the sum of its distances from two fixed points S_1 and S_2 is a constant, equal to $r_1 + r_2$. S thus traces out an ellipse with S_1 and S_2 as the two foci. As an exercise, you are urged to evaluate the eccentricity of this ellipse.

(b) The reader is urged to show, using a reasoning analogous to the one above, that the locus in this case will be a hyperbola with its foci as the centers of the two fixed circles.

S15. Assume $X(\text{on } C_1) \equiv (\cos\theta_1, \sin\theta_1)$ and $Y(\text{on } C_2) \equiv (10+3\cos\theta_2, 10+3\sin\theta_2)$. Further, assume M (the mid-point of XY) $\equiv (h, k)$. We thus have

$$h = 5 + \frac{\cos\theta_1 + 3\cos\theta_2}{2}, \quad k = 5 + \frac{\sin\theta_1 + 3\sin\theta_2}{2}$$

$$\Rightarrow \quad (h-5)^2 + (k-5)^2 = \frac{5+3\cos(\theta_1 - \theta_2)}{2} = r^2 \text{ (say)}$$

$$\Rightarrow \quad r_{\max} = 2, r_{\min} = 1$$

Therefore, the locus is a ring-shaped circular region with center at (5, 5) and inner and outer radii 1 and 2 respectively.

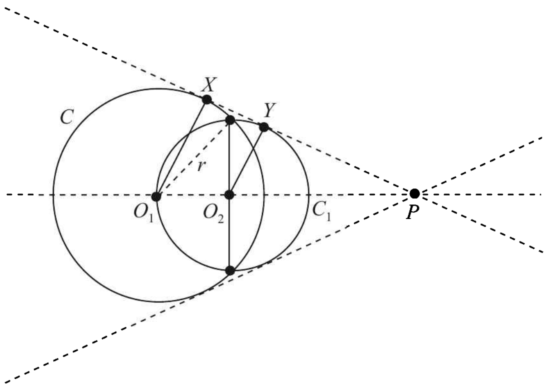
S16. The radius of C_1 is $O_1O_2 = \frac{r}{\sqrt{2}}$ (why ?). Since $\Delta PO_2Y \sim \Delta PO_1X$, we have

$$\frac{PO_2}{O_2Y} = \frac{PO_1}{O_1X} = \frac{PO_2 + O_1O_2}{O_1X}$$

$$\Rightarrow \quad \frac{PO_2}{r/\sqrt{2}} = \frac{PO_2 + r/\sqrt{2}}{r}$$

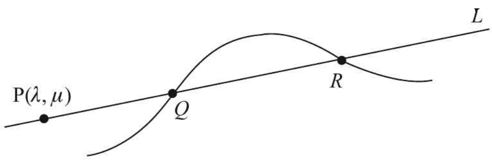
$$\Rightarrow \quad PO_2 = \frac{r}{\sqrt{2}-1}$$

$$\Rightarrow \quad PO_1 = r\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}-1}\right)$$



Thus P is at this distance from the center O_1 of C_1. The required locus is a circle with its center at O_1 and this distance as its radius.

S17. We assume an arbitrary shape of the curve to start with, and a point P not on this curve:



Any point T on the line L can be specified as

$$T \equiv (\lambda + r\cos\theta, \mu + r\sin\theta)$$

where r is the distance of T from P and θ specifies the inclination of L. If T lies on the curve $ax^2 + 2hxy + by^2 = 1$, then

$$a(\lambda + r\cos\theta)^2 + 2h(\lambda + r\cos\theta)(\mu + r\sin\theta) + b(\mu + r\sin\theta)^2 = 1$$

$$\Rightarrow \quad (a\cos^2\theta + h\sin 2\theta + b\sin^2\theta)r^2 + (2a\lambda\cos\theta + 2b\mu\sin\theta + 2h(\lambda\sin\theta + \mu\cos\theta))r$$
$$+ a\lambda^2 + 2h\lambda\mu + b\mu^2 - 1 = 0$$

This is a quadratic equation in r, which will have two roots r_1 and r_2, and from the figure, r_1 and r_2 correspond to PQ and PR. Thus,

$$PQ \cdot PR = r_1 \cdot r_2 = \frac{a\lambda^2 + 2h\lambda\mu + b\mu^2 - 1}{(a\cos^2\theta + h\sin 2\theta + b\sin^2\theta)}$$

is a constant. This means that

$$\begin{aligned} f(\theta) &= a\cos^2\theta + h\sin 2\theta + b\sin^2\theta \\ &= \frac{a}{2}(1+\cos 2\theta) + h\sin 2\theta + \frac{b}{2}(1-\cos 2\theta) \\ &= \frac{a+b}{2} + \left(\frac{a}{2} - \frac{b}{2}\right)\cos 2\theta + h\sin 2\theta \end{aligned}$$

has a constant value, regardless of θ. This can only happen if $a = b$ and $h = 0$. The equation $ax^2 + 2hxy + by^2 = 1$ thus reduces to $ax^2 + ay^2 = 1$, which represents a circle.

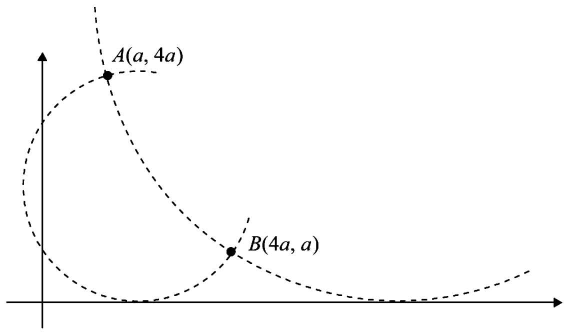
S18. An approximate figure of two distinct circles passing through A and B and touching the x-axis is shown below:

The equation of the circle with A and B as end points of a diameter is

$$S : (x-a)(x-4a) + (y-4a)(y-a) = 0,$$

while the line through A and B has the equation

$$L : x + y - 5a = 0$$



Thus, any circle C passing through A and B can be represented by the equation $C \equiv S + \lambda L = 0$ for some $\lambda \in \mathbb{R}$; this gives:

$$C \equiv x^2 + y^2 + (\lambda - 5a)x + (\lambda - 5a)y + 8a^2 - 5a\lambda = 0 \tag{1}$$

Since C touches the x-axis, the y-coordinate of its center must be equal to its radius; imposing this constraint and simplifying gives

$$\lambda^2 + 10a\lambda - 7a^2 = 0, \tag{2}$$

which, as expected, yields two values of λ, say λ_1 and λ_2. By differentiating (1), we can obtain the slopes of the two tangents to the two circles at (say) the point $A(a, 4a)$:

$$2x + 2y\left(\frac{dy}{dx}\right) + (\lambda - 5a) + (\lambda - 5a)\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{\lambda - 5a + 2x}{\lambda - 5a + 2y}\right)$$

Using $(x, y) \equiv (a, 4a)$, the two slopes are thus

$$m_1 = -\left(\frac{\lambda_1 - 3a}{\lambda_1 + 3a}\right) \text{ and } m_2 = -\left(\frac{\lambda_2 - 3a}{\lambda_2 + 3a}\right)$$

If θ is the angle of intersection, then using $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ and simplifying gives:

$$\tan\theta = \left|\frac{6a\sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2}}{2\lambda_1\lambda_2 + 18a^2}\right|$$

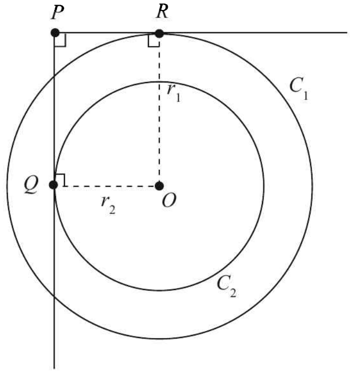
From (2), we have $\lambda_1 + \lambda_2 = -10a$ and $\lambda_1\lambda_2 = -7a^2$; using these values gives

$$\tan\theta = 12\sqrt{2}$$

S19. Note that $PQ = r_1$ and $PR = r_2$. This proves the assertion in part - (a).

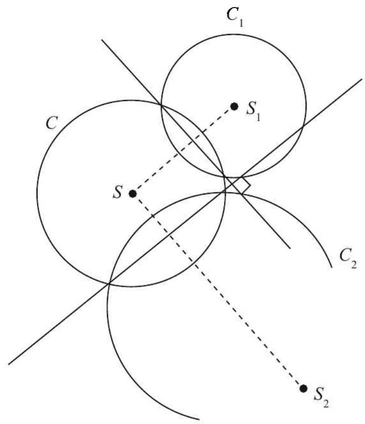
Also, $OP = \sqrt{r_1^2 + r_2^2}$:

$\Rightarrow$ The locus of P is a circle concentric with C_1 and C_2, with radius $\sqrt{r_1^2 + r_2^2}$.



S20. Note that $\angle S_1SS_2 = \frac{\pi}{2}$:

$\Rightarrow$ S lies on a circle with S_1S_2 as a diameter.



S21. Consider the accompanying figure carefully. The centers of the two fixed circles have been represented by C_1 and C_2. We have

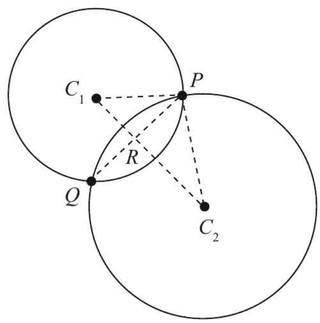
$$C_1P = \sqrt{a^2 - c}, \quad C_2P = \sqrt{a^2 + c} \quad C_1C_2 = \sqrt{2}a$$

Since $C_1P^2 + C_2P^2 = C_1C_2^2$, the circles intersect orthogonally:

$$\Rightarrow \quad \frac{1}{2} \times C_1P \times C_2P = \frac{1}{2} \times C_1C_2 \times PR$$

$$\Rightarrow \quad PR = \frac{\sqrt{a^4 - c^2}}{\sqrt{2}a}$$

This is the required radius.



S22. Let the three lines representing the sides of the triangle be $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$. The idea is to write down the equation of a general second degree curve passing through the points of intersection of these lines, and find the condition(s) for that curve to represent a circle-which must be the circumcircle of the triangle. That curve, as you may recall, can be written as

$$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0 \qquad (1)$$

Using $L_i \equiv a_i x + b_i y + c_i$, the curve will represent a circle if

(i) Coeff. of x^2 = Coeff. of y^2

$$\Rightarrow \quad \lambda(a_2a_3 - b_2b_3) + \mu(a_3a_1 - b_3b_1) + (a_1a_2 - b_1b_2) = 0 \tag{2}$$

(ii) Coeff. of xy = 0

$$\Rightarrow \quad (a_1b_2 + a_2b_1) + \lambda(a_2b_3 + a_3b_2) + \mu(a_3b_1 + a_1b_3) = 0 \tag{3}$$

From (1), (2) and (3), λ and μ can be easily eliminated to obtain

$$\begin{vmatrix} L_2L_3 & L_3L_1 & L_1L_2 \\ a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 & a_1a_2 - b_1b_2 \\ a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 & a_1b_2 + a_2b_1 \end{vmatrix} = 0$$

Note that this second degree equation *is* the equation representing the circle; we need to rearrange it to the form provided in the problem. To do that, we expand the determinant, whence we'll obtain (three) terms like $L_2L_3(a_1^2 + b_1^2)(a_2b_3 - a_3b_2)$, so that this determinant relation can be rewritten as

$$\begin{vmatrix} L_2L_3(a_1^2 + b_1^2) & L_3L_1(a_2^2 + b_2^2) & L_1L_2(a_3^2 + b_3^2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

Dividing the first row by $L_1L_2L_3$, we obtain the form we seek.

S23. This is a straightforward problem; you assume the equations for S_1 and S_2 as

$$x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0, i = 1, 2,$$

so that $r_i^2 = g_i^2 + f_i^2 - c_i$, and then write the equations for $\frac{S_1}{r_1} + \frac{S_2}{r_2} = 0$ and $\frac{S_1}{r_1} - \frac{S_2}{r_2} = 0$ separately. Finally, for these new circles, you prove the orthogonality condition:

$$2(G_1G_2 + F_1F_2) = C_1 + C_2$$

S24. Let S be a variable circle orthogonal to S_1 and S_2. The centre of S obviously lies on the radical axis of S_1 and S_2, which is $2x + 3y - 8 = 0$. If C is the centre of S, then $C \equiv (\lambda, \frac{8-2\lambda}{3})$ for some λ. The chord of contact from C to S_1 has the form

$$T = 0: \quad \lambda x + \left(\frac{8 - 2\lambda}{3}\right) y = 16 \tag{1}$$

If $P(h, k)$ is the mid-point of this chord, we can write another equation for the same chord using

$$T = S_1: \quad hx + ky = h^2 + k^2 \tag{2}$$

Since (1) and (2) represent the same line, we immediately have

$$\frac{\lambda}{h} = \frac{8 - 2\lambda}{3k} = \frac{16}{h^2 + k^2}$$

Eliminating λ and simplifying, we have

$$h^2 + k^2 - 4h - 6k = 0$$

Using $(h \to x, k \to y)$, the required locus is

$$x^2 + y^2 - 4x - 6y = 0$$

S25. If CD is bisected at $P(\lambda, 0)$, the equation of CD can be written as

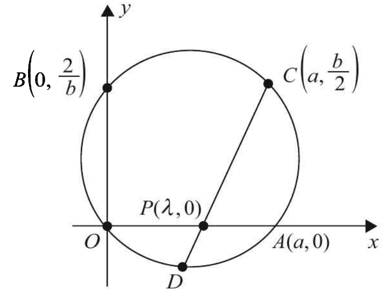


Figure is not accurate

$$T_{(\lambda,0)} = S_{1(\lambda,0)}$$

$$\Rightarrow \quad \lambda x - \frac{a}{2}(\lambda + x) - \frac{b}{4}y = \lambda^2 - a\lambda$$

$$\Rightarrow \quad \left(\lambda - \frac{a}{2}\right)x - \frac{b}{4}y = \lambda^2 - \frac{a\lambda}{2}$$

This passes through $C(a, \frac{b}{2})$, so that

$$\left(\lambda - \frac{a}{2}\right)a - \frac{b}{4}\cdot\frac{b}{2} = \lambda^2 - \frac{a\lambda}{2}$$

$$\Rightarrow \quad \lambda^2 - \frac{3a}{2}\lambda + \frac{4a^2 + b^2}{8} = 0 \qquad (1)$$

For two distinct values of λ to exist, the discriminant of (*) must be positive:

$$\frac{9a^2}{4} > \frac{4a^2 + b^2}{2} \quad \Rightarrow \quad a^2 > 2b^2 \quad \Rightarrow \quad |a| > \sqrt{2}\,|b|$$

S26. Let S_1 and S_2 represent the two (original) circles, while $S_0 \equiv x^2 + y^2 + 2g_0x + 2f_0y + c_0 = 0$ represent the circle passing through P, Q, R, S. Thus, $Ax + By + C = 0$ is a common chord for S_1 and S_0, while $A'x + B'y + C' = 0$ is a common chord for S_2 and S_0. Also, the radical axis for S_1 and S_2 is $(g - g')x + (f - f')y + c - c' = 0$. These three lines must be concurrent, so that

$$\begin{vmatrix} g - g' & f - f' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

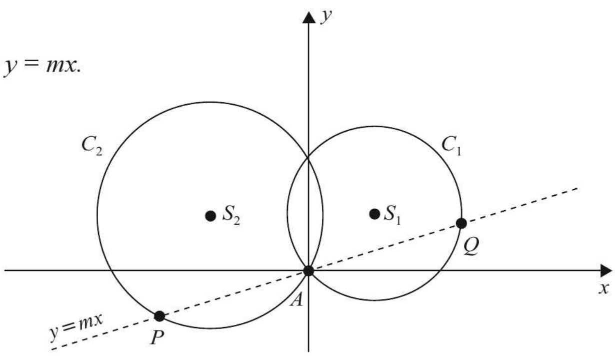
S27. An important step towards the solution to this problem is to use the appropriate reference axes. We use an axes such that A is the origin while the y-axis is the common chord of the two circles, as shown:

The centres of the two circles can be assumed as

$$S_1 : (-g_1, -f)$$
$$S_2 : (-g_2, -f)$$

while the variable line can be assumed to be $y = mx$.
Thus,

$$C_1 : x^2 + y^2 - 2g_1x - 2fy = 0$$
$$C_2 : x^2 + y^2 - 2g_2x - 2fy = 0$$



$$\Rightarrow \quad \left\{ \begin{array}{ll} P : (\lambda, m\lambda) & \text{where } \lambda = \dfrac{2(g_1 + mf)}{1 + m^2} \\ Q : (\mu, m\mu) & \text{where } \mu = \dfrac{2(g_2 + mf)}{1 + m^2} \end{array} \right\}$$

(These values are obtained by solving the line's equation with the equations of the circles)

If $R(h,k)$ is the mid-point of PQ, then

$$\begin{aligned} 2h &= \lambda + \mu \\ 2k &= m(\lambda + \mu) \end{aligned} \quad \Rightarrow \quad m = \frac{k}{h}$$

Using this value of m back in $2h = \lambda + \mu$, we obtain

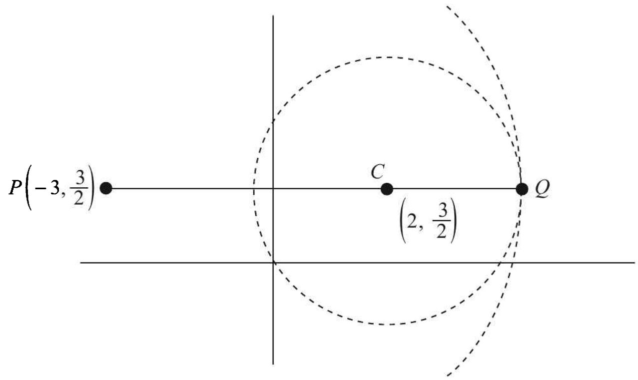
$$h^2 + k^2 = (g_1 + g_2)h + 2fk$$

Using $(h \to x,\ k \to y)$, the required locus for R is

$$x^2 + y^2 = (g_1 + g_2)x + 2fy$$

which is a circle that obviously passes through the origin A.

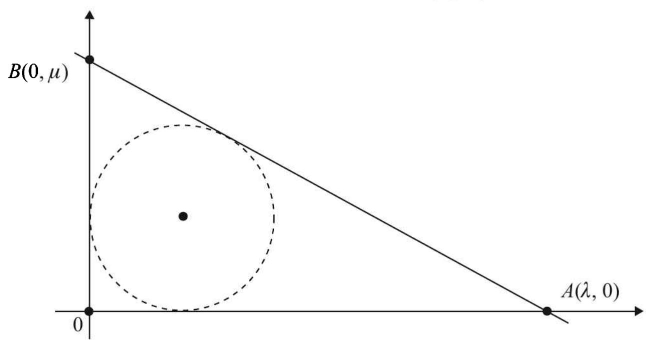
S28. The two normals are $x + 3 = 0$ and $x + 2y = 0$, whose point of intersection is $P(-3, \frac{3}{2})$, which is the center of the circle. The circle given to us is $x(x-4) + y(y-3) = 0$, so that (0, 0) and (4, 3) are the diametric end points of this circle, and its radius is $\frac{5}{2}$, while the center is $C(2, \frac{3}{2})$:



Therefore, the minimum radius of the containing circle is $PQ = PC + CQ = 5 + \frac{5}{2} = \frac{15}{2}$. The required equation of the containing circle is

$$(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$$

S29. The given circle has a radius of a units while its center is (a, a).



Assuming that the variable tangent intersects the x-axis at $(\lambda, 0)$ and the y-axis at $(0, \mu)$, its equation is $\mu x + \lambda y = \lambda\mu$, and its distance from (a, a) must be a units:

$$\frac{|a\mu + a\lambda - \lambda\mu|}{\mu^2 + \lambda^2} = a \qquad (1)$$

Note that the circumcenter of ΔOAB will be the point of intersection of the perpendicular bisectors of OA and OB and so it will have the coordinates $(h,k) \equiv (\frac{\lambda}{2}, \frac{\mu}{2})$. Using $(\lambda \to 2h, \mu \to 2k)$ in (1), and squaring, we have

$$(2ak + 2ah - 4hk)^2 = a^2(4k^2 + 4h^2)$$

Simplifying and finally using $(h \to x,\ k \to y)$ will yield the required locus as

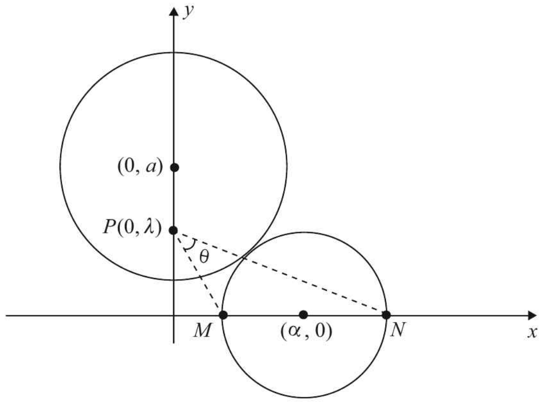
$$2xy - 2ax - 2ay + a^2 = 0$$

S30. Assuming the radius of the variable circle to be r, we make the following observations:

- $r = \sqrt{a^2 + \alpha^2} - b$
- $M \equiv (\alpha - r, 0)$; $N \equiv (\alpha + r, 0)$
- Slopes:

$$m_{MP} = \frac{-\lambda}{\alpha - r}$$

$$m_{NP} = \frac{-\lambda}{\alpha + r}$$



Thus,

$$\tan\theta = \left|\frac{m_{MP} - m_{NP}}{1 + m_{MP}m_{NP}}\right| = \left|\frac{2\lambda r}{\alpha^2 - r^2 + \lambda^2}\right| \qquad (1)$$

We want $\tan\theta$ to have a constant value for any value for α. Substituting $r = \sqrt{a^2 + \alpha^2} - b$ in (1), we want that

$$\tan\theta = \frac{2\lambda(\sqrt{a^2 + \alpha^2} - b)}{\alpha^2 - \left\{a^2 + \alpha^2 + b^2 - 2b\sqrt{a^2 + \alpha^2}\right\} + \lambda^2}$$

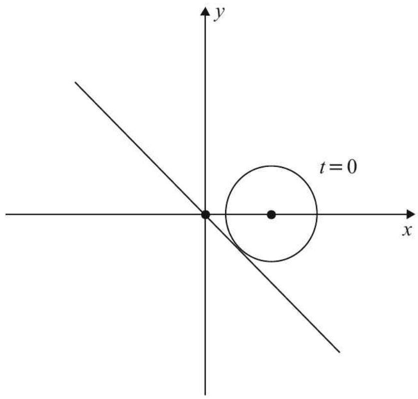
$$= \frac{2\lambda(\sqrt{a^2 + \alpha^2} - b)}{2b\left(\sqrt{a^2 + \alpha^2} - \left(\dfrac{a^2 + b^2 - \lambda^2}{2b}\right)\right)}$$

should be constant. For any value of α, this is constant if and only if

$$b = \frac{a^2 + b^2 - \lambda^2}{2b} \quad \Rightarrow \quad \lambda = \pm\sqrt{a^2 - b^2}$$

Thus, the point P is $(0, \pm\sqrt{a^2 - b^2})$.

Consider the following figure:



S31. The radius of the circle is $\frac{1}{\sqrt{2}}$, so it touches the line $y + x = 0$. At some time instant $t = t_1$, suppose that the line has rotated by an angle θ. This means that the center of the circle has undergone an angular displacement of 2θ, *i.e.*, its center is now $(\cos 2\theta, \sin 2\theta)$.

The equation of the line at $t = t_1$ is $y = x\tan(\theta - 45°)$. When the line and the circle meet again after $t = 0$, the line must be at a distance of $\frac{1}{\sqrt{2}}$ from the new center. Using this condition gives us the value of θ as 90°. Thus, the new equations of the line and the circle are

$$y = x, \quad x^2 + y^2 + 2x + \frac{1}{2} = 0$$

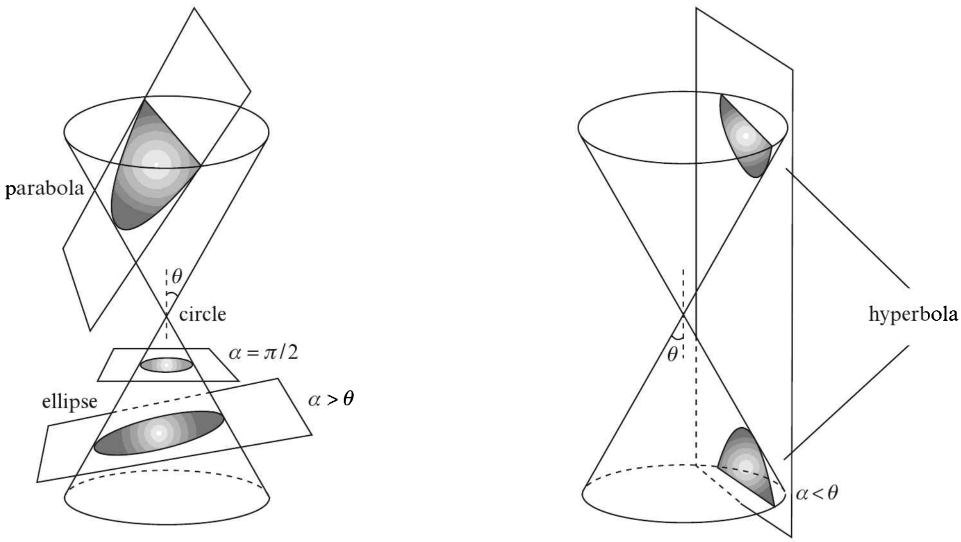
Perhaps this result may be evident to some students by observation.

Conic Section

PART-A: Summary of Important Concepts

1. What are Conics?

Conic sections, or conics in short, are geometrical figures obtained by the intersection of a plane with a three-dimensional double cone. In the following figure, the angle which the normal to the (variable) plane makes with the axis of the double cone is represented by α, while the semi-vertical angle of the cone is θ:



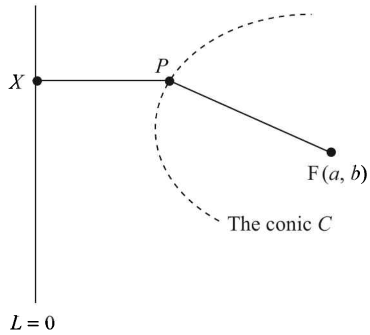
Depending on the orientation of the intersecting plane, different types of conic sections will be generated from the double cone.

Depending on the orientation of the plane, the following four types of conics are obtained:

(a) **Circles:** Circles are a special kind of conics with the intersecting plane at inclination $\pi/2$.
(b) **Parabola:** The intersecting plane for a parabola is parallel to the slant of the cone, *i.e.*, at an angle θ.
(c) **Ellipse:** The intersecting plane is at an angle $\alpha > \theta$ ($\alpha \neq \pi/2$ since then a circle will be formed).
(d) **Hyperbola:** The intersecting plane is at an angle $\alpha < \theta$; in this case, the plane cuts both the top and bottom halves of the cone.

A very important property that a conic section C satisfies this. It is the locus of a moving point P such that P's distance from a fixed point is always in a constant ratio to its (perpendicular) distance from a

fixed line. The fixed point is called the *focus* of C while the fixed line is called the *directrix* of C. The constant ratio is called the *eccentricity* of C and is denoted by e.



A conic section C is the locus of a moving point P such that

$$\frac{PF}{PX} = e \text{ (a constant)}$$

where PX is the perpendicular distance of P from the directrix $L = 0$ and F is the (fixed) focus.

For a circle, the eccentricity e is 0 because while the fixed point (the focus) is the centre of the circle, the fixed line is assumed to be at infinity. Thus, PX in the figure above always remains infinitely large so that $e = 0$.

For the remaining conic sections, we have

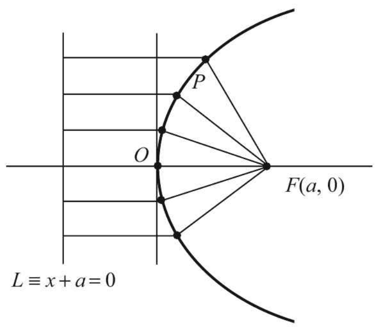
Parabola :	$e = 1$
Ellipse :	$e < 1$
Hyperbola:	$e > 1$

Since circles are a special type of conics ($e = 0$), we generally don't include circles in any discussion on conics; that is, from now on, whenever we talk of conics, we will be talking of parabolas, ellipses and hyperbolas.

2. Parabolas

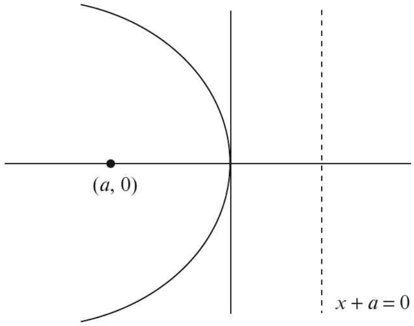
2.1 Basic equations

A parabola is a locus of a moving point P such that its distance from a fixed point (the focus F) is always equal to its distance from a fixed line $L = 0$ (the directrix). In other words, for a parabola, the eccentricity $e = 1$. The most basic equation representing a parabola is $y^2 = 4ax$,and the focus of this parabola is $F(a, 0)$,while the directrix is $L \equiv x + a = 0$.



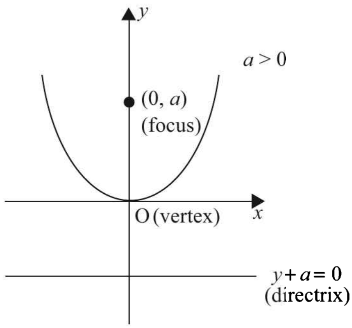
No matter where P may lie on $y^2 = 4ax$, its distance from $F(a, 0)$ is always equal to its distance from $L \equiv x + a = 0$

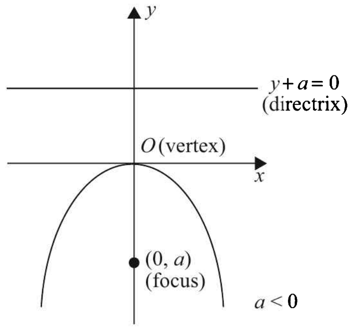
The line passing through F and perpendicular to $L = 0$ (in this case, $y = 0$) is termed the *axis* of the parabola, while its intersection point with the parabola (in this case, the origin) is termed the *vertex* of the parabola. For $a < 0$, the orientation of the parabola changes:



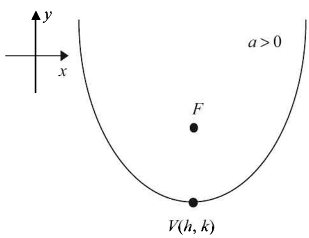
The curve $y^2 = 4ax$ when $a < 0$.

If we interchange x and y in the standard equation and instead write $x^2 = 4ay$, we get parabolas whose axes are the line $x = 0$:

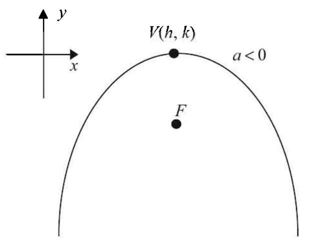




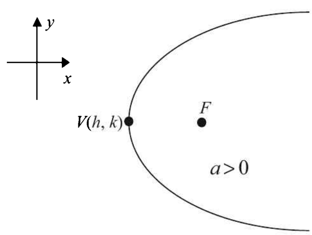
In fact, using these standard equations, we can write the equation for any parabola with its vertex at $V(h, k)$ and the axis parallel to the x-axis or the y-axis:



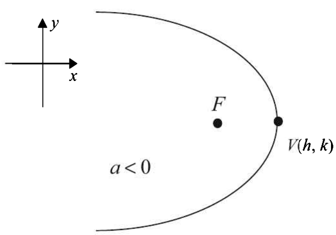
Equation: $(x-h)^2 = 4a(y-k)$



Equation: $(x-h)^2 = 4a(y-k)$



Equation: $(y-k)^2 = 4a(x-h)$



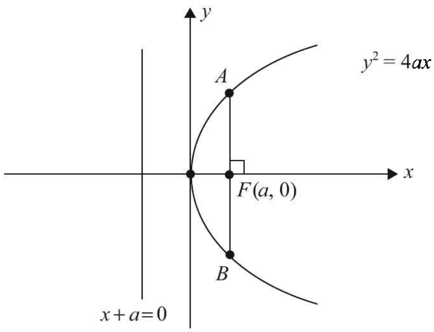
Equation: $(y-k)^2 = 4a(x-h)$

Note that in these parabolas, the focus lies at a distance of $|a|$ units from the vertex, along the axis of the parabola.

2.2 Important Parameters in a Parabola

(a) Latus Rectum (LR)

This is a unique chord for a given parabola. It is the chord passing through the focus and perpendicular to the axis of the parabola.



AB is the latus rectum for this parabola.

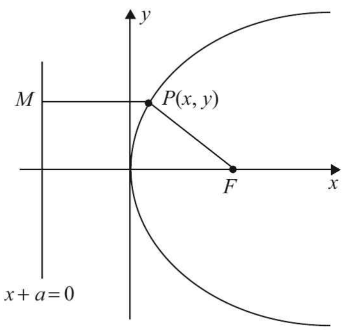
The length of the LR of $y^2 = 4ax$ is $4a$.

(b) Focal chord

Any chord passing through the focus of the parabola will be its focal chord. The LR is a particular focal chord.

(c) Focal Distance

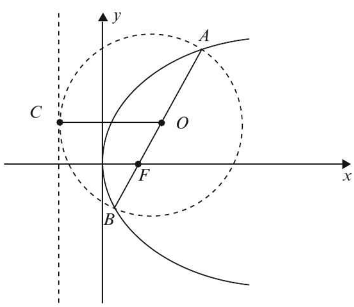
This, as the name suggests, is the distance of any point on the parabola from its focus, and which will by definition, be equal to its distance from the directrix.



The focal distance of $P(x, y)$ is PF which equals PM. (the $\perp$ distance of P from the directrix) The numerical value of PM can be seen to be $|x+a|$

The focal distance of any point $P(x, y)$ on $y^2 = 4ax$ is equal to $|x + a|$.

A very important property of a parabola is that a circle described on any focal chord as a diameter will touch the directrix:



2.3 General equation of a Parabola

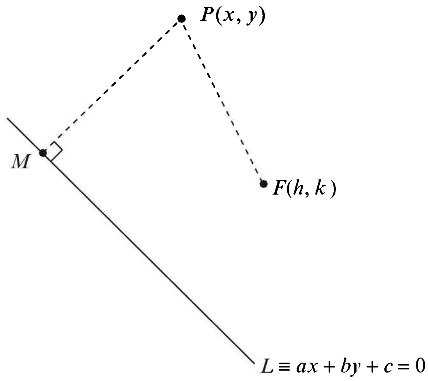
The most general equation of a parabola has the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where the following conditions must be satisfied:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0,\ h^2 = ab$$

This can be proved by considering a point P which moves so that it is equidistant from a fixed point F and a fixed line $L = 0$:



Any point $P(x, y)$ lying on the parabola must be equidistant from F and L.

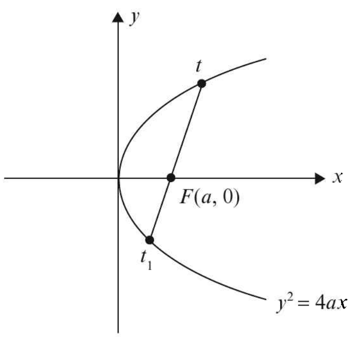
2.4 Parametric Form

The parametric form of a parabola $y^2 = 4ax$ is:

$$x = at^2, \quad y = 2at$$

The point $(at^2, 2at)$ is sometimes simply referred to as the point t. As t is varied, different points on the parabola are obtained. Using the parametric form, some very useful results can be obtained. For example:

(a) Let the focal chord through t intersect the parabola again in t_1:



We have $tt_1 = -1$

(b) The length of the focal chord through t is $l = a(t + \frac{1}{t})^2$. This is minimum when $t = 1$, which corresponds to the LR. That is, the LR is the shortest focal chord in any parabola.

2.5 Tangents

The equations of tangents are generally discussed only for the standard form of a parabola, $y^2 = 4ax$.

(a) The tangent at a point $P(x_1, y_1)$ lying on the parabola is

$$yy_1 = 2a(x + x_1)$$

This is sometimes written concisely as $T(x_1, y_1) = 0$.

(b) The tangent at the point $P(t)$, that is, $(at^2, 2at)$, is

$$ty = x + at^2$$

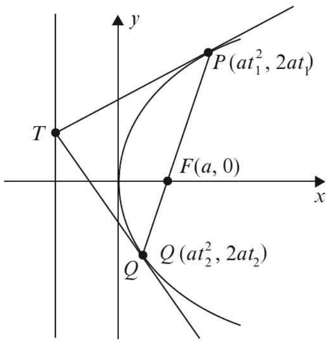
(c) The tangent of slope m is

$$y = mx + \frac{a}{m}$$

Any line with this equation will always be a tangent to $y^2 = 4ax$, no matter what the value of m is.
We now mention some of the important properties related to tangents to a parabola:

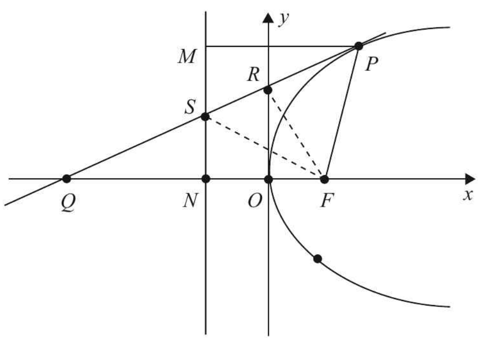
(a) For a parabola $y^2 = 4ax$, the point of intersection of the tangents at t_1 and t_2 is $(at_1t_2, a(t_1 + t_2))$.

(b) The tangents at the extremities of any focal chord intersect at right angles on the directrix:



The tangents at P and Q intersect at right angles on the directrix, *i.e.*, $\angle PTQ = 90°$

(c) The tangent at any point P bisects the angle between the line joining P to the focus, and the perpendicular dropped from P onto the directrix. This means that in the figure below, $\angle FPS = \angle MPS$:



(d) The portion of the tangent to a parabola cut-off between the directrix and the curve subtends a right angle at the focus (for example, $\angle PFS$ in the figure above is a right angle).

(e) The perpendicular dropped from the focus onto any tangent to a parabola is concurrent with that tangent and the tangent at the vertex. This means, for example, that $\angle FRP$ in the figure above is a right angle.

2.6 Normals

As in the case of tangents, the equations of normals are generally discussed only for the standard form of a parabola, $y^2 = 4ax$.

(a) The normal at a point $P(x_1, y_1)$ lying on the parabola is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) The normal at a point $P(t)$, *i.e.*, $(at^2, 2at)$, is

$$y + tx = 2at + at^3$$

(c) The normal of slope m is

$$y = mx - 2am - am^3$$

and the point of contact of this normal is $(am^2, -2am)$.

These are the three forms of a normal's equation. We now mention some of the important properties related to normals to a parabola:

(a) The normal at a point t to $y^2 = 4ax$ intersects the parabola again at the point t_1 given by

$$t_1 = -t - \frac{2}{t}$$

(b) For any focal chord, the tangent at one end-point and the normal at the other must be parallel.

(c) If the normals at two points t_1 and t_2 to $y^2 = 4ax$ intersect at a point on the parabola itself, then $t_1 t_2 = 2$.

(d) The normal at any point to a parabola is equally inclined to the focal chord passing through that point and the axis of the parabola. Refer to the last figure and draw a normal at P to understand this result better.

2.7 Miscellaneous Results

(a) Let $y^2 = 4ax$ be a parabola and $P(x_1, y_1)$ be a point in the plane. The position of P with respect to the parabola is governed by the following conditions:

$y_1^2 - 4ax_1 > 0$:	P lies 'outside' the parabola
$y_1^2 - 4ax_1 = 0$:	P lies on the parabola
$y_1^2 - 4ax_1 < 0$:	P lies 'inside' the parabola

(b) From any external point, two distinct tangents can be drawn to a parabola, while no tangent can be drawn to it from any internal point. Of course, for a point lying on the parabola, exactly one tangent can be drawn.

(c) From an external point $P(h, k)$, the joint equation of the pair of tangents drawnto $y^2 = 4ax$ is

$$T^2(h, k) = S(x, y)S(h, k)$$

or $\quad T^2 = SS_1 \qquad$ in brief

Here,

$T(h,k) \equiv ky - 2a(x+h)$ (Expression which occurs in the equation of a tangent)
$S(x,y) \equiv y^2 - 4ax$ (Expression which occurs in the standard equation of a parabola)

(d) The chord of contact from $P(h, k)$ to the parabola $y^2 = 4ax$ has the equation

$$T(h, k) = 0$$

(e) The chord of $y^2 = 4ax$ bisected at an interior point $P(h, k)$ has the equation

$$T(h, k) = S(h, k)$$

3. Ellipses

3.1 Basic Equations and Parameters

An ellipse can be defined in the following two ways:

Definition 1:

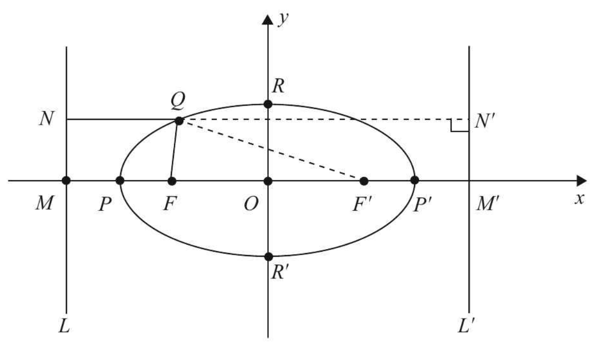
An ellipse is the locus of a moving point such that the ratio of its distance from a fixed point to its distance from a fixed line is a constant less than unity. This constant is termed the *eccentricity* of the ellipse. The fixed point is the *focus* while the fixed line is the *directrix*. The symmetrical nature of the ellipse ensures that there will be two foci and two directrices.

Definition 2:

An ellipse is the locus of a moving point such that the sum of its distances from two fixed points is constant. The two fixed points are the two foci of the ellipse.
The standard equation of an ellipse is

$$S(x, y) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and it is positioned in the plane as follows:



The important aspects and parameters of this ellipse are summarized as:

Standard Equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

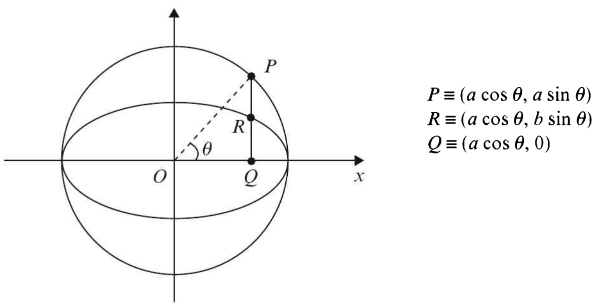
	If $a > b$	**If $a < b$**
Vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Major axis	$2a$ (along x-axis)	$2b$ (along y-axis)
Minor axis	$2b$ (along y-axis)	$2b$ (along x-axis)
Directrices	$x=\dfrac{a}{e}$ and $x=-\dfrac{a}{e}$	$x=\dfrac{b}{e}$ and $y=-\dfrac{b}{e}$
Eccentricity e	$\sqrt{1-\dfrac{b^2}{a^2}}$	$\sqrt{1-\dfrac{a^2}{b^2}}$
Latus-rectum	$\dfrac{2b^2}{a}$	$\dfrac{2a^2}{b}$
Focal distances of (x, y)	$a \pm ex$	$b \pm ey$

As in the cases of circles and parabolas, we can determine the position of a point $P(x_1, y_1)$ with respect to the ellipse as follows:

$$S(x_1, y_1) < 0 \Rightarrow P \text{ lies inside the ellipse}$$
$$S(x_1, y_1) = 0 \Rightarrow P \text{ lies on the ellipse}$$
$$S(x_1, y_1) > 0 \Rightarrow P \text{ lies outside the ellipse}$$

3.2 Parametric Form of an Ellipse

To define an ellipse in parametric form, we first define its *auxiliary circle*. This is a circle described on the major axis of the ellipse as diameter:



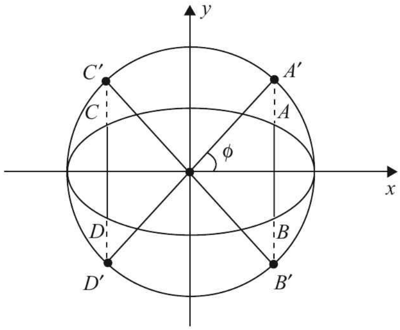
We see that any point R on the ellipse can be specified in the form $R \equiv (a\cos\theta, b\sin\theta)$, where θ is called the *eccentric angle* of the point R ; θ is the angle which the line joining the origin to the corresponding point on the auxiliary circle (P), makes with the x-axis, as shown. Thus, the parametric form of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$x = a\cos\theta, \quad y = b\sin\theta$$

Once again, we emphasize that θ is *not* the angle that $(a\cos\theta, b\sin\theta)$ makes with the horizontal; it is the angle which the corresponding point on the auxiliary circle $(a\cos\theta, a\sin\theta)$, makes with the horizontal. As θ is varied, $(a\cos\theta, b\sin\theta)$ gives us different points on the circumference of the ellipse. The point $(a\cos\theta, b\sin\theta)$ is sometimes simply referred to as the point θ.

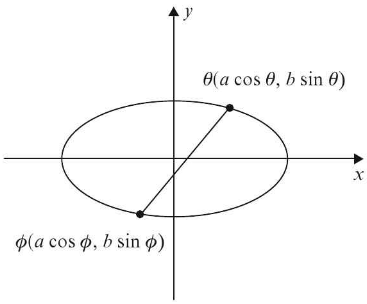
Listed below are some important results based on the parametric form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) The eccentric angle of the four extremities of the two latus-recta of the ellipse are given by $\tan\phi = \pm\frac{b}{ae}$:



The two latus-recta *AB* and *CD* meet the auxiliary circle in *A′*, *B′*, *C′*, and *D′*, The slopes of the lines joining the origin to these four points give us the eccentric angles of the four extremities. Here, only one possible value of ϕ has been shown: the eccentric angle of point *A*.

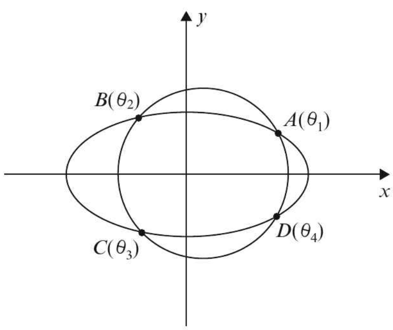
(b) Consider a chord of the ellipse joining the two points θ and ϕ:



The equation of this chord will be

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

(c) A circle intersects the ellipse in four points as shown below:



The sum $\theta_1+\theta_2+\theta_3+\theta_4$ will be an integral multiple of 2π.

3.3 Tangents

The equations of tangents are generally discussed only for the standard form of an ellipse, given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) The tangent to the ellipse at a point $P(x_1, y_1)$ lying on the ellipse is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

As in the case of circles and parabolas, this equation can be written concisely as $T(x_1, y_1)=0$ where

$$T(x_1, y_1)=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

(b) The tangent to the ellipse at the point θ, that is, the point $(a\cos\theta, b\sin\theta)$, has the equation

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

(c) The (two) tangents to the ellipse of slope m are given by

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

Any line with an equation of this form will be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, regardless of the value of m.

(d) The tangents drawn to the ellipse at two points $A(\theta_1)$ and $B(\theta_2)$ on the ellipse intersect in P. The coordinates of P are given by

$$P\equiv\left(\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}\right)$$

3.4 Normals

As in the case of tangents, the equation of normals are discussed only for the case of the ellipse being a standard one, that is, with the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) The normal to the ellipse at a point $P(x_1, y_1)$ lying on the ellipse is given by

$$\frac{a^2}{x_1}x-\frac{b^2}{y_1}y=a^2-b^2$$

(b) The normal at a point $P(\theta)$ on the ellipse is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$$

(c) The (two) normals of slope m are given by

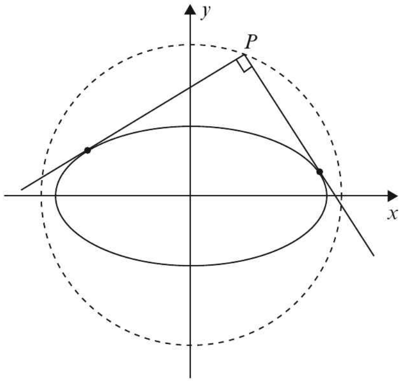
$$y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

We observe that any line with an equation of this form will be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ regardless of the value of m.

3.5 Miscellaneous Results

We now mention some of the more important miscellaneous results on ellipses:

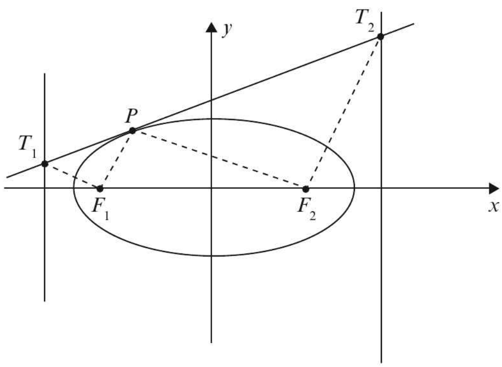
(a) *Director Circle*: For any given ellipse, the director circle is that circle such that tangents drawn from any point on it to the ellipse are perpendicular:



From any point on the director circle of an ellipse, the two tangents drawn to the ellipse are perpendicular.

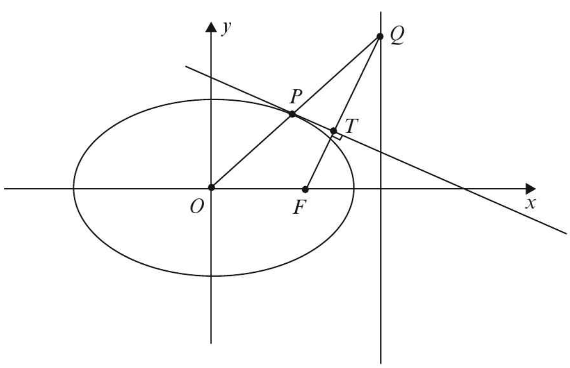
For an ellipse with the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the director circle has the equation $x^2+y^2=a^2+b^2$.

(b) The portion of the tangent to any ellipse intercepted between the curve and a directrix subtends a right angle at the *corresponding* focus:



$\angle PF_1T_1 = \angle PF_2T_2 = \frac{\pi}{2}$

(c) For any ellipse, the perpendicular from a focus upon any tangent and the line joining the center of the ellipse to the point of contact meet on the *corresponding* directrix:



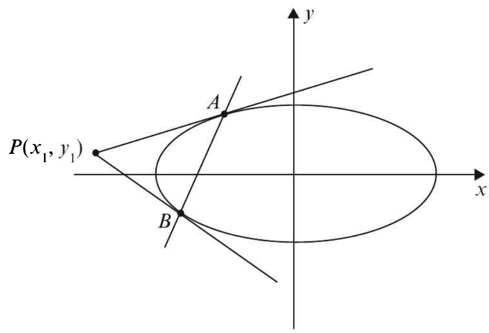
(d) As in the case of parabolas, the joint equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y2}{b^2}=1$ is given by

$$T^2(x_1, y_1) = S(x, y)\, S(x_1, y_1),$$

where

$$T(x_1, y_1) = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1, \quad S(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

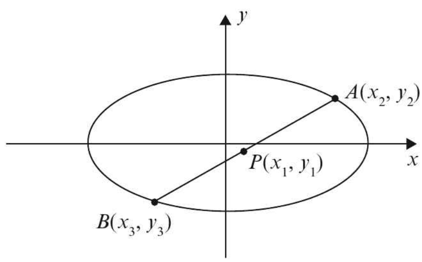
(e) Consider the chord of contact for a point $P(x_1, y_1)$ external to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:



AB is the chord of contact for the tangents drawn from *P* to the ellipse.

The equation of the chord of contact is $T(x_1, y_1) = 0$.

(f) Let *AB* be a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bisected at the point $P(x_1, y_1)$:



The equation of *AB* is given by

$$T(x_1, y_1) = S(x_1, y_1)$$

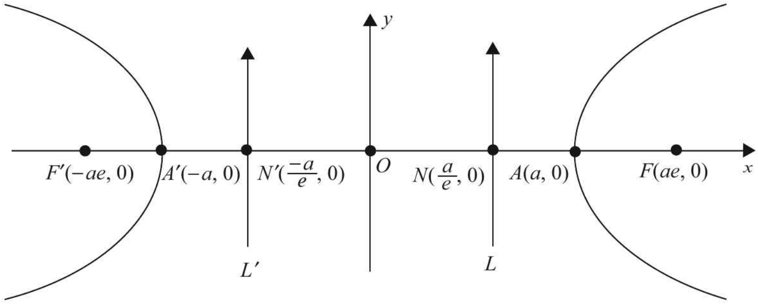
4. Hyperbolas

4.1 Basic Equations and Parameters

A hyperbola is the locus of a moving point such that the ratio of its distance from a fixed point to its distance from a fixed line is constant, that is, it is a conic section with eccentricity $e > 1$. The standard form (which we generally consider) of a hyperbola is

$$S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } e^2 = 1 + \frac{b^2}{a^2}$$

As in the case of an ellipse, a hyperbola will have *two foci* and *two directries*:



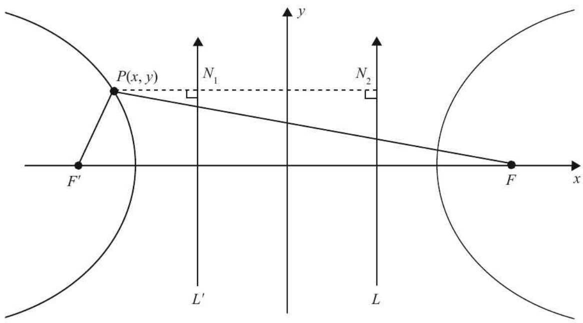
We see that a hyperbola has two disjoint curves. For any point P on either segment of the hyperbola, we will have

$$\frac{PF}{\text{Distance of } P \text{ from } L} = \frac{PF'}{\text{Distance of } P \text{ from } L'} = e$$

Here is a brief summary on the terminology we use for a hyperbola (refer to the figure above):

Centre: The point O
Vertices: The points $A(a, 0)$ and $A'(-a, 0)$
Transverse axis: The segment joining the vertices; AA' above. Its length is $2a$.
Conjugate axis: The line through the centre perpendicular to the transverse axis (the y-axis above). Although the hyperbola never intersects the conjugate axis, we still say that the 'length' of the conjugate axis is $2b$. You should not get worried about this; it is said just to maintain uniformity.

A very significant property of any hyperbola pertains to focal distances of an arbitrary point on it. Consider a point $P(x, y)$ on a hyperbola, with its focal distances as PF and PF':



We will have

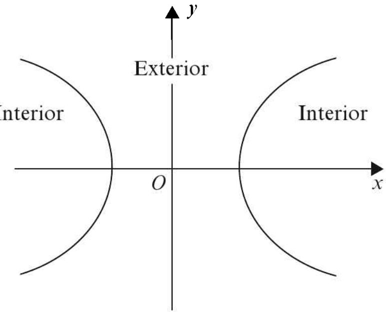
$$|PF - PF'| = 2a = \text{Length of the transverse axis}$$

This leads us to another definition of a hyperbola: A hyperbola is the locus of a moving point such that the *difference* of its distances from two fixed points is always constant. The two fixed points are the foci of the hyperbola.

As in the case of ellipses, we have:

(a) Focal distances of $P(x, y)$: $ex - a, ex + a$
(b) Length of Latus Rectum (chord(s) of the hyperbola: $\frac{2b^2}{a}$
passing through a focus and perpendicular to the transverse axis.)

The interior and exterior regions of a hyperbola are defined as follows:

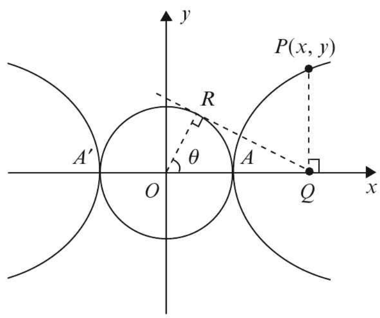


If the equation of the hyperbola is $S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, we can determine the position of a poiont $P(x_1, y_1)$ with respect to the hyperbola as in the earlier cases:

$$S(x_1, y_1) < 0 \quad \Rightarrow \quad P \text{ lies 'inside' the hyperbola}$$
$$S(x_1, y_1) = 0 \quad \Rightarrow \quad P \text{ lies on the hyperbola}$$
$$S(x_1, y_1) > 0 \quad \Rightarrow \quad P \text{ lies 'outside' the hyperbola}$$

4.2 Parametric Form of a Hyperbola

To define a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in parametric form, we first define its *auxiliary circle*, which is the circle described on the transverse axis of the hyperbola as diameter:



We see that for any point $P(x, y)$, we can relate its position to the angle θ constructed as shown above. We note that

$$x = a\sec\theta \;\Rightarrow\; y = b\tan\theta$$

Thus, any point P on this hyperbola can be specified in the form $P \equiv (a\sec\theta, b\tan\theta)$,where θ is called the *eccentric angle* of the point P. It is strongly emphasized at this juncture that the relation of the eccentric angle to its corresponding point must be very well understood. Specifically speaking, to obtain the eccentric angle θ for a point P on the hyperbola:

(1) Drop the perpendicular PQ onto the transverse axis.
(2) Draw the tangent QR to the auxiliary circle.
(3) Join OR, and measure the angle it makes with the x-axis.

We conclude that the parametric form of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x = a\sec\theta, \; y = b\tan\theta$$

We take $\theta \in [0, 2\pi)$ by convention. As θ is varied in this interval, different points on the hyperbola will be obtained. The point $(a\sec\theta, b\tan\theta)$ is sometimes simply referred to as the point θ.

Listed below are some important results based on the parametric form of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) The eccentric angles of the four extremities of the two latus-recta of the hyperbola are given by $\sin\phi = \pm\frac{b}{ae}$.

(b) The equation of the chord joining the points $P(\theta_1)$ and $Q(\theta_2)$ on the hyperbola is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

4.3 Tangents

The equations of tangents are generally discussed only for the standard form of a hyperbola, given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) The tangent to the hyperbola at the point $P(x_1, y_1)$ lying on the hyperbola is given by

$$T(x_1, y_1) \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$$

(b) The tangent to the hyperbola at the point θ, that is, the point $(a\sec\theta, b\tan\theta)$, has the equation $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

(c) The (two) tangents to the hyperbola of slope m are given by

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

Any line with an equation of this form will be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, regardless of the value of m.

(d) Suppose that the tangents drawn to the hyperbola at two points $A(\theta_1)$ and $B(\theta_2)$ lying on the hyperbola intersect in P. The coordinates of P are given by

$$P \equiv \left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

4.4 Normals

As in the case of tangents, the equations of normals are discussed only for the case of the hyperbola being a standard one, that is, with the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) The normal to the hyperbola at a point $P(x_1, y_1)$ lying on the hyperbola is given by

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

(b) The normal at a point $P(\theta)$ on the hyperbola is

$$ax\sin\theta + by = (a^2+b^2)\tan\theta$$

(c) The (two) normals of slope m are given by

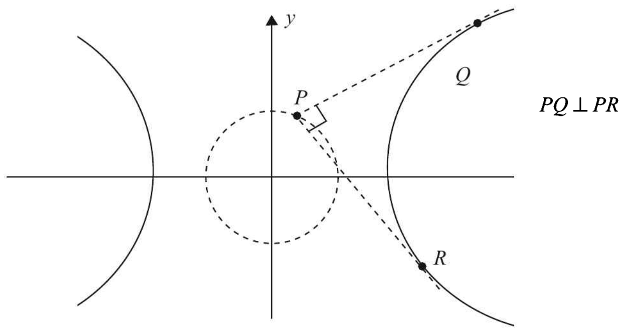
$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

Any line with an equation of this form will always be a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, for any value of m.

4.5 Miscellaneous Results

Listed below are some important results on hyperbolas:

(a) *Director Circle*: For any given hyperbola, the director circle is that circle such that tangents drawn from any point on it to the hyperbola are perpendicular:



For a hyperbola with the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the director circle has the equation $x^2+y^2=a^2-b^2$

Consider a standard hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The following three results are analogous to the earlier results:

(b) Joint equation of pair of tangents from $P(x_1, y_1)$:

$$T^2(x_1, y_1) = S(x, y)S(x_1, y_1),$$

where

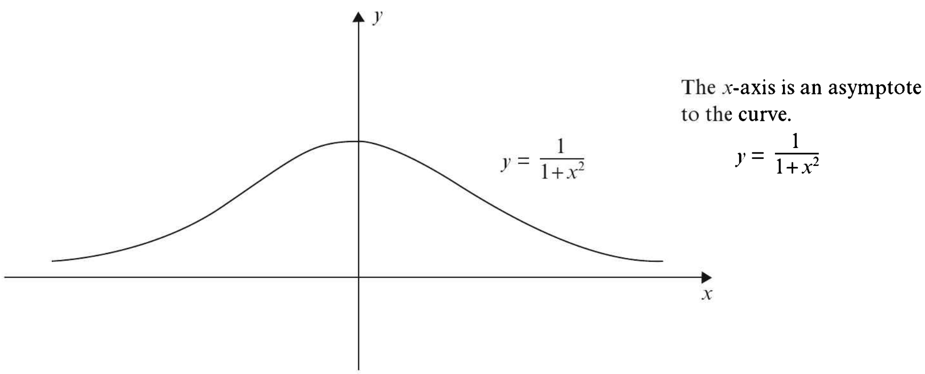
$$T(x_1, y_1) = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1, \quad S(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$$

(c) Chord of contact from $P(x_1, y_1) : T(x_1, y_1) = 0$

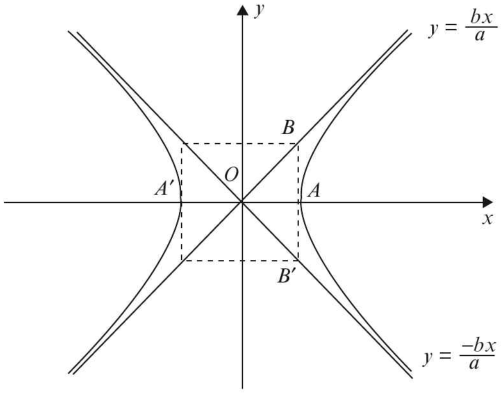
(d) Chord bisected at $P(x_1, y_1) : T(x_1, y_1) = S(x_1, y_1)$

4.6 Rectangular Hyperbolas

To understand the concept of rectangular hyperbolas, it is important to understand the concept of *asymptotes*. An asymptote to a curve is a straight line, to which the tangent to the curve tends as the point of contact goes to infinity. If this sounds confusing, you can think of an asymptote as follows: an asymptote to a curve is a straight line such that the perpendicular distance of a point $P(x, y)$ on the curve from this line tends to zero as the point P goes to infinity (along some branch of the curve). For example, the line $y = 0$ is an asymptote to the curve $y = \frac{1}{1+x^2}$ as shown below:

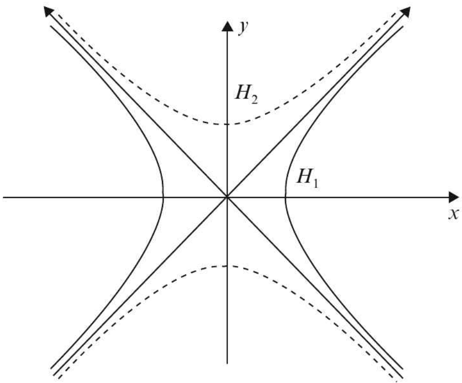


For a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, two asymptotes exist, given by $\frac{x}{a}\pm\frac{y}{b}=0$. These are two lines passing through the origin and inclined to each other at angle θ given by $\tan\theta=\frac{2ab}{a^2-b^2}$:



The dotted rectangle gives a better perspective on the placement of these asymptotes. Note that
$AA' = 2a$
$BB' = 2b$
Thus, given a and b, the dotted rectangle gives us a way to sketch the asymptotes by drawing its diagonals.

From the figure above, you might be able to infer that we can draw another hyperbola with the same pair of asymptotes, but with its transverse axis being the conjugate axis of the original hyperbola and vice-versa.



Hyperbola H_1 (represented) by the solid curve) and hyperbola H_2 (represented by the dotted curve) have the same pair of asymptotes.

H_2 is called the *conjugate hyperbola* of H_1. The transverse axis of H_2 is the conjugate axis of H_1 and vice-versa. It should therefore be evident that if the equation of H_1 is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the equation of its conjugate H_2 will be

$$\boxed{H_2 : \frac{y^2}{b^2}-\frac{x^2}{a^2}=1}$$ **Conjugate hyperbola**

The pair of asymptotes for both H_1 and H_2 is $y=\pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$ which can be specified jointly

as $$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0}$$ **Pair of asymptotes**

One very important point you must notice is that H_1 and H_2 do not have the same eccentricity. In fact,

$$e_{H_1} = \sqrt{1+\frac{b^2}{a^2}}$$

$$e_{H_2} = \sqrt{1+\frac{a^2}{b^2}}$$

We can deduce another very important and useful result from this discussion: the equation of a hyperbola and the equation of its pair of asymptotes differ by just a constant. The equation of the conjugate hyperbola differs from that of the asymptotes by the same constant. This will always hold, irrespective of what coordinate system we use to write the equations.

Let us now examine the concept of *rectangular hyperbolas*. A hyperbola is said to be rectangular if its transverse and conjugate axis are equal, *i.e.*, if

$$2a = 2b$$

$$\Rightarrow \quad a = b$$

Thus, the equation of a rectangular hyperbola is of the form

$$x^2 - y^2 = a^2$$

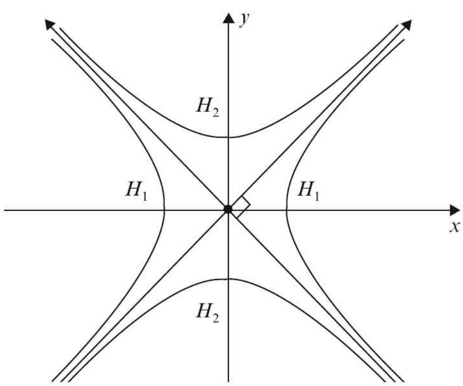
The eccentricity of any rectangular hyperbola will be

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$

The asymptotes of a rectangular hyperbola will be

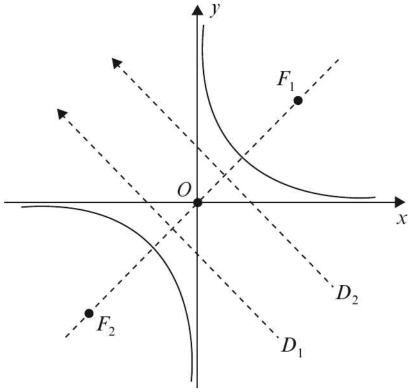
$$y = \pm x \text{ or } x^2 - y^2 = 0$$

This means that the *asymptotes of a rectangular hyperbola are perpendicular*. It should be obvious that the conjugate of a rectangular hyperbola $x^2 - y^2 = a^2$ will also be rectangular with the equation $x^2 - y^2 = -a^2$.



H_1 and H_2 are rectangular hyperbolas.

By rotating the coordinate axes (by 45°), we can write the equation of a rectangular hyperbola in the alternate form $xy = c^2$. The position of this hyperbola, along with its foci and directrices, will be as follows:



The foci obviously lie along the line $y=x$ while the two directrices are perpendicular to $y=x$.

To summarize:

$$xy = c^2 \Rightarrow \begin{Bmatrix} \text{Foci at} & (\pm\sqrt{2}c, \pm\sqrt{2}c) \\ \text{Eccentricity} & \sqrt{2} \\ \text{Transverse axis} & 2\sqrt{2}c \\ \text{Directrices} & x + y = \pm\sqrt{2}c \\ \text{Asymptotes} & x = 0,\ y = 0 \end{Bmatrix}$$

The rectangular hyperbola $xy = c^2$ can be specified in parametric form as

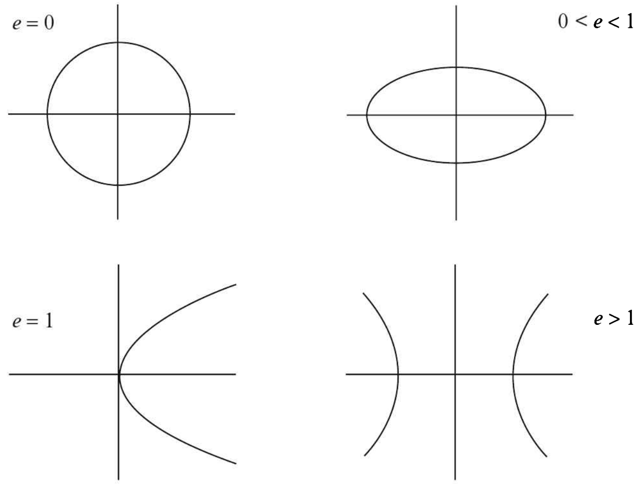
$$x = ct,\ y = \frac{c}{t}$$

For a rectangular hyperbola $xy = c^2$, the equations of tangents and normals will have the following forms:

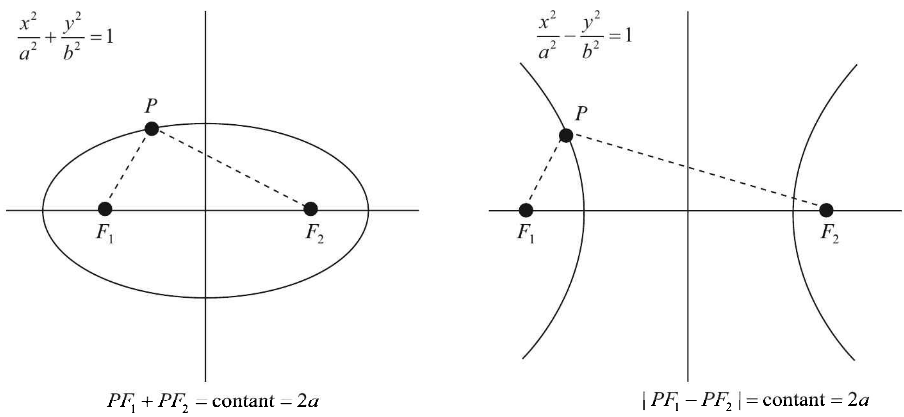
(a) Tangent at $P(x_1, y_1)$: $\frac{x}{x_1} + \frac{y}{y_1} = 2$
(b) Tangent at $P(ct, \frac{c}{t})$: $x + yt^2 = 2ct$
(c) Normal at $P(x_1, y_1)$: $xx_1 - yy_1 = x_1^2 - y_1^2$
(d) Normal at $P(ct, \frac{c}{t})$: $xt^3 - yt = ct^4 - c$

IMPORTANT IDEAS AND TIPS

1. *Basic Definitions*: It is important to always remember the basic definitions of the three conics we study: Parabola, Ellipse and Hyperbola. The common characteristic of these three is that each is the locus of a moving point which moves in such a way so that the ratio of its distance from a fixed point to a fixed line is constant, and that constant is the eccentricity e. Also, we have $e = 1$, $e < 1$ and $e > 1$ respectively for the three types of conics. This unifying characteristic of the three conics should not be forgotten. Think of conics as follows to help visualize the connection between them. Start from $e = 0$, which gives you a circle. As you increase e, you obtain ellipses of increasing eccentricity. When e is exactly 1, the curve you get is no longer a *closed* ellipse, its a *just open* parabola. As e is increased beyond 1, you get *more open* hyperbolas.



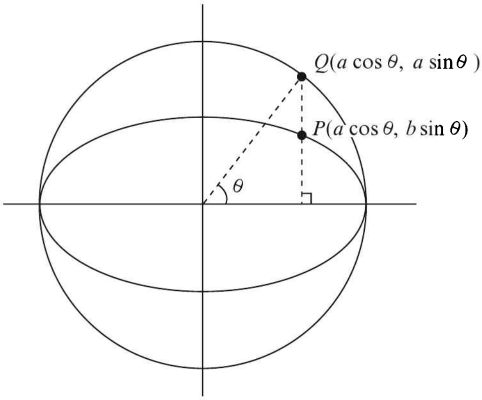
2. *An Important Property*: One of the most important properties satisfied by both ellipses and hyperbolas is that for an ellipse, the sum of the distances of a point lying on it from the two foci is constant; for a hyperbola, the difference of the distances of a point lying on it from the two foci is constant:



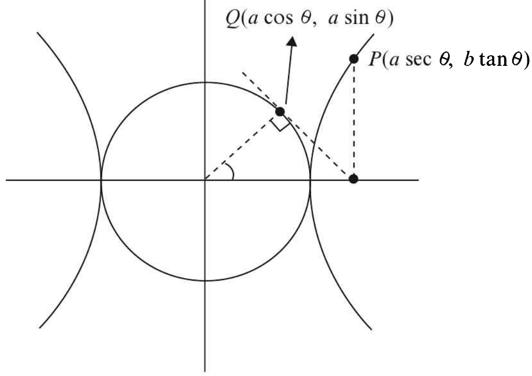
This fact is frequently used in problem solving.

3. *Conic Formulae*: Many of the formulae which we have seen for conics apply only to the standard conic equations $y^2 = 4ax, \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Take special care that you do not apply one of these standard equations to a non-standard situation.
4. *Parametric Form*: In problem-solving, preference should be given to assuming unknown points in parametric (rather than coordinate) form. This helps because your equations will then contain only a single variable, the unknown parameter.
5. *Auxiliary Circles*: One of the most widely prevalent mistakes in conics pertain to the auxiliary circles of ellipses and hyperbolas and the way those circles lead to the parametric forms of points on these curves. The right way to think is as follows: every point P on an ellipse/hyperbola can be mapped to a corresponding point Q on the auxiliary circle. The angle θ which you encounter in the parametric form is the polar angle of Q and *not* of P. Note how for any point P on an ellipse/hyperbola, the corresponding point Q is obtained (note the difference for the two types of conics):

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



6. *Geometric Facts*: Most students, when working with conics, tend to see everything from a coordinates point of view. However, there are many (pure) geometrical results which apply to conics irrespective of the coordinate system, and it will be very useful if you can remember the more important of these results. Many of them are mentioned in the theory. For example, a circle drawn using any focal chord of a parabola as diameter will touch its directrix. The importance of remembering such results stems from the fact that they can become your guiding points on which to build your solutions to advanced problems.
7. *A multitude of Results*: You have encountered a lot of results in this chapter, and almost all of them are important. The best way to internalize all these results is to make a table on a large chart in which you summarize the results for each type of conic side by side. Here are a couple of sample entries from such a table:

			Hyperbola	
	Parabola	**Ellipse**	**Standard form**	**Rectangular form**
			$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric form	$(at^2, 2at)$	$(a\cos\theta, b\sin\theta)$	$(a\sec\theta, b\tan\theta)$	$\left(ct, \frac{c}{t}\right)$
Equation of tangent at a general (parametric point)	$ty=x+at^2$	$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$	$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$	$x+yt^2=2ct$

Once you've made such a chart, pin it up where you can see it frequently; eventually all these relations and formulae will become easier to retain in your mind.

8. *Rectangular Hyperbolas*: It is important to remember in what sense a rectangular hyperbola is a special case of a general hyperbola: the asymptotes in a rectangular hyperbola are perpendicular. From this, all other properties of rectangular hyperbolas follow. And since the asymptotes are perpendicular, we generally refer such hyperbolas to the x - y coordinate axes (so that the axes themselves become the asymptotes). Thus, the equation of a rectangular hyperbola in standard form ($x^2-y^2=a^2$) becomes the equation $xy=c^2$ when the axes are taken as the asymptotes. You should in addition note that the reciprocal function which you study in the chapter on Functions, $y=f(x)=\frac{1}{x}$, corresponds to the rectangular hyperbola $xy=1$.

Conic Section

PART-B: Illustrative Examples

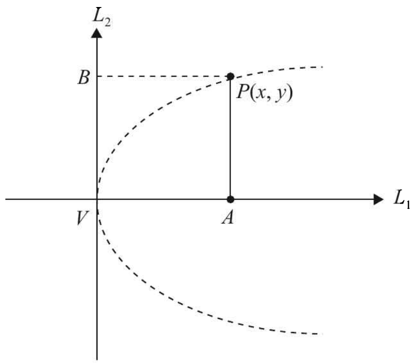
OBJECTIVE TYPE EXAMPLES

Example 1

The axis of a parabola is the line $L_1: 3x + 4y - 4 = 0$ and the tangent to it at the vertex is $L_2: 4x - 3y + 7 = 0$. The LR is 4 units in length. The equations of the parabolas are

(A) $(3x+4y-4)^2 = \pm 16(4x-3y+7)$ (C) $(4x-3y+7)^2 = \pm 16(3x+4y-4)$ (E) None of these
(B) $(3x+4y-4)^2 = \pm 20(4x-3y+7)$ (D) $(4x-3y+7)^2 = \pm 20(3x+4y-4)$

Solution: Consider for a moment the co-ordinate axes system formed by L_1 and L_2.



$L_1: 3x + 4y - 4 = 0$
$L_2: 4x - 3y + 7 = 0$

If L_1 and L_2 were truly the actual co-ordinate axes, the equation of the parabola would have been

$$y^2 = 4ax$$

$$\Rightarrow \quad PA^2 = (\text{Length of LR}) \times PB \qquad (1)$$

Now, even if we use some other co-ordinate system (here the $L_1 - L_2$ system), the relation (1) will still hold since that only depends on lengths, which are invariant with respect to the co-ordinate system chosen. Thus, simply applying (1) will give us the required equation :

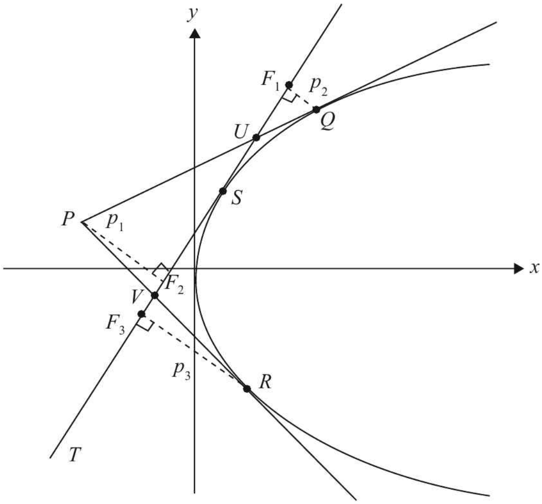
$$\left(\frac{|3x+4y-4|}{5}\right)^2 = (4) \times \frac{|4x-3y+7|}{5}$$

$$\Rightarrow \quad (3x+4y-4)^2 = \pm 20(4x-3y+7)$$

Thus, as might have been expected, we'll obtain two parabolas with the given property, one opening to the 'left' and one to the 'right' in the $L_1 - L_2$ co-ordinate system. The correct option is (B). ■

Example 2

(a) Tangents PQ and PR are drawn to $y^2 = 4ax$ from an external point P. Another tangent is drawn at a point S on the parabola. Perpendiculars from P, Q, and R are dropped onto this tangent, and their lengths are p_1, p_2 and p_3. Prove that $p_1^2 = p_2 p_3$.



The tangent drawn at S is the line T. Perpendiculars from P, Q and R onto T are of length p_1, p_2 and p_3 respectively. We need to prove that

$$p_1^2 = p_2 p_3$$

(b) Referring to the figure above, the value of $\frac{PU}{PQ} + \frac{PV}{PR}$ is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution: (a) We assume the points Q and R to be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, so that P is $(at_1t_2, a(t_1+t_2))$. Let S be the point t, so that the tangent at S is

$$ty = x + at^2$$

We now evaluate the lengths of the three perpendiculars, *i.e.*, p_1, p_2, and p_3.

$$p_1 = \frac{|at_1t_2 - at(t_1+t_2) + at^2|}{\sqrt{1+t^2}} = \frac{a|(t-t_1)(t-t_2)|}{\sqrt{1+t^2}}$$

$$p_2 = \frac{|at_1^2 - 2att_1 + at^2|}{\sqrt{1+t^2}} = \frac{a(t-t_1)^2}{\sqrt{1+t^2}}$$

$$p_3 = \frac{|at_2^2 - 2att_2 + at^2|}{\sqrt{1+t^2}} = \frac{a(t-t_2)^2}{\sqrt{1+t^2}}$$

It is evident that $p_1^2 = p_2 p_3$.

(b) Note that

$$\Delta PUF_2 \sim \Delta QUF_1$$

$$\Rightarrow \frac{PF_2}{PU} = \frac{QF_1}{QU}$$

$$\Rightarrow \frac{PF_2}{QF_1} = \frac{PU}{QU} = \frac{p_1}{p_2} \quad (\because PF_2 = p_1, QF_1 = p_2)$$

$$\Rightarrow \frac{PU}{PQ} = \frac{p_1}{p_1 + p_2}$$

Similarly, $\Delta PVF_2 \sim \Delta RVF_3$ so that

$$\frac{PV}{PR} = \frac{p_1}{p_1 + p_3}$$

Thus,

$$\frac{PU}{PQ} + \frac{PV}{PR} = \frac{p_1}{p_1 + p_2} + \frac{p_1}{p_1 + p_3}$$

$$= \frac{p_1(2p_1 + p_2 + p_3)}{p_1^2 + p_1p_2 + p_1p_3 + p_2p_3}$$

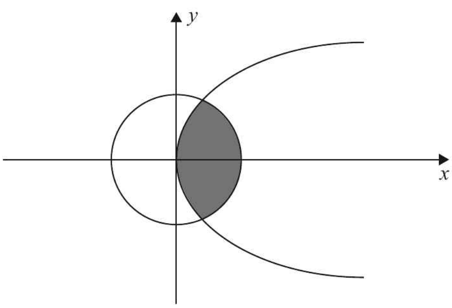
$$= \frac{2p_2p_3 + p_1p_2 + p_1p_3}{2p_2p_3 + p_1p_2 + p_1p_3} \quad \begin{Bmatrix}\text{Using } p_1^2 = p_2p_3 \text{ from} \\ \text{the part (a)}\end{Bmatrix}$$

$$= 1$$

The correct option is (B). ■

Example 3

What are the values that a can take so that the point $P(-2a, a+1)$ lies inside the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$ as shown?



For what values of a will $(-2a, a+1)$ lie inside the shaded region ?

(A) $(-1, -5+2\sqrt{6})$ (C) $(-1, -4+5\sqrt{6})$ (E) None of these

(B) $(-2, -3+\sqrt{6})$ (D) $(-2, -5+3\sqrt{6})$

Solution: Note that the shaded region represents the interior of *both* the circle and the parabola. Thus,

$$\underbrace{(-2a)^2 + (a+1)^2 - 4 < 0}_{\substack{\text{so that } P \text{ lies in the} \\ \text{interior of the circle}}} \text{ and } \underbrace{(a+1)^2 - 4(-2a) < 0}_{\substack{\text{so that } P \text{ lies in the} \\ \text{interior of the parabola}}}$$

$$\Rightarrow \quad 5a^2 + 2a - 3 < 0 \quad \text{and} \quad a^2 + 10a + 1 < 0$$

$$\Rightarrow \quad -1 < a < \frac{3}{5} \quad \text{and} \quad -\sqrt{5} - 2\sqrt{6} < a < -5 + 2\sqrt{6}$$

The intersection of the two constraints gives the possible values of a as

$$a \in (-1, -5+2\sqrt{6})$$

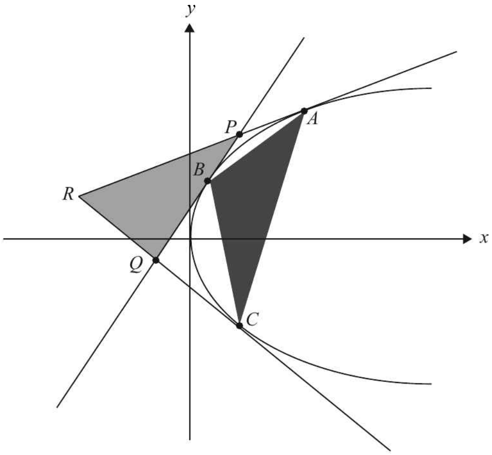
The correct option is (A). ■

Example 4

The ratio of the area of the triangle formed by three points on a parabola to the area of the triangle formed by the tangents at these points is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution:



We need to find the ratio of area (ΔABC) to area (ΔPQR).

Let the parabola be $y^2 = 4ax$. Assume the points A, B and C to be t_1, t_2, and t_3 respectively. Thus, P, Q, and R are respectively $(at_1t_2, a(t_1+t_2))$, $(at_2t_3, a(t_2+t_3))$ and $(at_3t_1, a(t_3+t_1))$. Now we simply use the determinant formula to find the area of the two triangles:

$$\text{area } (\Delta ABC) = \frac{1}{2}\begin{vmatrix} at_1^2 & at_2^2 & at_3^2 \\ 2at_1 & 2at_2 & 2at_3 \\ 1 & 1 & 1 \end{vmatrix} = a^2\begin{vmatrix} t_1^2 & t_2^2-t_1^2 & t_3^2-t_1^2 \\ t_1 & t_2-t_1 & t_3-t_1 \\ 1 & 0 & 0 \end{vmatrix} \begin{Bmatrix} \text{Using } C_2 \to C_2 - C_1; \\ C_3 \to C_3 - C_1 \end{Bmatrix}$$

$$= a^2(t_2-t_1)(t_3-t_1)\begin{vmatrix} t_1^2 & t_2+t_1 & t_3+t_1 \\ t_1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$$

Similarly,

$$\text{area } (\Delta PQR) = \frac{1}{2}\begin{vmatrix} at_1t_2 & at_2t_3 & at_3t_1 \\ a(t_1+t_2) & a(t_2+t_3) & a(t_3+t_1) \\ 1 & 1 & 1 \end{vmatrix} = \frac{a^2}{2}\begin{vmatrix} t_1t_2 & t_2(t_3-t_1) & t_1(t_3-t_2) \\ (t_1+t_2) & t_3-t_1 & t_3-t_2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{a^2}{2}(t_3-t_2)(t_3-t_1)\begin{vmatrix} t_1t_2 & t_2 & t_1 \\ (t_1+t_2) & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \frac{a^2}{2}(t_2-t_1)(t_2-t_3)(t_3-t_1)$$

Clearly, $|\text{area }(\Delta ABC)| = 2\,|\text{area }(\Delta PQR)|$. The required answer is 2. Therefore, the correct option is (C).

This example should show you that the solution would have been much lengthier had we not directly made use of the formula for the coordinates of the intersection point of two tangents to a parabola. This is one subject where some things are to be learnt! ■

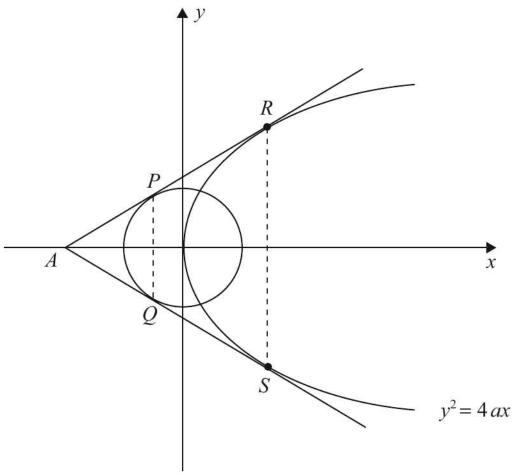
Example 5

From a point A, common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. The area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola is

(A) $\frac{15a^2}{4}$ sq. units (C) $\frac{17a^2}{4}$ sq. units (E) None of these

(B) $4a^2$ sq. units (D) $\frac{9a^2}{2}$ sq. units

Solution: Consider the figure below carefully:



The chord of contact of the circle is PQ while that of the parabola is RS. We need to find the area of the quadrilateral $PQSR$.

Any tangent to $y^2 = 4ax$ can be written in terms of its slope m as

$$y = mx + \frac{a}{m}$$

or

$$m^2x - my + a = 0$$

If this is to touch the circle $x^2 + y^2 = \frac{a^2}{2}$, its distance from the circle's centre (0, 0) must be equal to the circle's radius, which is $\frac{a}{\sqrt{2}}$. Thus,

$$\frac{a}{\sqrt{m^4 + m^2}} = \frac{a}{\sqrt{2}}$$

$$\Rightarrow \quad m^4 + m^2 = 2$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad m = \pm 1$$

We therefore conclude that the two tangents from A are $y = \pm(x + a)$ so that the point A is $(-a, 0)$. The chord of contact PQ is

$$T_C(-a,0)=0$$

$$\Rightarrow \quad -ax+0y=\frac{a^2}{y}$$

$$\Rightarrow \quad x=-\frac{a}{2}$$

The chord of contact RS is

$$T_P(-a,0)=0$$

$$\Rightarrow \quad 0y=2a(x-a)$$

$$\Rightarrow \quad x=a$$

The length PQ is now simply a (verify) whereas RS becomes the Latus-Rectum (since $x = a$ for RS) so that its length is $4a$. The distance between PQ and RS is $\frac{a}{2}+a=\frac{3a}{2}$. Thus, the area of $PQSR$ (note that it is a trapezium) is

$$\Delta=\frac{1}{2}(a+4a)\times\frac{3a}{2}=\frac{15a^2}{4} \text{ sq. units}$$

The correct option is (A). ■

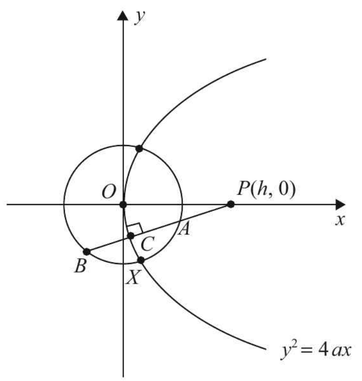
Example 6

The range of the x-coordinates of all the points on the x-axis from which exactly three distinct chords of the circle $x^2+y^2=a^2$ can be drawn which are bisected by the parabola $y^2=4ax$ $(a>0)$ is

(A) $(2a, (\sqrt{5}+3)a)$ (C) $(3a, (\sqrt{5}+2)a)$ (E) None of these

(B) $(3a, (\sqrt{5}+1)a)$ (D) $(4a, (\sqrt{5}+2)a)$

Solution: Let such a point be $P(h, 0)$. We need to find the possible values that h can take.



We need to find all points P such that three chords of the circle can be drawn passing through P which are bisected by the parabola.

Note that one such chord will always be simply along the x-axis because it is bisected by the parabola at the origin. Referring to the figure above, let C be the point t so that its co-ordinates are $(at^2, 2at)$. We can write the equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at C as

$$T(at^2, 2at) = S(at^2, 2at)$$

$$\Rightarrow \quad tx + 2y = at^3 + 4at$$

Since this passes through $P(h, 0)$, we have

$$th = at^3 + 4at$$

$$\Rightarrow \quad t(at^2 + (4a - h)) = 0 \tag{1}$$

One of the roots of (1) is $t = 0$ which corresponds to the case already mentioned, the diameter along the x-axis. The other two roots are real and distinct from zero if

$$\frac{h}{a} - 4 > 0 \Rightarrow \quad h > 4a \tag{2}$$

If you think carefully, you will realise that an additional constraint has to be imposed, namely, a limit on the value of t so that $(at^2, 2at)$ lies inside the circle, since only then will a chord be formed. Thus,

$$(at^2)^2 + (2at)^2 < a^2$$

$$\Rightarrow \quad t^2 < \sqrt{5} - 2 \tag{3}$$

From (1), we can see that (3) can equivalently be written as

$$\frac{h}{a} - 4 < \sqrt{5} - 2$$

$$\Rightarrow \quad h < (\sqrt{5} + 2)a$$

From (2) and (3), we see that the range of the possible values of h is

$$h \in (4a, (\sqrt{5} + 2)a)$$

The correct option is (D). ■

Example 7

The locus of an external point P from which exactly two distinct normals can be drawn to the parabola $y^2 = 4ax$ is

(A) $16ay = (x - 2a)^2$ (C) $16ay^2 = 2(x - 2a)$ (E) None of these

(B) $27ay^3 = 8(x - 2a)^2$ (D) $27ay^2 = 4(x - 2a)^3$

Solution: The equation of the normal in terms of slope is a cubic, from which we can infer that in general, three normals can be drawn to the parabola from a given point. Thus, we could either have one real and two imaginary solutions of the cubic (implying only one normal) or all the three roots of the cubic could be real (implying three real normals). In this question, we have exactly two distinct normals which can only happen if all the three roots of the cubic are real and *two of these three roots are identical*. This is the insight we use for solving this question.

The equation of an arbitrary normal to the parabola is

$$y = mx - 2am - am^3$$

If this passes through $P(h, k)$, we have

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + (2a - h)m + k = 0 \qquad (1)$$

Let the roots of this cubic be m_1, m_1 and m_2, so that

$$2m_1 + m_2 = 0 \quad \Rightarrow \quad m_2 = -2m_1$$

$$m_1^2 m_2 = -\frac{k}{a}$$

These two relations give

$$m_1^3 = \frac{k}{2a} \quad \Rightarrow \quad m_1 = \left(\frac{k}{2a}\right)^{\frac{1}{3}}$$

We now substitute this value of m_1 back in (1) to obtain a relation between h and k:

$$\frac{ak}{2a} + (2a - h)\left(\frac{k}{2a}\right)^{\frac{1}{3}} + k = 0$$

$$\Rightarrow \quad 27ak^2 = 4(h - 2a)^3$$

Using (x, y) instead of (h, k) gives the required locus as

$$27ay^2 = 4(x - 2a)^3$$

The correct option is (D). ■

Example 8

The locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other is

(A) $x^2 - y + 6 = 0$ (B) $x^2 - 2y + 12 = 0$ (E) None of these

(C) $x^2 - 3y + 15 = 0$ (D) $x^2 - 4y + 20 = 0$

Solution: We observe that the equation to the parabola has been given not in the standard form $y^2 = 4ax$ but in the form $x^2 = 4ay$: this means that in all the formulae that we use, x and y must be interchanged. This means that the equation of an arbitrary normal to $x^2 = 8y$ can be written as

$$x = my - 4m - 2m^3$$

If this passes through $P(h, k)$, we obtain

$$h = km - 4m - 2m^3 \qquad (1)$$

This cubic has three roots, say m_1, m_2 and m_3. Since two of the normals are perpendicular, we can take $m_1 m_2$ equal to -1. From (1), we have

$$m_1 + m_2 + m_3 = 0$$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -\frac{k}{2} + 2$$

$$m_1 m_2 m_3 = -\frac{h}{2}$$

Using $m_1 m_2 = -1$ and these three equations, a relation involving only h and k can easily be obtained:

$$h^2 - 2k + 12 = 0$$

Thus, the required locus is

$$x^2 - 2y + 12 = 0$$

The correct option is (B). ■

Example 9

Normals are drawn from the point P with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If the locus of P with $m_1 m_2 = \alpha$ is a part of the parabola itself, then the value of α is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: If we let P be the point (h, k), we have

$$m^3 + (2-h)m + k = 0 \quad (1)$$

Thus,

$$m_1 m_2 m_3 = -k$$

$$\Rightarrow \quad m_3 = -\frac{k}{\alpha} \quad (\text{since } m_1 m_2 = \alpha)$$

Substituting this back into (1), we obtain

$$-\frac{k^3}{\alpha^3} - \frac{k}{\alpha}(2-h) + k = 0$$

$$\Rightarrow \quad k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3 \quad (2)$$

Also, since P lies on the parabola itself, we have

$$k^2 = 4h \quad (3)$$

From (2) and (3), we have

$$\alpha^2 = 4 \quad \text{and} \quad \alpha^3 - 2\alpha^2 = 0$$

$$\Rightarrow \quad \alpha = 2$$

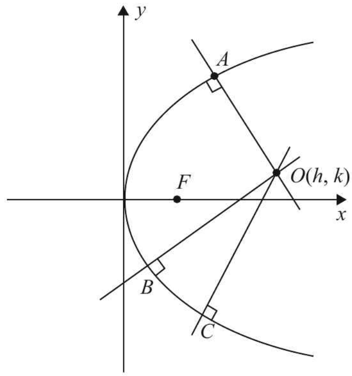
The correct option is (B). ■

Example 10

The normals at three points A, B, and C on a parabola intersect at O. F is the focus of the parabola. The value of $\frac{FA \cdot FB \cdot FC}{FO^2}$ in terms of l, the length of the latus-rectum, is

(A) l (B) $\frac{l}{2}$ (C) $\frac{l}{3}$ (D) $\frac{l}{4}$ (E) None of these

Solution: Assume the parabola to be $y^2 = 4ax$, and the point O to be (h, k). F is the point $(a, 0)$.



Any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3,$$

and since this passes through $O(h, k)$, we have

$$k = mh - 2am - am^3 \tag{1}$$

The feet of the three normal correspond to A, B and C. Thus, if m_1, m_2 and m_3 are the roots of (1), the coordinates of A, B and C are

$$(am_1^2, -2am_1), (am_2^2, -2am_2) \text{ and } (am_3^2, -2am_3)$$

We have now,

$$FA = \sqrt{(a - am_1^2)^2 + (2am_1)^2} = a\left(1 + m_1^2\right)$$

Similarly, $FB = a\left(1 + m_2^2\right)$ and $FC = a\left(1 + m_3^2\right)$.

Thus,

$$FA \cdot FB \cdot FC = a^3\left(1 + m_1^2\right)\left(1 + m_2^2\right)\left(1 + m_3^2\right)$$

$$= a^3\left(1 + m_1^2 + m_2^2 + m_3^2 + m_1^2m_2^2 + m_2^2m_3^2 + m_3^2m_1^2 + (m_1m_2m_3)^2\right) \tag{2}$$

From (1), we can infer the following:

$$\left.\begin{aligned} m_1 + m_2 + m_3 &= 0 \\ m_1m_2 + m_2m_3 + m_3m_1 &= \left(\frac{2a - h}{a}\right) \\ m_1m_2m_3 &= -\frac{k}{a} \end{aligned}\right\} \tag{3}$$

Our task is to express the relation (2) in terms of the known quantities given by (3). This can be done as follows :

$$m_1^2 + m_2^2 + m_3^3 = (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)$$

$$m_1^2m_2^2 + m_2^2m_3^2 + m_3^2m_1^2 = (m_1m_2 + m_2m_3 + m_3m_1)^2 - 2m_1m_2m_3(m_1 + m_2 + m_3)$$

Substituting the appropriate values gives

$$FA\cdot FB\cdot FC = a^3\left\{1+\left(\frac{2a-h}{a}\right)^2-2\left(\frac{2a-h}{a}\right)+\frac{k^2}{a^2}\right\}$$

$$= a\{(h-a)^2+k^2\}$$

which evidently equals $a\cdot FO^2$. Since $l = 4a$, we have

$$\frac{FA\cdot FB\cdot FC}{FO^2}=\frac{l}{4}$$

The correct option is (D). ■

Example 11

The equation of the ellipse whose major and minor axes lie along the lines $x-3y+3=0$ and $3x+y-1=0$ and whose lengths are 6 and $2\sqrt{6}$ respectively is

(A) $19x^2-2xy+24y^2+13x-44y+72=0$

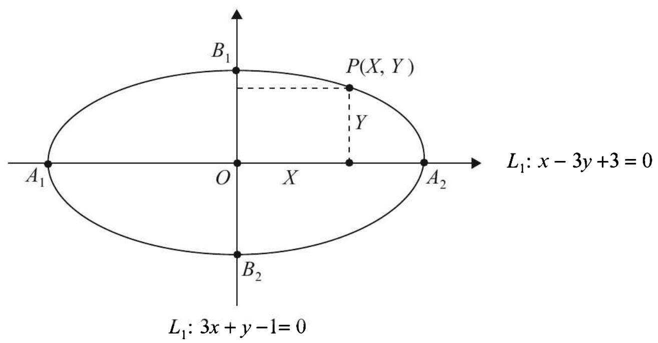
(B) $17x^2-3xy+32y^2+17x-31y+99=0$

(C) $21x^2-6xy+29y^2+6x-58y-151=0$

(D) $24x^2-8xy+17y^2-8x+34y-46=0$

(E) None of these

Solution: The equation to the ellipse will obviously not be in the standard form since the axes are not along the coordinate axes. However, we can use the coordinate axes formed by these two lines as our reference frame:



$A_1A_2 = 6$
$B_1B_2 = 2\sqrt{6}$
Assume any point on the ellipse as $P(x, y)$ referred to the original axes or $P(X, Y)$ referred to the new axes

Consider an arbitrary point P on the ellipse whose coordinates are (x, y) with respect to the original axes (not shown) and (X, Y) with respect to the new axes, the $L_1 - L_2$ system. In this new system, the equation of the ellipse is simply

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

where $a = 3$ and $b=\sqrt{6}$. Thus, the equation is

$$\frac{X^2}{9}+\frac{Y^2}{6}=1 \qquad (1)$$

Now, we wish to write the equation of the ellipse in the x-y coordinate system. For this purpose, we make the following observations from the figure: What is X? It is simply the perpendicular distance of P from L_2:

$$X = \frac{|3x + y - 1|}{\sqrt{10}}$$

Similarly, Y is simply the perpendicular distance of P from L_1 :

$$Y = \frac{|x - 3y + 3|}{\sqrt{10}}$$

Thus, using (1), the equation of the ellipse in x-y form is

$$\frac{(3x + y - 1)^2}{90} + \frac{(x - 3y + 3)^2}{60} = 1$$

$$\Rightarrow \quad 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

The correct option is (C). As an exercise find the centre and the foci of this ellipse.
Hint: The centre is simply the intersection of L_1 and L_2. The foci are at $(\pm ae, 0)$ in the X -Y system. To find the foci in the x-y system, find the two points along L_1 which are at a distance of ae from O on either side of it.

On a sidenote: given an arbitrary fixed point $P(h, k)$ and an arbitrary fixed straight line $lx + my + n = 0$ as the focus and directrix of an ellipse with eccentricity e, its equation can be written by using the definition of an ellipse. Let (x, y) be any point on the ellipse:

$$\frac{\text{Distance of } (x, y) \text{ from } P}{\text{Distance of } (x, y) \text{ from the line}} = e$$

$$\Rightarrow \quad (x - h)^2 + (y - k)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2}$$

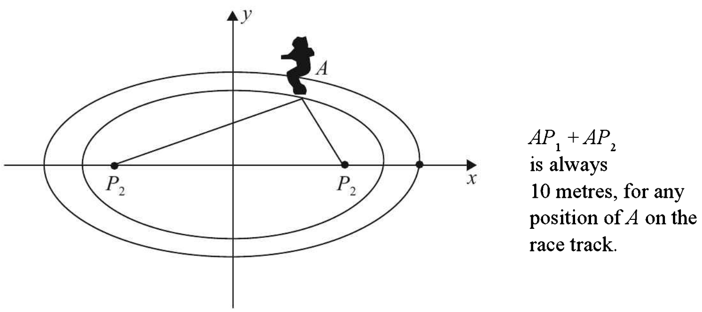
This gives us the general equation of an ellipse with a given eccentricity, focus and directrix. However, we will almost always be using the (simple!) standard form of the equation. ■

Example 12

An athlete running around a race track finds that the sum of his distances from two flag posts is always 10 metres, while the distance between the flag posts is 8 meters. In square metres, what is the area that the race track encloses?

(A) 9π (B) 12π (C) 15π (D) 18π

Solution:



From the situation described, the race track must be an ellipse. The eccentricity is simply

$$e = \frac{8}{10} = \frac{4}{5}$$

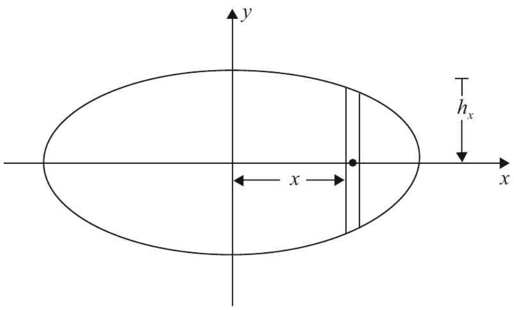
If the major axis is of length $2a$, we have

$$P_1P_2 = 2ae = 8$$
$$\Rightarrow \quad a = 5$$

Thus, $b = a\sqrt{(1-e^2)} = 3.$ The equation of the elliptical race track is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

To evaluate the area enclosed, we solve the general problem: What is the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$? To find the area, we divide the ellipse into elemental strips of width dx: one such strip is shown below.



h_x represents the height of an elemental strip at distance x from the centre. Note from the equation of the ellipse that

$$h_x = \frac{b}{a}\sqrt{a^2 - x^2}$$

The area of the elemental strip shown is

$$dA = 2h_x dx \; = \frac{2b}{a}\sqrt{a^2 - x^2}\,dx$$

The area of the right half of the ellipse is therefore

$$A_{\text{half}} = \frac{2b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi}{2}ab$$

The total area of the ellipse is thus

$$A = 2 \times A_{\text{half}} = \pi ab$$

For the current example, the area becomes

$$A = \pi \times 5 \times 3 = 15\pi \quad \text{sq. mt.}$$

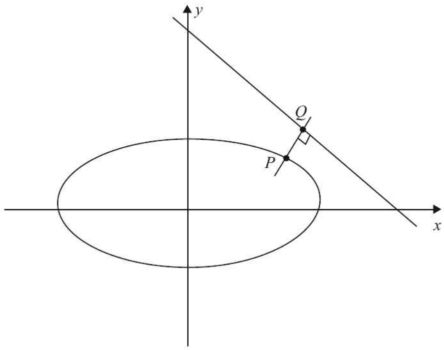
The correct option is (C). ■

Example 13

Let (h,k) be the point on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ whose distance from the line $x + y = 7$ is minimum. The value of $h + k$ is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: Any point on the given ellipse can be assumed to be $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$. From the following figure, observe that for the distance of P from the given line to be minimum, the normal at P must be perpendicular to the given line.



If P is the point of the minimum distance from the given line, the normal at P must be perpendicular to the given line.

The equation of the normal at P, using parametric form, is

$$(\sqrt{6}\sec\theta)x - (\sqrt{3}\operatorname{cosec}\theta)y = 3$$

whose slope is

$$m_N = \sqrt{2}\tan\theta$$

If the normal is perpendicular to $x + y = 7$, we have

$$m_N = 1 \Rightarrow \quad \tan\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{3}} \text{ and } \cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

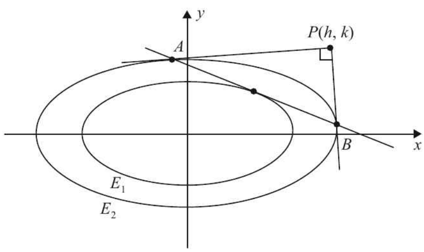
Thus, the point P is $(a\cos\theta, b\sin\theta) \equiv (2,1)$. The required value of $h+k$ is 3. The correct option is (B). ■

Example 14

A tangent is drawn to the ellipse $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which cuts the ellipse $E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ at the points A and B. Tangents to this second ellipse at A and B intersect at right angles. The value of $\frac{a^2}{c^2} + \frac{b^2}{d^2}$ is

(A) 1 (B) $\frac{1}{4}$ (C) $\frac{1}{9}$ (D) $\frac{1}{16}$

Solution: Let the point of intersection of the two tangents be $P(h, k)$.



Note that since AB is the chord of contact for the tangents drawn from P to E_2, we have the equation of AB as

$$T(h, k) = 0$$

$$\Rightarrow \quad \frac{hx}{c^2} + \frac{ky}{d^2} = 1 \quad \Rightarrow \quad y = \left(\frac{-d^2h}{c^2k}\right)x + \frac{d^2}{k}$$

If AB is to touch the inner ellipse E_1, the condition of tangency must be satisfied:

$$\frac{d^4}{k^2} = a^2 \cdot \frac{d^4h^2}{c^4k^2} + b^2$$

$$\Rightarrow \quad c^4d^4 = a^2d^4h^2 + b^2c^4k^2 \tag{1}$$

Since PA and PB intersect at right angles, P must lie on the director circle of the ellipse E_2. Thus,

$$h^2 + k^2 = c^2 + d^2 \tag{2}$$

(1) and (2) can be considered a system of equations in the variables h^2 and k^2:

$$(a^2d^4)h^2 + (b^2c^4)k^2 = c^4d^4$$

$$h^2 + k^2 = c^2 + d^2$$

If these relations are to hold for the variables h and k, they must in fact be identical. Thus, these variables can now easily be eliminated to obtain:

$$\frac{a^2d^4}{c^4d^4} = \frac{1}{c^2+d^2}; \quad \frac{b^2c^4}{c^4d^4} = \frac{1}{c^2+d^2}$$

$$\Rightarrow \quad \frac{a^2}{c^2} = \frac{c^2}{c^2+d^2}; \quad \frac{b^2}{d^2} = \frac{d^2}{c^2+d^2}$$

$$\Rightarrow \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$$

The correct option is (A). ■

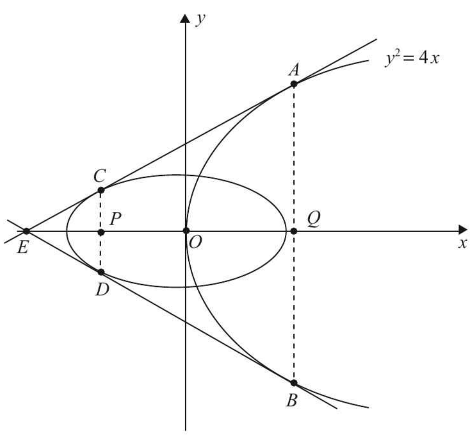
Example 15

Common tangents are drawn to the parabola $y^2 = 4x$ and the ellipse $\frac{x^2}{16} + \frac{y^2}{6} = 1$, touching the parabola at A and B and the ellipse at C and D. What is the area of the quadrilateral $ABDC$?

(A) $55\sqrt{2}$ sq. units (C) $63\sqrt{2}$ sq. units

(B) $60\sqrt{2}$ sq. units (D) $72\sqrt{2}$ sq. units (E) None of these

Solution:



An approximate figure showing the common tangents AC and BD intersecting in E (which will lie on the axis due to the symmetry of the problem).

Any tangent to the parabola $y^2 = 4x$ can be written in the form

$$y = mx + \frac{1}{m}$$

This line touches the ellipse if the condition for tangency, $c^2 = a^2m^2 + b^2$ is satisfied, i.e., if

$$\frac{1}{m^2} = 10m^2 + 6,$$

giving

$$m = \pm\frac{1}{2\sqrt{2}}$$

Thus, the two tangents AE and BE are

$$y = \pm\left(\frac{1}{2\sqrt{2}}x + 2\sqrt{2}\right)$$

which evidently intersect at $E(-4, 0)$. The point of contact for the parabola $y^2 = 4ax$ is given by $(\frac{a}{m^2}, \frac{2a}{m})$. Thus A and B have the coordinates $(8, \pm 4\sqrt{2})$ so that

$$AB = 8\sqrt{2}$$

The point of contact for the ellipse will be $(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}})$. Thus, C and D will have the coordinates $(-2, \pm\frac{3}{\sqrt{2}})$ so that

$$CD = 3\sqrt{2}$$

Finally, PQ can now easily be seen to be 8 + 2 = 10. The area of quadrilateral $ABDC$ (which is actually a trapezium) is

$$\Delta = \frac{1}{2}\times(AB + CD)\times PQ$$

$$= 55\sqrt{2} \text{ sq. units}$$

The correct option is (A). ■

Example 16

What is the farthest distance at which a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can lie from the centre of the ellipse?

(A) $\sqrt{|a^2 - b^2|}$ (B) $\sqrt{|2(a^2 - b^2)|}$ (C) $|a - b|$ (D) $\sqrt{2}\,|a - b|$ (E) None of these

Solution: Any normal to the ellipse is of the form

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$$

The distance of this normal from the centre (0, 0) is

$$d = \frac{|a^2 - b^2|}{\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}}$$

We need to find the maximum value of d, or equivalently, the minimum value of

$$f(\theta) = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$$

We have

$$f'(\theta) = 2a^2 \sec^2\theta \tan\theta - 2b^2 \operatorname{cosec}^2\theta \cot\theta$$

$$\Rightarrow \quad f'(\theta) = 0 \text{ when } \tan^2\theta = \tfrac{b}{a}$$

$$\Rightarrow \quad \tan\theta = \pm\sqrt{\frac{b}{a}}$$

Verify that at this value of $\theta, f''(\theta)$ is positive so that this θ indeed gives us the minimum value of $f(\theta)$.

Now, $\quad f_{\min}(\theta) = a^2 \sec^2\theta_{\min} + b^2 \operatorname{cosec}^2\theta_{\min}$

$$= a^2(1+\tan^2\theta_{\min}) + b^2(1+\cot^2\theta_{\min}) = (a+b)^2$$

$$\Rightarrow \quad d_{\max} = \frac{|a^2-b^2|}{\sqrt{f_{\min}(\theta)}} = \frac{|a^2-b^2|}{(a+b)} = |a-b|$$

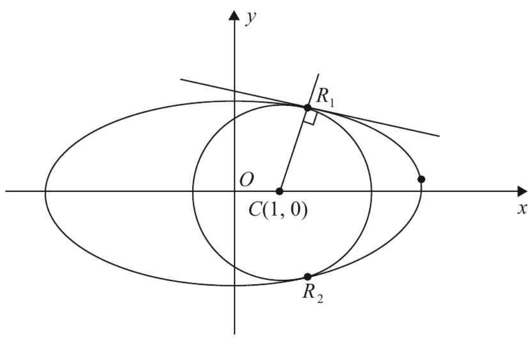
The correct option is (C). ■

Example 17

The radius of the largest circle with centre (1, 0) that can be inscribed inside the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ is

(A) $\sqrt{3}$ (B) $\sqrt{\frac{10}{3}}$ (C) $\sqrt{\frac{11}{3}}$ (D) 2 (E) None of these

Solution: The following diagram shows the largest such circle. Observe it carefully:



Note that the largest possible circle lying completely inside the ellipse must touch it, say, at the points R_1 and R_2, as shown. At these points, it will be possible to draw common tangents to the circle and the ellipse.

Let point R_1 be $(4\cos\theta,\ 2\sin\theta)$. The equation of the tangent at R_1 is

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{2} = 1 \quad \Rightarrow \quad x\cos\theta + 2y\sin\theta = 4$$

If CR_1 is perpendicular to this tangent (which must happen if this tangent is to be common to both the ellipse and the circle), we have

$$\frac{2\sin\theta - 0}{4\cos\theta - 1} \times \frac{-\cos\theta}{2\sin\theta} = -1 \quad \Rightarrow \quad \cos\theta = \frac{1}{3}$$

Thus, R_1 is $(\frac{4}{3}, \frac{4\sqrt{2}}{3})$. The largest possible radius is therefore

$$r_{\max} = CR_1 = \sqrt{\left(\frac{4}{3}-1\right)^2 + \left(\frac{4\sqrt{2}}{3}-0\right)^2} = \sqrt{\frac{11}{3}}$$

The correct option is (C). ■

Example 18

The equations of the transverse and conjugate axes of a hyperbola are $L_1: x+2y-3=0$ and $L_2: 2x-y+4=0$ respectively, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. What is the equation of the hyperbola?

(A) $x^2-3xy-y^2+8x+6y=0$

(B) $x^2-4xy-2y^2+10x+4y=0$

(C) $x^2-2xy-y^2-6x+2y=0$

(D) $x^2-5xy+3y^2-8x+4y=0$

(E) None of these

Solution: We have,

$$2a=\sqrt{2} \Rightarrow \quad a=\frac{1}{\sqrt{2}}$$

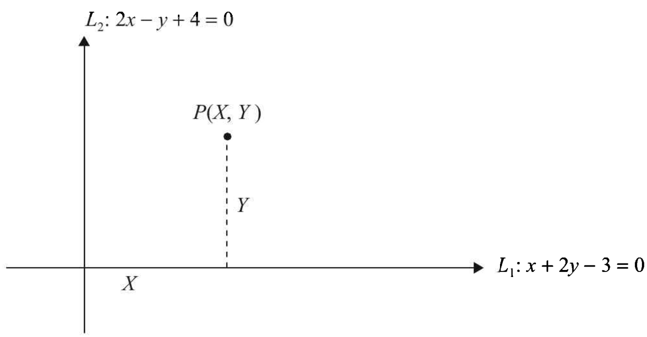
$$2b=\frac{2}{\sqrt{3}} \Rightarrow \quad b=\frac{1}{\sqrt{3}}$$

We first find out the equation of the hyperbola in the L_1-L_2 system (in which the coordinates of any point can be represented by (X, Y)). This equation is simply

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

$$\Rightarrow \quad 2X^2-3Y^2=1 \tag{1}$$

Now, what do X and Y represent? As we've already discussed in other similar problems, X and Y represent the perpendicular distances of the point P(X, Y) from L_2 and L_2 respectively:



Thus, if in the original x - y system, P has the coordinates (x, y), we have,

$$X=\frac{|2x-y+4|}{\sqrt{5}}; \quad Y=\frac{|x+2y-3|}{\sqrt{5}} \tag{2}$$

Using (2) in (1), the equation of the hyperbola is

$$\frac{2}{5}(2x-y+4)^2-\frac{3}{5}(x+2y-3)^2=1$$

which simplifies to

$$x^2-4xy-2y^2+10x+4y=0$$

The correct option is (B). ■

Example 19

The equation of the hyperbola with asymptotes $3x-4y+7=0$ and $4x+3y+1=0$ and which passes through the origin is

(A) $12x^2-7xy-12y^2+31x+17y=0$ (C) $12x^2-7xy-12y^2+19x+28y=0$

(B) $12x^2-7xy-12y^2+17x-31y=0$ (D) $12x^2-7xy-12y^2+28x-19y=0$

(E) None of these

Solution: The joint equation of the asymptotes is

$$(3x-4y+7)(4x+3y+1)=0$$

Since the equation of a hyperbola and that of the joint equation of its asymptotes differ by just a constant, the equation of the hyperbola must be

$$(3x-4y+7)(4x+3y+1)+k=0$$

We can obtain k by exploiting the fact that this hyperbola passes through the origin (0, 0); thus

$$(7)(1)+k=0$$

$$\Rightarrow \quad k=-7$$

The equation of the hyperbola is

$$(3x-4y+7)(4x+3y+1)-7=0$$

$$\Rightarrow \quad 12x^2-7xy-12y^2+31x+17y=0$$

The correct option is (A). ■

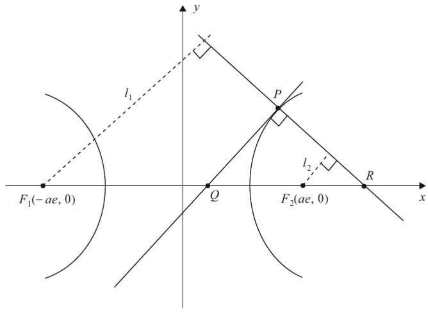
Example 20

A tangent drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point P intersects its transverse axis in Q. The lengths of the perpendiculars drawn from the two foci F_1 and F_2 upon the normal at P are l_1 and l_2 respectively. Which of the following statements is true?

(A) PQ is the arithmentic mean between l_1 and l_2 (C) PQ is the harmonic mean between l_1 and l_2

(B) PQ is the geometric mean between l_1 and l_2. (D) None of these

Solution:



Let P be the point $(a\sec\theta, b\tan\theta)$. The equation of the tangent at P is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

To obtain Q, we put $y = 0$ in this equation; thus,

$$Q \equiv (a\cos\theta, 0)$$

The normal at P has the equation

$$ax\cos\theta + by\cot\theta = a^2 + b^2$$

Thus, the point R is

$$R \equiv \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$$

From the figure, observe that

$$\frac{l_2}{RF_2} = \frac{PQ}{RQ} = \frac{l_1}{RF_1} = \lambda \quad \Rightarrow \quad l_1 = \lambda(RF_1) = \lambda\left(\frac{a^2+b^2}{a}\sec\theta + ae\right)$$

$$\Rightarrow \quad l_2 = \lambda(RF_2) = \lambda\left(\frac{a^2+b^2}{a}\sec\theta - ae\right)$$

$$\Rightarrow \quad l_1 l_2 = \lambda^2\left(\frac{(a^2+b^2)^2}{a^2}\sec^2\theta - a^2e^2\right)$$

$$= \lambda^2\left(\frac{(a^2+b^2)^2}{a^2}\sec^2\theta - (a^2+b^2)\right) = \frac{\lambda^2(a^2+b^2)}{a^2}(a^2\tan^2\theta + b^2\sec^2\theta)$$

$$= \lambda^2 e^2 (a^2\tan^2\theta + b^2\sec^2\theta)$$

$$\text{and } l_1 + l_2 = \frac{2\lambda\sec\theta}{a}(a^2+b^2) = 2\lambda a e^2 \sec\theta$$

Thus,

$$\frac{2l_1l_2}{l_1+l_2} = \frac{\lambda(a^2\tan^2\theta + b^2\sec^2\theta)}{a\sec\theta} \tag{1}$$

Also,

$$PQ = \lambda(RQ) = \lambda\left(\frac{a^2+b^2}{a}\sec\theta - a\cos\theta\right)$$

$$= \lambda\left(\frac{(a^2+b^2)\sec\theta - a^2\cos\theta}{a}\right) = \lambda\left(\frac{a^2\tan^2\theta + b^2\sec^2\theta}{a\sec\theta}\right) \tag{2}$$

From (1) and (2), we have

$$PQ = \frac{2l_1 l_2}{l_1 + l_2}$$

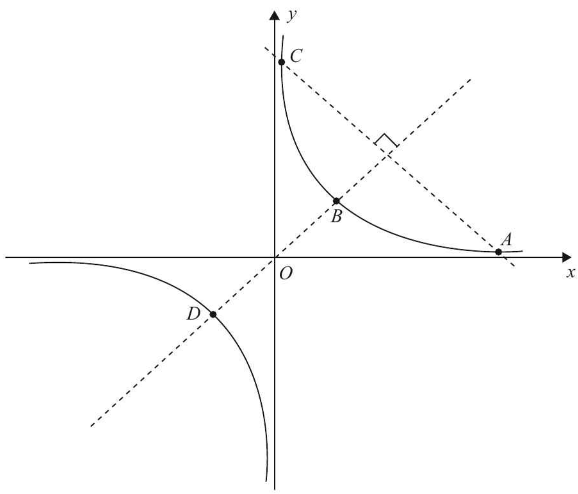
which means that PQ is the HM between l_1 and l_2. The correct option is (C). ■

Example 21

Four points A, B, C, D lie on a rectangular hyperbola $xy = c^2$ such that $AC \perp BD$. Let O be the the centre of the hyperbola. The slopes of OA, OB, OC and OD are m_1, m_2, m_3, and m_4 respectively. The value of $m_1\, m_2\, m_3\, m_4$ is

(A) 1 (B) 2 (C) 4 (D) 8

Solution:



The coordinates of A, B, C, D can be assumed to be $(ct_i, \frac{c}{t_i}), i = 1,2,3,4$. Thus,

$$m_i = \frac{\frac{c}{t_i} - 0}{ct_i - 0} = \frac{1}{t_i^2}$$

$$\Rightarrow \quad m_1\, m_2\, m_3\, m_4 = \frac{1}{t_1^2 t_2^2 t_3^2 t_4^2} \tag{1}$$

Since $AC \perp BD$, we have

$$\frac{\frac{c}{t_3} - \frac{c}{t_1}}{ct_3 - ct_1} \times \frac{\frac{c}{t_2} - \frac{c}{t_4}}{ct_2 - ct_4} = -1$$

$$\Rightarrow \quad t_1\, t_2\, t_3\, t_4 = -1 \tag{2}$$

Using (1) and (2), we have

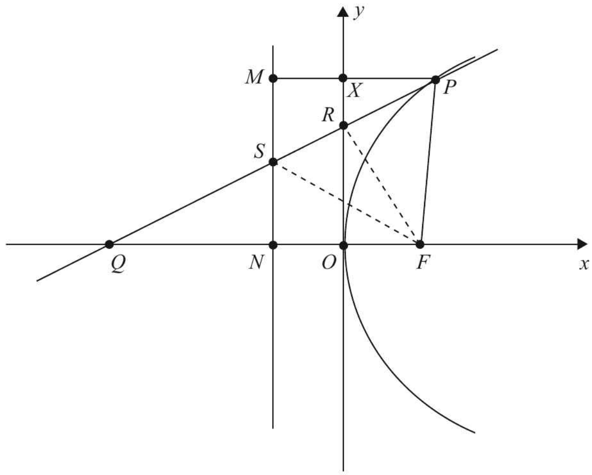
$$m_1\, m_2\, m_3\, m_4 = 1$$

Thus, the correct option is (A). ■

SUBJECTIVE TYPE EXAMPLES

Example 22

Let P be any point on the parabola with focus F. A perpendicular PM is dropped from P onto the directrix as shown:



(a) Prove that the tangent at P to the parabola bisects the angle FPM.
(b) Prove that angle PFS is a right angle.
(c) Show that angle FRP is a right angle.

Solution: (a) Let the equation of the parabola be $y^2 = 4ax$, and let P be the point t. Thus, F is the point $(a, 0)$ while N is $(-a, 0)$. By definition, we have

$$PF = PM$$

But PM is $PX + XM$, *i.e.*, $at^2 + a$. Thus $PF = a + at^2$. Now, the equation of the tangent at P is

$$ty = x + at^2$$

This intersects the x-axis at the point $Q(-at^2, 0)$. Thus,

$$FQ = FO + OQ = a + at^2$$

We see that in ΔPFQ,

$$FP = FQ$$

$$\Rightarrow \quad \angle FPQ = \angle FQP \tag{1}$$

But since PM is parallel to the x-axis, we also have

$$\angle FQP = \angle QPM \quad \text{(alternate interior angles)} \tag{2}$$

From (1) and (2), we obtain $\angle FPQ = \angle QPM$ which means that the tangent at P bisects $\angle FPM$.

(b) The equation of the tangent at $P(at^2, 2at)$ is

$$ty = x + at^2$$

This intersects the directrix at a point given by $(-a, \frac{a(t^2-1)}{t})$. The slope of PF is

$$m_{PF} = \frac{2at}{at^2 - a} = \frac{2t}{t^2 - 1}$$

The slope of SF is

$$m_{SF} = \frac{\frac{a(t^2-1)}{t}}{-2a} = \frac{t^2 - 1}{-2t}$$

Since, $m_{PF} \times m_{SF} = -1$, angle PFS is a right angle.

(c) Once again, we use the equation of the tangent at $P(at^2, 2at)$:

$$ty = x + at^2$$

This intersects the y-axis at the point $R(0, at)$. The slope of PR is simply the slope of the tangent at P, *i.e.*, $m_{PR} = \frac{1}{t}$. The slope of RF is

$$m_{RF} = \frac{at - 0}{0 - a} = -t$$

Since $m_{PR} \times m_{RF} = -1$, angle FRP is a right angle.

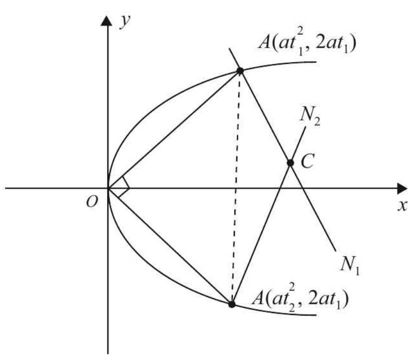
The results of these three parts are important, so we summarize them here:

- The tangent at any point on a parabola bisects the angle between the focal chord through that point and the perpendicular on the directrix from that point.
- The portion of the tangent to a parabola cut-off between the directrix and the curve subtends a right angle at the focus.
- The perpendicular dropped from the focus onto any tangent to a parabola is concurrent with that tangent and the tangent at the vertex.
  Along with these, we include a fourth important result here.
- The tangents at the extremities of any focal chord of a parabola intersect at right angles on the directrix. ■

Example 23

Find the locus of the point of intersection of normals drawn to the parabola $y^2 = 4ax$ at the extremities of a chord which subtends a right angle at the vertex of the parabola.

Solution: Let $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ be the extremities of a chord which subtends a right angle at the vertex (0, 0):



AB is a chord which subtends a right angle at O. The normals at A and B, N_1 and N_2, intersect at C. We need to find the locus of C.

Since $OA \perp OB$, we have

$$\underbrace{\left(\frac{2at_1 - 0}{at_1^2 - 0}\right)}_{\text{slope of } OA} \times \underbrace{\left(\frac{2at_2 - 0}{at_2^2 - 0}\right)}_{\text{slope of } OB} = -1$$

$$\Rightarrow \quad t_1 t_2 = -4$$

The equation to N_1 and N_2 can be written using the standard form of a normal at a point t:

$$N_1\colon y + t_1 x = 2at_1 + at_1^3$$

$$N_2\colon y + t_2 x = 2at_2 + at_2^3$$

Let the intersection of N_1 and N_2 be the point $C(h, k)$. The coordinates of C can be evaluated by solving the equations of N_1 and N_2 simultaneously:

$$h = 2a + a\left(t_1^2 + t_2^2 + t_1 t_2\right) = -2a + a\left(t_1^2 + t_2^2\right)$$

$$\text{and } k = -at_1 t_2 (t_1 + t_2) = 4a(t_1 + t_2)$$

We thus have, by eliminating t_1 and t_2, a relation in h and k:

$$\frac{k^2}{16a^2} = \frac{h + 2a}{a} - 8$$

$$\Rightarrow \quad k^2 = 16a(h - 6a)$$

Using (x, y) instead of (h, k), the required locus is

$$y^2 = 16a(x - 6a)$$

■

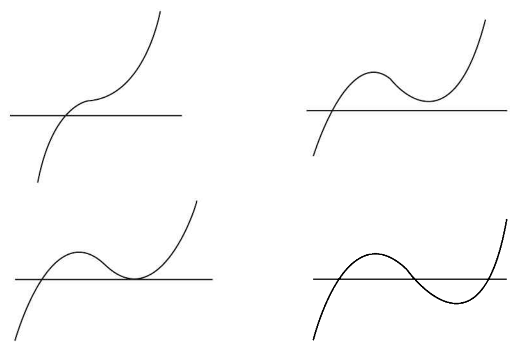
Example 24

Find the condition for all the three normals drawn from a given point $P(h, k)$ to the parabola $y^2 = 4ax$ to be real and distinct. Assume $a > 0$.

Solution: The given requirement is equivalent to the equation

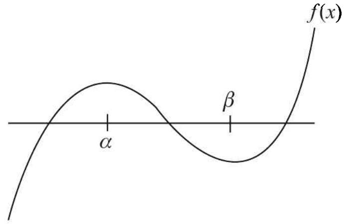
$$k = mh - 2am - am^3$$

having three real and distinct roots. A cubic will always have at least one real root. Here are some configurations that the graph of a cubic can take (assuming the coefficient of m^3 is positive):



Some configurations that the graph of a cubic can take.

Since we want three real roots, we focus on the last graph above. Observe that it has two extremas, say at α and β.



A cubic $f(x)$ with three real zeroes.

Thus, $f'(x) = 0$ must have two distinct real roots (since $f'(\alpha) = f'(\beta) = 0$). Also, we must have $f(\alpha)f(\beta) < 0$. Convince yourself that these two constraints will ensure that the cubic $f(x)$ will have three real and distinct zeroes.

We use this for the current cubic $f(m) = am^3 + (2a-h)m + k$:

- $f'(m) = 0$ has two real and distinct roots:

$$\Rightarrow \quad f'(m) = 3am^2 + (2a-h)$$

$$\Rightarrow \quad f'(m) = 0 \text{ for } m = \pm\sqrt{\frac{h-2a}{3a}}$$

$$\Rightarrow \quad h - 2a > 0 \text{ or } h > 2a$$

The two real and distinct roots are

$$\alpha = \sqrt{\frac{h-2a}{3a}} \quad \text{and} \quad \beta = -\sqrt{\frac{h-2a}{3a}} \tag{1}$$

- $f(\alpha)f(\beta) < 0$:

$$\Rightarrow \quad (a\alpha^3 + (2a-h)\alpha + k)(a\beta^3 + (2a-h)\beta + k) < 0$$

$$\Rightarrow \quad a^2(\alpha\beta)^3 + a(2a-h)\alpha\beta(\alpha^2+\beta^2) + ak(\alpha^3+\beta^3) + k^2 < 0$$

Simplifying this using (1), we obtain

$$27ak^2 < 4(h-2a)^3$$

Thus, if we are to be able to draw three real and distinct normals form $P(h, k)$ to $y^2 = 4ax$, h and k must satisfy

$$h > 2a \quad \text{and} \quad 27ak^2 < 4(h-2a)^3$$ ■

Example 25

From a variable point P on a fixed normal to the parabola $y^2 = 4ax$, two more normals are drawn to the parabola to intersect it at Q and R. Show that the variable chord QR will have a fixed slope.

Solution: Let P be the point (h, k). An arbitrary normal to the given parabola can be written as

$$y = mx - 2am - am^3,$$

and if this passes through P, we have

$$k = mh - 2am - am^3 \tag{1}$$

Now, this cubic has three roots, say m_1, m_2, and m_3 of which one, say m_1, is fixed, equal to the slope of the fixed normal. The other two slopes are variable and thus the points of intersection of the other two normals through P $\{Q(am_2^2, -2am_2)$ and $R(am_3^2, -2am_3)\}$ are also variable. From (1), we have

$$m_1 + m_2 + m_3 = 0 \tag{2}$$

The slope of QR is

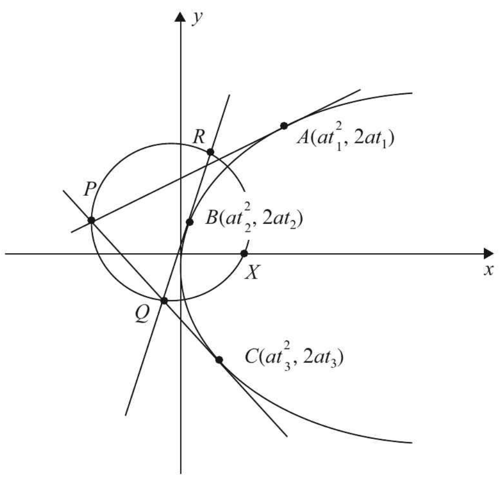
$$m = \frac{2a(m_2 - m_3)}{a(m_3^2 - m_2^2)} = \frac{2}{-(m_2 + m_3)} = \frac{2}{m_1} \quad \text{(from (2))}$$

This shows that m, the slope of QR, is fixed since m_1 is fixed. ■

Example 26

Three tangents to a parabola form the triangle PQR. Prove that the circumcircle of PQR passes through the focus of the parabola.

Solution: We assume the parabola to be $y^2 = 4ax$ and three points on it to be $A(at_1^2, 2at_1), B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$. Tangents at A, C and B meet at P, Q and R, as shown:



We need to show that the circumcircle of PQR passes through the focus F, or equivalently, the point X is the same as F.

The co-ordinates of the points P, Q and R are respectively $(at_1t_3, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$ and $(at_1t_2, a(t_1 + t_2))$. To show that the circumcircle passes through F, it would suffice to prove that the chord PQ subtends the same angle on F as it does on R. Since a chord of a circle subtends equal angles anywhere on the circumference, this will prove that F also lies on the circumference of the circle.

To evaluate $\angle PRQ$, we need the slopes of PR and RQ, which are simply the slopes of the tangents at A and B respectively, *i.e.*,

$$m_{PR} = \frac{1}{t_1} \quad \text{and} \quad m_{RQ} = \frac{1}{t_2}$$

Thus,

$$\tan(\angle PRQ) = \frac{m_{PR} - m_{RQ}}{1 + m_{PR}m_{RQ}} = \frac{t_2 - t_1}{1 + t_1t_2} \tag{1}$$

Now, to evaluate the angle that the chord PQ subtends at F, we need to find the slopes m_{PF} and m_{QF}:

$$m_{PF} = \frac{a(t_1+t_3)}{at_1t_3 - a} \quad \text{and} \quad m_{QF} = \frac{a(t_2+t_3)}{at_2t_3 - a}$$

$$\Rightarrow \quad m_{PF} = \frac{t_1+t_3}{t_1t_3-1} \quad \text{and} \quad m_{QF} = \frac{t_2+t_3}{t_2t_3-1}$$

Thus,

$$\tan(\angle PFQ) = \frac{m_{PF}-m_{QF}}{1+m_{PF}m_{QF}} = \frac{(t_1+t_3)(t_2t_3-1)-(t_2+t_3)(t_1t_3-1)}{(t_1t_3-1)(t_2t_3-1)+(t_1+t_3)(t_2+t_3)}$$

$$= \frac{t_2-t_1}{1+t_1t_2} \tag{2}$$

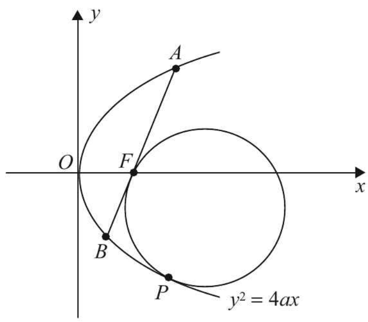
Comparing (1) and (2) gives $\angle PRQ = \angle PFQ$, which confirms that F does indeed lie on the circumference of $\Delta PQR's$ circumcircle.

We could alternatively have done this question by explicitly evaluating the equation of the circle passing through P, Q, and R and showing the F satisfies that equation. ■

Example 27

Let AB be a fixed chord passing through the focus of a parabola. Prove that three circles can be drawn which touch the parabola and AB at the focus.

Solution: Let the parabola be $y^2 = 4ax$; its focus is then $F(a, 0)$. The following diagram shows one such circle which touches AB at F and also the parabola:



Let AB have a (fixed) slope m. The equation of AB can be written using the point-slope form :

$$y - 0 = m(x-a)$$

$$\Rightarrow \quad y = mx - am$$

Any circle touching this line at $F(a, 0)$ can be written in terms of a real variable λ as:

$$(x-a)^2 + y^2 + \lambda(y - mx + am) = 0$$

$$\Rightarrow \quad x^2 + y^2 - (2a+m\lambda)x + \lambda y + a(a+m\lambda) = 0 \tag{1}$$

We want those values of λ for which this circle *touches* the parabola. Assume the point of contact of the circle and the parabola to be $P(at^2, 2at)$. Thus, at P, the circle and the parabola should have a common tangent. The tangent at P to the parabola is

$$ty = x + at^2$$

Since this line is a tangent to the circle as well (at the same point $P(at^2, 2at)$), we can write the equation of the same circle using another real parameter α:

$$(x-at^2)^2+(y-2at)^2+\alpha(ty-x-at^2)=0$$

$$\Rightarrow \quad x^2+y^2-(\alpha+2at^2)x+(\alpha t-2at)y+a^2t^4+4a^2t^2-a\alpha t^2=0 \qquad (2)$$

The circles given by (1) and (2) being the same, we can compare the coefficients to obtain the following equations:

$$2a+m\lambda=\alpha+2at^2$$
$$\lambda=\alpha t-2at$$
$$a+m\lambda=at^4+4at^2-\alpha t^2$$

We can rearrange these equations to make them look more *systematic*:

$$(m)\lambda+(-1)\alpha+(2a-2at^2)=0$$
$$(1)\lambda+(-t)\alpha+(2at)=0$$
$$(m)\lambda+(t^2)\alpha+(a-4at^2-at^4)=0$$

The variables λ and α can now be eliminated to obtain a relation purely in terms of t:

$$\begin{vmatrix} m & -1 & 2a(1-t^2) \\ 1 & -t & 2at \\ m & t^2 & -a(t^4+4t^2-1) \end{vmatrix}=0$$

Expanding along C_3 and simplifying, we obtain

$$mt(t^4-2t^2-3)+1-2t^2-3t^4=0$$
$$\Rightarrow \quad mt(t^2+1)(t^2-3)=(3t^2-1)(t^2+1)$$
$$\Rightarrow \quad mt(t^2-3)=3t^2-1$$
$$\Rightarrow \quad mt^3-3t^2-3mt+1=0 \qquad (3)$$

This is a cubic in t which will have in general three roots. This will imply that three possible points of contact $(at^2, 2at)$, and therefore three possible circles exist satisfying the given property. But something is missing! We still have to prove that the cubic *will* actually yield three real values of t. For that, we proceed as follows. Let $f(t)=mt^3-3t^2-3mt+1$. First, we show that $f'(t)$ has two real roots, say t_1 and t_2.

$$f'(t)=3mt^2-6t-3m$$
$$f'(t)=0 \quad \Rightarrow \quad mt^2-2t-m=0 \quad \Rightarrow \quad D=4+4m^2>0$$

Thus, $f'(t)$ has two real and distinct root t_1 and t_2.

$$\Rightarrow \quad t_1+t_2=\frac{2}{m}, \quad t_1t_2=-1 \qquad (4)$$

Now,

$$f(t_1)f(t_2)=\left(mt_1^3-3t_1^2-3mt_1+1\right)\left(mt_2^3-3t_2^2-3mt_2+1\right)$$

which upon simplification yields (using (4))

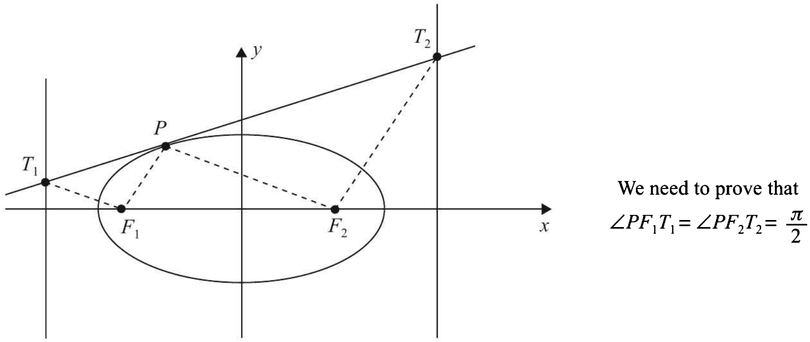
$$f(t_1)f(t_2)=-4m^2-\frac{4}{m^2}-2,$$

which is evidently always negative. Thus, the cubic in (3) will *always* give three real values of t, and hence three corresponding circles. ■

Example 28

Prove that the portion of the tangent to any ellipse intercepted between the curve and a directrix subtends a right angle at the *corresponding* focus.

Solution: The following diagram makes the phrase 'corresponding focus' clear:



Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and assume P to be $(a\cos\theta, b\sin\theta)$. F_1 and F_2 are $(-ae, 0)$ and $(ae, 0)$ respectively. The equation of the tangent at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

T_1 and T_2 can now be evaluated since we know their x-coordinates as $\frac{-a}{e}$ and $\frac{a}{e}$ respectively.

$$T_1 : x = \frac{-a}{e} \Rightarrow \quad y = \frac{b(e+\cos\theta)}{e\sin\theta}$$

$$\Rightarrow \quad T_1 \equiv \left(\frac{-a}{e}, \frac{b(e+\cos\theta)}{e\sin\theta}\right)$$

Similarly,

$$T_2 \equiv \left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

Now we evaluate the appropriate slopes:

$$m_{F_1T_1} = \frac{\frac{b(e+\cos\theta)}{e\sin\theta}}{ae - \frac{a}{e}} = \frac{-b(e+\cos\theta)}{a\sin\theta(1-e^2)}$$

$$m_{PF_1} = \frac{b\sin\theta}{a\cos\theta + ae} = \frac{b\sin\theta}{a(e+\cos\theta)}$$

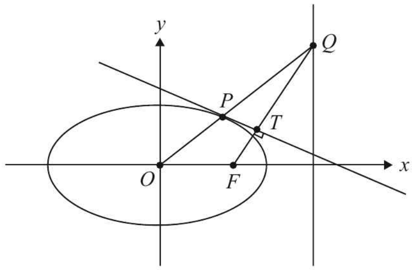
$$\Rightarrow \quad m_{F_1T_1} \times m_{PF_1} = \frac{-b^2}{a^2(1-e^2)} = \frac{-b^2}{b^2} = -1$$

which implies $F_1T_1 \perp PF_1$. Similarly, we can show that $F_2T_2 \perp PF_2$. Thus, the two intercepts subtend right angles at their *corresponding* foci.

Example 29

Prove that in any ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to point of contact meet on the corresponding directrix.

Solution: Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let a tangent be drawn to it at an arbitrary point $P(a\cos\theta, b\sin\theta)$ as shown:



We need to show that the perpendicular from F onto this tangent, i.e., FT, and the line joining the centre to the point of contact, i.e., OP, intersect on the corresponding directrix; in other words, we need to show that the x-coordinate of Q as in the figure above is $x=\frac{a}{e}$.

The equation of the tangent at P is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. The slope of this tangent is $m_T=\frac{-b}{a}\cot\theta$. Therefore, the slope of FT is

$$m_{FT}=\frac{a}{b}\tan\theta$$

The equation of FT is

$$FT: y=\left(\frac{a}{b}\tan\theta\right)(x-ae) \quad (1)$$

The equation of OP is simply

$$OP: y=\left(\frac{b}{a}\tan\theta\right)x \quad (2)$$

Solving (1) and (2) gives $x=\frac{a}{e}$, which proves the stated assertion. ■

Example 30

Consider three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $P(\theta_1)$, $Q(\theta_2)$ and $R(\theta_3)$. What is the area of ΔPQR? When is this area maximum?

Solution: The three points have the coordinates

$$P\equiv(a\cos\theta_1,\ b\sin\theta_1);\quad Q\equiv(a\cos\theta_2, b\sin\theta_2);\quad R\equiv(a\cos\theta_3, b\sin\theta_3)$$

The area of this triangle, by the determinant formula, is

$$\Delta=\frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1\\ a\cos\theta_2 & b\sin\theta_2 & 1\\ a\cos\theta_3 & b\sin\theta_3 & 1\end{vmatrix} \quad (1)$$

$$=\frac{ab}{2}\{\cos\theta_1(\sin\theta_2-\sin\theta_3)+\sin\theta_1(\cos\theta_3-\cos\theta_2)+(\cos\theta_2\sin\theta_3-\sin\theta_2\cos\theta_3)\}$$

$$=\frac{ab}{2}\{\sin(\theta_2-\theta_1)+\sin(\theta_1-\theta_3)+\sin(\theta_3-\theta_2)\}$$

$$= \frac{ab}{2}\left\{\sin(\theta_2-\theta_1)+2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)\right\}$$

$$= ab\sin\left(\frac{\theta_1-\theta_2}{2}\right)\left\{\cos\left(\theta_3-\frac{\theta_1+\theta_2}{2}\right)-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right\}$$

$$= 2ab\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)$$

This is the area of the triangle PQR. To find its maximum value, we use a rather indirect route. Suppose we had to calculate the area Δ' of a triangle inscribed in the circle $x^2+y^2=a^2$ with the same polar angles as P, Q, and R. The only difference between Δ' and Δ will be that in the determinant expression for Δ in (1), we will have all 'a' instead of 'b' in the terms of the second column. This means that Δ and Δ' will always be in a constant ratio:

$$\frac{\Delta}{\Delta'}=\frac{b}{a}$$

Thus, the maximum for Δ will be achieved in the same configuration as the one in which the maximum of Δ' be achieved ! Since the area of a triangle inscribed in a circle has the maximum value when that triangle is equilateral (this should be intuitively obvious but can also be easily proved), Δ' and hence Δ will be maximum when

$$|\theta_1-\theta_2| = |\theta_2-\theta_3| = |\theta_3-\theta_1| = \frac{2\pi}{3}$$

Thus, the three eccentric angles must be equally spaced apart at $\frac{2\pi}{3}$. ■

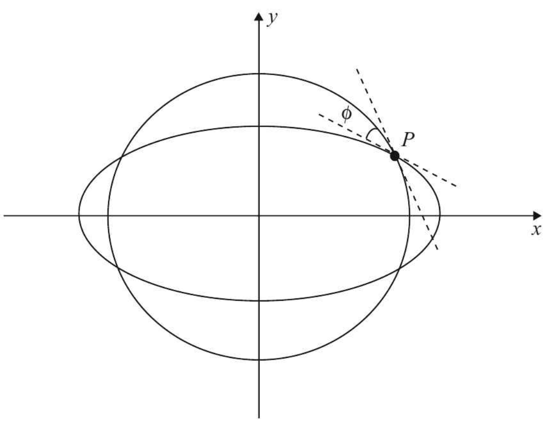
Example 31

Find the angle of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=ab$.

Solution: The semi-major and semi-minor axis of the ellipse are of lengths a and b respectively whereas the radius of the circle is $\sqrt{ab}$. Note that

$$b<\sqrt{ab}<a$$

Thus, the circle will intersect (symmetrically) the ellipse in four points.



We need to find ϕ.

Consider any point of intersection, say P, the one in the first quadrant. The coordinates of P can be assumed to be $(a\cos\theta, b\sin\theta)$. Since P also lies on the circle, we have

$$a^2\cos^2\theta + b^2\sin^2\theta = ab$$
$$\Rightarrow \quad (a^2 - b^2)\cos^2\theta = b(a-b)$$
$$\left.\begin{aligned} &\Rightarrow \quad \cos^2\theta = \frac{b}{a+b} \\ &\Rightarrow \quad \sin^2\theta = \frac{a}{a+b} \end{aligned}\right\} \qquad (1)$$

At P, the tangent to the ellipse has the slope

$$m_E = \frac{-b}{a}\cot\theta = -\left(\frac{b}{a}\right)^{\frac{3}{2}} \qquad \text{(Using (1))}$$

while the tangent to the circle has the slope

$$m_C = \frac{-a}{b}\cot\theta = -\left(\frac{a}{b}\right)^{\frac{1}{2}} \qquad \text{(Again, using (1))}$$

Thus, the angle of intersection is given by

$$\tan\phi = \left|\frac{m_C - m_E}{1 + m_C m_E}\right| = \left|\frac{\left(\frac{a}{b}\right)^{\frac{1}{2}} - \left(\frac{b}{a}\right)^{\frac{3}{2}}}{1 + \frac{b}{a}}\right| = \frac{a+b}{\sqrt{ab}}$$

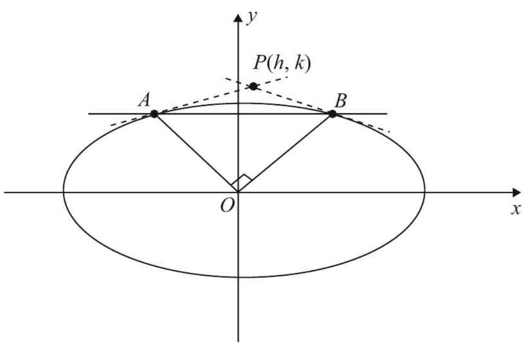
$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$$

■

Example 32

A variable chord AB of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtends a right angle at its centre. Tangents drawn at A and B intersect at P. Find the locus of P.

Solution: Let P be the point (h, k).



Since AB is the chord of contact for the tangents drawn from P, the equation of AB will be

$$T(h,k)=0$$

$$\Rightarrow \quad \frac{hx}{a^2}+\frac{ky}{b^2}=1 \tag{1}$$

We can now write the joint equation of OA and OB by homogenizing the equation of the ellipse using the equation of the chord AB obtained in (1):

$$\textbf{Joint equation of } \text{AB}\textbf{:} \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2 \tag{2}$$

Since, OA and OB are perpendicular, we must have (in (2))

$$\text{Coeff. of } x^2+\text{Coeff. of } y^2=0$$

$$\Rightarrow \quad \frac{1}{a^2}-\frac{h^2}{a^4}+\frac{1}{b^2}-\frac{k^2}{b^4}=0 \Rightarrow \quad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

The locus of P is therefore

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

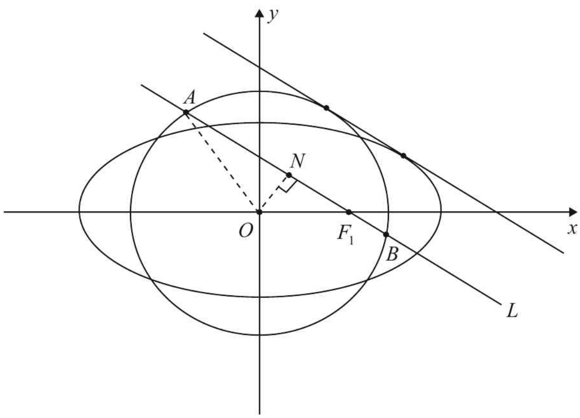
■

Example 33

A straight line L touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=r^2$ (where $b<r<a$). A focal chord of the ellipse parallel to L meets the circle in A and B. Find the length of AB.

Solution: L is a common tangent to the ellipse and the circle. We can assume the equation of L to be (using the form of an arbitrary tangent to an ellipse):

$$y=mx+\sqrt{a^2m^2+b^2}$$



Since L is a tangent to the circle too, its distance from (0, 0) must equal r. Thus,

$$\frac{\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}=r$$

$$\Rightarrow \quad m=\pm\sqrt{\frac{r^2-b^2}{a^2-r^2}} \tag{1}$$

The equation of AB (which passes through $F_1(ae, 0)$) can now be written as

$$y - 0 = m(x - ae)$$

$$\Rightarrow \quad mx - y = mae \qquad (2)$$

To evaluate the length AB, one alternative is to find the intercept that the circle $x^2 + y^2 = r^2$ makes on the line AB given by (2). However, a speedier approach would be to use the Pythagoras Theorem in ΔOAN. OA is simply the radius r. ON is the perpendicular distance of O from the line given by (2). Thus,

$$ON = \frac{mae}{\sqrt{m^2+1}}$$

Finally, we have

$$AB = 2AN = 2\sqrt{OA^2 - ON^2} = 2\sqrt{r^2 - \frac{m^2a^2e^2}{m^2+1}}$$

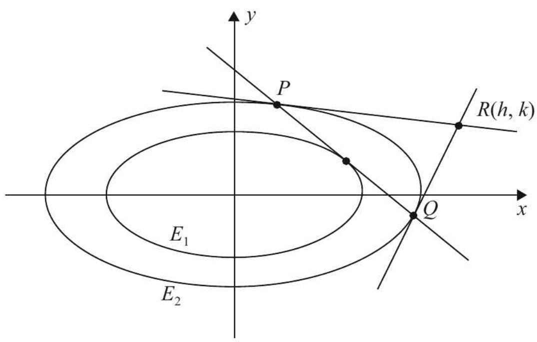
$$= 2\sqrt{r^2 - \left(\frac{r^2-b^2}{a^2-r^2}\right)\frac{a^2e^2(a^2-r^2)}{(a^2-b^2)}} = 2b$$

The length of the chord AB is equal to $2b$, the same as the minor axis of the ellipse. ■

Example 34

A tangent drawn to the ellipse $E_1: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersects the ellipse $E_2: \frac{x^2}{a} + \frac{y^2}{b} = a + b$ at P and Q. Prove that the tangents drawn to E_2 at P and Q intersect at right angles.

Solution: Let the point of intersection be $R(h, k)$.



PQ is the chord of contact for the tangents drawn from $R(h, k)$ to E_2. Thus, the equation of PQ is

$$T(h, k) = 0$$

$$\Rightarrow \quad \frac{hx}{a} + \frac{ky}{b} = a + b$$

$$\Rightarrow \quad y = \left(\frac{-bh}{ak}\right)x + \frac{b}{k}(a+b)$$

PQ touches the inner ellipse E_1 if the condition for tangency for ellipses $(c^2 = a^2m^2 + b^2)$ is satisfied. Thus,

$$\frac{b^2}{k^2}(a+b)^2 = a^2\left(\frac{b^2h^2}{a^2k^2}\right) + b^2$$

$$\Rightarrow \quad h^2 + k^2 = (a+b)^2 \qquad (1)$$

We also note that any point (h', k') lying on the director circle of E_2 must satisfy

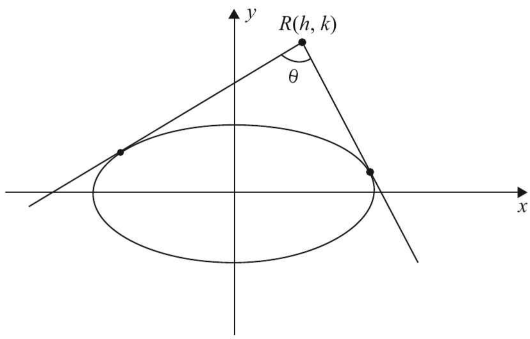
$$h'^2 + k'^2 = a(a+b) + b(a+b) \quad \text{(why?)}$$
$$= (a+b)^2 \tag{2}$$

From (1) and (2), it is evident that the point $R(h, k)$ itself lies on the director circle of E_2, and thus, by definition of a director circle, $\angle PRQ = 90°$. ■

Example 35

Find the locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are inclined at an angle θ.

Solution: Let $P(h, k)$ be the point of intersection; we need to find the locus of P.



Since we are dealing with the angle between the two tangents, it would be best to use the slope form for the tangent. Any arbitrary tangent of slope m to this ellipse can be written as

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If this tangent passes through $P(h, k)$, we have

$$k = mh + \sqrt{a^2m^2 + b^2}$$
$$\Rightarrow \quad (h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$$

As expected, a quadratic in m is formed, which will give two roots (both real if $P(h, k)$ is external to the ellipse) m_1 and m_2, where

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}, \quad m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2} \tag{1}$$

The angle θ is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \Rightarrow \quad (1 + m_1m_2)^2 \tan^2\theta = (m_1 - m_2)^2$$
$$= (m_1 + m_2)^2 - 4m_1m_2 \tag{2}$$

Using (1) in (2) and simplifying, we obtain a relation in h and k:

$$h^2 + k^2 = (a^2 + b^2) + 4\cot^2\theta(b^2h^2 + a^2k^2 - a^2b^2)$$

Thus, the locus of P is

$$x^2+y^2=(a^2+b^2)+4\cot^2\theta(b^2x^2+a^2y^2-a^2b^2) \tag{3}$$

From (3), we observe that the particular case of $\theta=\frac{\pi}{2}$ gives the locus of P as

$$x^2+y^2=a^2+b^2$$

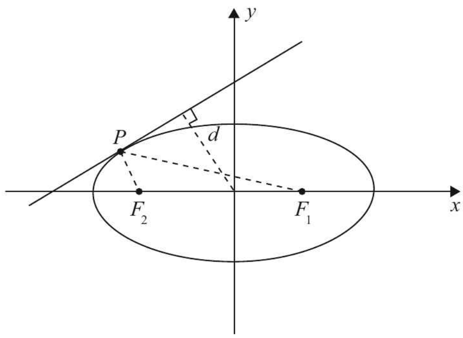
which is the equation of the director circle. ■

Example 36

Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, prove that

$$(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$$

Solution: Let P be the point $(a\cos\theta,\ b\sin\theta)$ whereas F_1 and F_2 are given by $(\pm ae, 0)$.



By definition, the focal distance of any point on an ellipse is e times the distance of that point from the corresponding directrix. Thus,

$$PF_1=\left|e\left(a\cos\theta-\frac{a}{e}\right)\right|=|ae\cos\theta-a|=a-ae\cos\theta$$

$$PF_2=e\left(a\cos\theta+\frac{a}{e}\right)=a+ae\cos\theta$$

$$\Rightarrow\quad (PF_1-PF_2)^2=4a^2e^2\cos^2\theta \tag{1}$$

Now, the equation of the tangent at P is

$$bx\cos\theta+ay\sin\theta-ab=0$$

The distance of $(0,0)$ from this tangent is d. Thus,

$$d=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\ \Rightarrow\ \frac{b^2}{d^2}=\sin^2\theta+\frac{b^2}{a^2}\cos^2\theta$$

$$\Rightarrow\quad 1-\frac{b^2}{d^2}=\left(1-\frac{b^2}{a^2}\right)\cos^2\theta=e^2\cos^2\theta$$

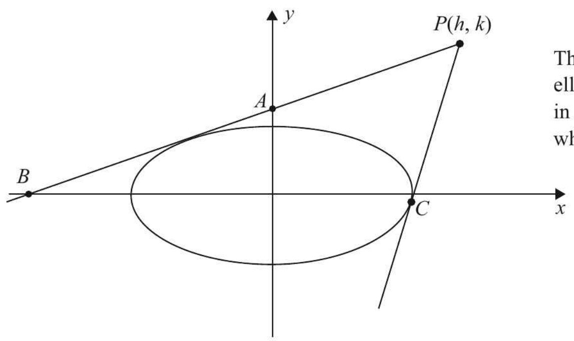
$$\Rightarrow\quad 4a^2\left(1-\frac{b^2}{d^2}\right)=4a^2e^2\cos^2\theta \tag{2}$$

From (1) and (2), we see that the equality stated in the question does indeed hold.

Example 37

Find the locus of the point P such that tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the coordinate axes in concyclic points.

Solution:



The tangents from $P(h, k)$ to the ellipse meet the coordinate axes in A, B, C, and D (not shown) which are concyclic.

The pair of tangents PA and PC has the joint equation

$$J_1 : T^2(h,k) = S(x,y)S(h,k)$$

$$\Rightarrow \quad J_1 : \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2 = 0$$

The coordinate axes has the joint equation

$$J_2 : xy = 0$$

We can treat J_1 and J_2 as two curves, which intersect in four different points A, B, C, D. Any second degree curve through these four points can be written in terms of a parameter λ as

$$J_1 + \lambda J_2 = 0$$

We now simply find that λ for which this represents a circle, since A, B, C, D are given to be concyclic. We have the parametric equation as

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2 + \lambda xy = 0$$

This represents a circle if

$$\textbf{Coeff. of } x^2 = \textbf{Coeff. of } y^2 \Rightarrow \quad \frac{k^2}{a^2b^2}-\frac{1}{a^2}=\frac{h^2}{a^2b^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad h^2 - k^2 = a^2 - b^2 \qquad (1)$$

$$\textbf{Coeff. of } xy = 0 \Rightarrow \quad \lambda = \frac{2hk}{a^2b^2} \qquad (2)$$

(1) itself gives the locus of $P(h, k)$ as

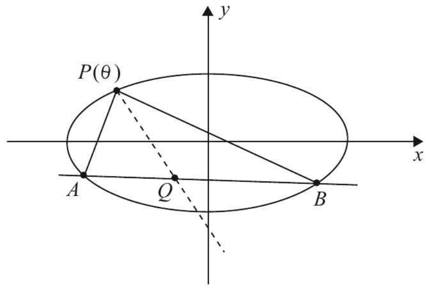
$$x^2 - y^2 = a^2 - b^2$$

■

Example 38

Through any arbitrary fixed point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, chords at right angles are drawn, such that the line joining the extremities of these chords meets the normal through P at the point Q. Prove that Q is fixed for all such chords.

Solution:



We can assume the eccentric angles of A and B as θ_1 and θ_2. The normal at P has the equation:

$$PQ: ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2 \tag{1}$$

The chord AB has the equation

$$AB: \frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{2}$$

Also, since $PA \perp PB$, we have

$$\underbrace{\frac{-b\cos(\frac{\theta+\theta_1}{2})}{a\sin(\frac{\theta+\theta_1}{2})}}_{\text{slope of chord } PA} \times \underbrace{\frac{-b\cos(\frac{\theta+\theta_2}{2})}{a\sin(\frac{\theta+\theta_1}{2})}}_{\text{slope of chord } PB} = -1$$

$$\Rightarrow \quad a^2\sin\left(\frac{\theta+\theta_1}{2}\right)\sin\left(\frac{\theta+\theta_2}{2}\right)+b^2\cos\left(\frac{\theta+\theta_1}{2}\right)\cos\left(\frac{\theta+\theta_2}{2}\right)=0$$

Using trignometric formulae, this expression can be rearranged to

$$\cos\left(\theta+\frac{\theta_1+\theta_2}{2}\right)=\frac{a^2+b^2}{a^2-b^2}\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{3}$$

From (2) and (3), we have

$$AB: \frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\frac{a^2-b^2}{a^2+b^2}\cos\left(\theta+\frac{\theta_1+\theta_2}{2}\right) \tag{4}$$

The point Q can now be obtained as the intersection of the lines represented by (1) and (4). Let us write them as a system and solve for Q using the Cramer's rule:

$$PQ: (a\sec\theta)x+(-b\operatorname{cosec}\theta)y+(b^2-a^2)=0$$

$$AB: (b\cos\alpha)x+(a\sin\alpha)y-\frac{ab(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha)=0$$

where $\alpha = \frac{\theta_1+\theta_2}{2}$ has been substituted for convenience. We now have

$$\frac{x}{\frac{ab^2(a^2-b^2)}{a^2+b^2}\operatorname{cosec}\theta\cos(\theta+\alpha) - a\sin\alpha(b^2-a^2)} = \frac{y}{b(b^2-a^2)\cos\alpha + \frac{a^2b(a^2-b^2)}{a^2+b^2}\sec\theta\cos(\theta+\alpha)}$$

$$= \frac{1}{a^2\sec\theta\sin\alpha + b^2\operatorname{cosec}\theta\cos\alpha}$$

$$\Rightarrow \quad x = \frac{\frac{ab^2(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha) - a\sin\theta\sin\alpha(b^2-a^2)}{\sin\theta(a^2\sec\theta\sin\alpha + b^2\operatorname{cosec}\theta\cos\alpha)}$$

$$= \frac{a(a^2-b^2)}{a^2+b^2}\cos\theta \qquad (5)$$

and $$y = \frac{b(b^2-a^2)\cos\alpha\cos\theta + \frac{a^2b(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha)}{\cos\theta(a^2\sec\theta\sin\alpha + b^2\operatorname{cosec}\theta\cos\alpha)}$$

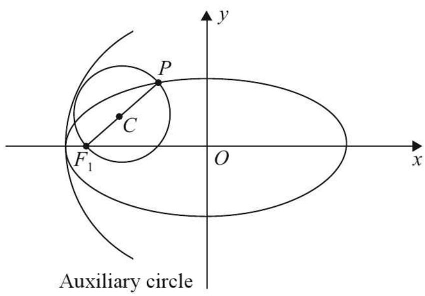
$$= \frac{b(b^2-a^2)\sin\theta}{a^2+b^2} \qquad (6)$$

Thus, the point Q, whose x and y coordinates are given by (5) and (6) respectively, can be seen to be independent of α or $\frac{\theta_1+\theta_2}{2}$. Q is the therefore fixed for such pairs of chords PA and PB and depends only on the eccentric angle of P. ■

Example 39

Prove that the circle on any focal distance as diameter touches the auxiliary circle of the ellipse.

Solution: Let $P(\theta)$ be an arbitrary point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let F_1 be one of its foci.



The radius of the auxiliary circle is a. The circle on PF_1 as diameter will touch the auxiliary circle (internally) if:

$$OC + (\text{radius of this circle}) = a$$

C, being the mid-point of PF_1, has the coordinates

$$C \equiv \left(\frac{a\cos\theta - ae}{2}, \frac{b\sin\theta}{2}\right)$$

Thus,

$$OC = \sqrt{\left(\frac{a\cos\theta - ae}{2}\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta + a^2e^2 - 2a^2e\cos\theta}$$

$$= \frac{a}{2}\sqrt{\cos^2\theta + (1-e^2)\sin^2\theta + e^2 - 2e\cos\theta}$$

$$= \frac{a}{2}\sqrt{1 + e^2\cos^2\theta - 2e\cos\theta} = \frac{a}{2}(1 - e\cos\theta)$$

Also, the radius of the inner circle is

$$CF_1 = \sqrt{\left(\frac{a\cos\theta - ae}{2} + ae\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$$

$$= \frac{a}{2}\sqrt{\cos^2\theta + (1-e^2)\sin^2\theta + e^2 + 2\cos\theta} = \frac{a}{2}(1 + e\cos\theta)$$

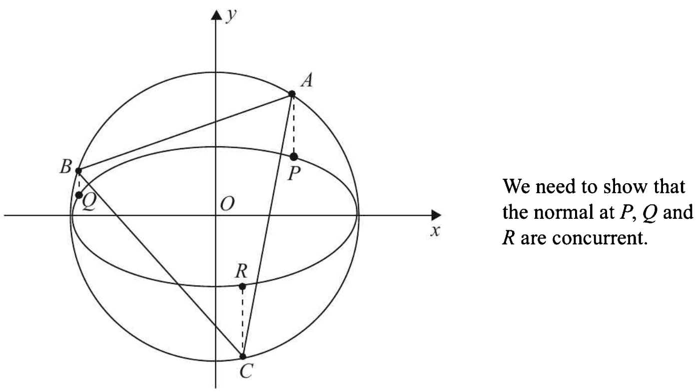
This gives

$OC + CF_1 = a$ which proves the stated assertion. ■

Example 40

Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b$) meet the ellipse respectively at P, Q, R so that P, Q and R lie on the same side of the major axis of A, B and C respectively. Prove that the normals to the ellipse at P, Q and R are concurrent.

Solution: The following figure portrays the situation described in the question :



The entire content of the the problem statement can be reduced to this single piece of significant information: the polar angles of A, B and C, and hence, the eccentric angles of P, Q and R, will be evenly spaced at $\frac{2\pi}{3}$, by virtue of ΔABC being equilateral. Thus, we can assume the eccentric angles of P, Q, and R to be θ, $\theta + \frac{2\pi}{3}$, $\theta - \frac{2\pi}{3}$. Now, the equation of a normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at eccentric angle ϕ is given by

$$ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2.$$

$$\Rightarrow \quad ax\sin\phi - by\cos\phi = \frac{(a^2 - b^2)\sin 2\phi}{2}.$$

Thus, the normals at P, Q, and R are respectively given by

$$N_P: ax\sin\theta - by\cos\theta = \frac{(a^2-b^2)\sin 2\theta}{2}$$

$$N_Q: ax\sin\left(\theta+\frac{2\pi}{3}\right) - by\cos\left(\theta+\frac{2\pi}{3}\right) = \frac{(a^2-b^2)}{2}\sin\left(2\theta+\frac{4\pi}{3}\right)$$

$$N_R: ax\sin\left(\theta-\frac{2\pi}{3}\right) - by\cos\left(\theta-\frac{2\pi}{3}\right) = \frac{(a^2-b^2)}{2}\sin\left(2\theta-\frac{4\pi}{3}\right)$$

Let us evaluate Δ, the determinant of the coefficients of these three equations:

$$\Delta = \frac{ab(a^2-b^2)}{2}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Using the row operation $R_1 \to R_1 + R_2 + R_3$, the first row reduces to zero, which means that

$$\Delta = 0$$

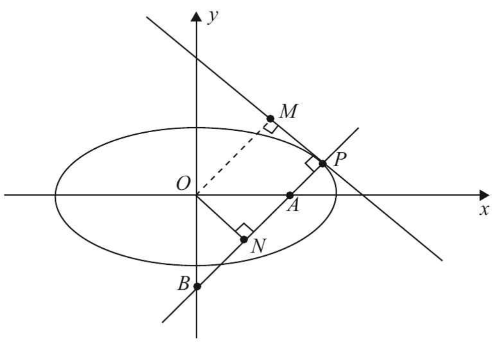
Thus, the normals at P, Q, and R must be concurrent. ■

Example 41

The normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major and minor axes at A and B respectively. ON is the perpendicular upon this normal from the centre O of the ellipse. Show that

$$PA\cdot PN = b^2 \quad \text{and} \quad PB\cdot PN = a^2$$

Solution:



Assume the point P to be $(a\cos\theta, b\sin\theta)$. The normal at P has the equation

$$(a\sec\theta)x - (b\operatorname{cosec}\theta)y = a^2 - b^2 \tag{1}$$

The coordinates of A are therefore

$$A \equiv \left(\frac{a^2-b^2}{a\sec\theta}, 0\right) \equiv (ae^2\cos\theta, 0)$$

Similarly, B is

$$B \equiv \left(0, \frac{b^2 - a^2}{b \operatorname{cosec} \theta}\right) \equiv \left(0, \frac{-a^2 e^2}{b} \sin\theta\right)$$

PA and PB can now be evaluated using the distance formula:

$$PA = \sqrt{(a\cos\theta - ae^2\cos\theta)^2 + (b\sin\theta)^2}$$

$$= \sqrt{\frac{b^4}{a^2}\cos^2\theta + b^2\sin^2\theta}$$

$$= \frac{b}{a}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \qquad (2)$$

$$PB = \sqrt{(a\cos\theta)^2 + \left(b\sin\theta + \frac{a^2e^2}{b}\sin\theta\right)^2}$$

$$= \frac{a}{b}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \qquad (3)$$

PN can be evaluated either using the perpendicular distance of O from the normal at $P\,(PN^2 = OP^2 - ON^2)$ or simply as the perpendicular distance of O from the tangent at P. The tangent at P has the equation

$$bx\cos\theta + ay\sin\theta - ab = 0$$

Thus,

$$PN = \frac{|ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \qquad (4)$$

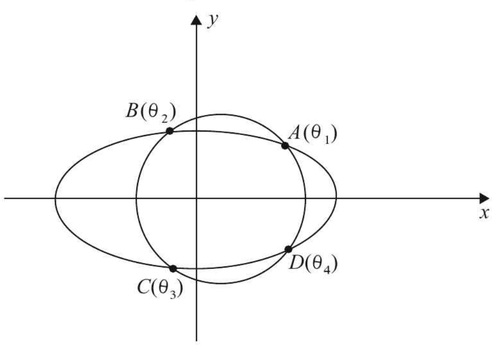
From (2), (3) and (4), we have

$$PA \cdot PN = b^2 \quad \text{and} \quad PB \cdot PN = a^2$$ ■

Example 42

A circle intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in four points A, B, C and D whose eccentric angles are respectively $\theta_1, \theta_2, \theta_3$ and θ_4. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4$ will be an integral multiple of 2π.

Solution: Suppose that the circle cuts the ellipse as shown:



Using the general equation of a chord joining two arbitrary points on an ellipse derived earlier, we can write $L_1 = 0$ and $L_2 = 0$, the equation of AB and CD respectively. Doing this has the advantage that we can now write (using a family of circles approach) any circle passing through the four point A, B, C and D as

$$S + \lambda L_1 L_2 = 0$$

where $S = 0$ is the equation of the ellipse. Imposing the necessary condition on λ for this equation to represent a circle will finally yield the constraint we are actually looking for. Thus, any curve through A, B, C, D has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda \underbrace{\left\{ \frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) - \cos\left(\frac{\theta_1-\theta_2}{2}\right) \right\}}_{\text{Equation of } AB} \underbrace{\left\{ \frac{x}{a}\cos\left(\frac{\theta_3+\theta_4}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_3+\theta_4}{2}\right) - \cos\left(\frac{\theta_3-\theta_4}{2}\right) \right\}}_{\text{Equation of } CD} = 0$$

This represents a circle if

$$\text{Coeff. of } x^2 = \text{Coeff. of } y^2$$

$$\text{Coeff. of } xy = 0$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{\lambda}{a^2}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_4}{2}\right) = \frac{1}{b^2} + \frac{\lambda}{b^2}\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_3+\theta_4}{2}\right)$$

$$\text{and} \quad \cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_3+\theta_4}{2}\right) + \cos\left(\frac{\theta_3+\theta_4}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}\right) = 0$$

The second relation gives

$$\sin\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right) = 0$$

$$\Rightarrow \quad \frac{\theta_1+\theta_2+\theta_3+\theta_4}{2} = n\pi, \quad n \in \mathbb{Z}$$

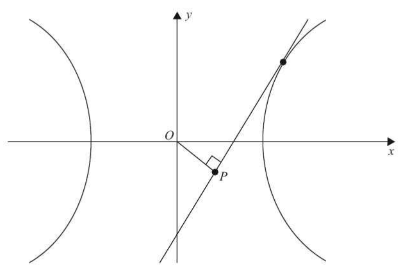
$$\Rightarrow \quad \theta_1+\theta_2+\theta_3+\theta_4 = 2n\pi, \quad n \in \mathbb{Z}$$

This is the desired result; it is quite significant and has a good amount of use. It would therefore be advantageous to remember this result. ■

Example 43

From the centre O of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, perpendicular OP is dropped upon any tangent to the hyperbola. Find the locus of P.

Solution:



Assume the coordinates of P to be (h, k). Any tangent to the given hyperbola has the form

$$y = mx + \sqrt{a^2m^2 - b^2}$$

Since P lies on this tangent, we have

$$k = mh + \sqrt{a^2m^2 - b^2} \qquad (1)$$

Also, since OP is perpendicular to this tangent, we have

$$\frac{k}{h} \times m = -1$$

$$\Rightarrow \quad m = -\frac{h}{k} \qquad (2)$$

Using (2) in (1), we obtain a relation in h and k:

$$(h^2 + k^2)^2 = a^2h^2 - b^2k^2$$

Thus, the locus of P is

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2$$

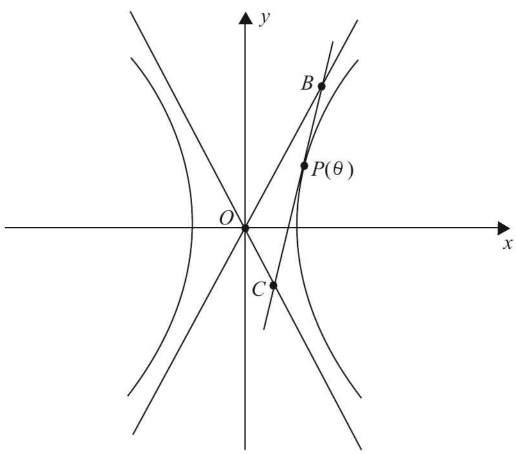
■

Example 44

On any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a tangent is drawn intersecting the asymptotes in B and C. Let O be the centre of the hyperbola. Prove that

(a) area (ΔOBC) is constant (b) $PB = PC$

Solution:



Assume P to be the point $(a\sec\theta, b\tan\theta)$. The equation of the tangent BPC is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \qquad (1)$$

The two asymptotes have the equation

$$y = \pm\frac{b}{a}x \qquad (2)$$

The points B and C can therefore be determined by simultaneously solving (1) and (2):

$$B \equiv (a(\sec\theta + \tan\theta),\ b(\sec\theta + \tan\theta)),\quad C \equiv (a(\sec\theta - \tan\theta),\ -b(\sec\theta - \tan\theta))$$

The area of ΔOBC can be evaluated using the determinant formula:

$$\Delta = \frac{1}{2}\begin{vmatrix} a(\sec\theta + \tan\theta) & b(\sec\theta + \tan\theta) & 1 \\ a(\sec\theta - \tan\theta) & -b(\sec\theta - \tan\theta) & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have

$$\Delta = \frac{1}{2}\left(2ab\left(\sec^2\theta - \tan^2\theta\right)\right) = ab$$

which is constant. That P is the mid-point of BC is straightaway evident by simple inspection of the coordinates of B and C. ■

Example 45

If the normals at the four points (x_i, y_i), $i = 1,2,3,4$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, find the values of $(\Sigma\, x_i)(\Sigma \frac{1}{x_i})$ and $(\Sigma\, y_i)(\Sigma \frac{1}{y_i})$.

Solution: The nature of the problem might be able to give you a hint that we must try to form a biquadratic equation in x (or y) to obtain the desired result. Let the point of concurrency of the four normals be $P(h,\ k)$. Now, the normal to the hyperbola at $(X,\ Y)$ has the form

$$\frac{a^2x}{X} + \frac{b^2y}{Y} = a^2 + b^2$$

If this passes through P, we have

$$\frac{a^2h}{X} + \frac{b^2k}{Y} = a^2 + b^2 \tag{1}$$

(1) must be satisfied by four values of $(X,\ Y)$, namely $(x_i,\ y_i)$, $i = 1,2,3,4$. Since $(X,\ Y)$ lies on the hyperbola, we have

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \tag{2}$$

From (1) and (2), it should be evident that we can form a biquadratic equation in either X or Y, by eliminating the other. Let us form a biquadratic in X. From (1), we have

$$\frac{b^2k}{Y} = a^2 + b^2 - \frac{a^2h}{X} = \frac{(a^2+b^2)X - a^2h}{X}$$

$$\Rightarrow\quad Y = \frac{b^2kX}{(a^2+b^2)X - a^2h} \tag{3}$$

Using (3) in (2), we have

$$\frac{X^2}{a^2} - \frac{b^4k^2X^2}{b^2\{(a^2+b^2)X - a^2h\}^2} = 1$$

$$\Rightarrow\quad (\lambda X - a^2h)^2X^2 - a^2b^2k^2X^2 = a^2\{\lambda X - a^2h\}^2 \quad \left[\begin{array}{l}\text{where } \lambda = a^2 + b^2 \text{ has} \\ \text{been substituted for} \\ \text{convenience}\end{array}\right]$$

$$\Rightarrow\quad \lambda^2X^4 - 2a^2h\lambda X^3 + (a^4h^2 - a^2b^2k^2 - a^2\lambda^2)X^2 + 2a^4h\lambda X - a^6h^2 = 0$$

This biquadratic yields four X values, namely x_1, x_2, x_3, x_4. We have

$$\sum x_i = \frac{2a^2h\lambda}{\lambda^2} = \frac{2a^2h}{\lambda}$$

$$\sum \frac{1}{x_i} = \frac{2a^4h\lambda}{a^6h^2} = \frac{2\lambda}{a^2h}$$

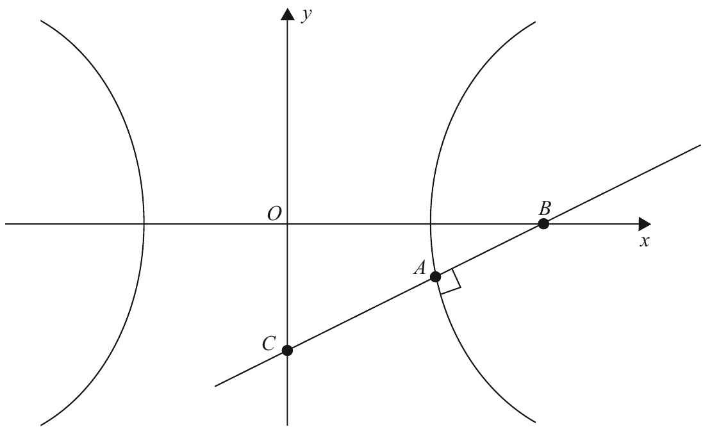
This immediately yields

$$\left(\sum x_i\right)\left(\sum \frac{1}{x_i}\right) = 4$$

We can analogously prove that the other expression also has a value of 4. ■

Example 46

The normal at a point A on a hyperbola of eccentricity e intersects its transverse and conjugate axes at B and C respectively, as shown in the figure below. Show that the locus of the mid-point of BC is a hyperbola of eccentricity $\frac{e}{\sqrt{e^2-1}}$.



Solution: Let the point A be $(a\sec\theta, b\tan\theta)$. Then the equation for the normal at A is

$$ax\sin\theta + by = (a^2+b^2)\tan\theta$$

The points B and C can easily be determined using this equation:

Point B: Put $y = 0 \Rightarrow \quad x = \dfrac{(a^2+b^2)}{a\cos\theta} \quad \Rightarrow \quad B \equiv \left(\dfrac{a^2+b^2}{a\cos\theta}, 0\right)$

Point C: Put $x = 0 \Rightarrow \quad y = \dfrac{(a^2+b^2)\tan\theta}{b} \quad \Rightarrow \quad C \equiv \left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$

Let the mid point of BC be $P(h, k)$, whose locus we wish to determine. We have,

$$h = \frac{(a^2+b^2)}{2a\cos\theta}, \quad k = \frac{(a^2+b^2)\tan\theta}{2b}$$

$$\Rightarrow \quad \sec\theta = \frac{2ah}{a^2+b^2}, \quad \tan\theta = \frac{2bk}{a^2+b^2}$$

Eliminating θ using the relation $\sec^2\theta - \tan^2\theta = 1$, we have

$$\frac{4a^2h^2}{(a^2+b^2)^2} - \frac{4b^2k^2}{(a^2+b^2)^2} = 1$$

Thus, the locus of P is

$$\frac{x^2}{\left(\dfrac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\dfrac{a^2+b^2}{2b}\right)^2} = 1$$

This is obviously a hyperbola, with eccentricity E given by

$$E^2 = 1 + \frac{(\frac{a^2+b^2}{2b})^2}{(\frac{a^2+b^2}{2a})^2} = 1 + \frac{a^2}{b^2} = 1 + \frac{1}{\frac{b^2}{a^2}}$$

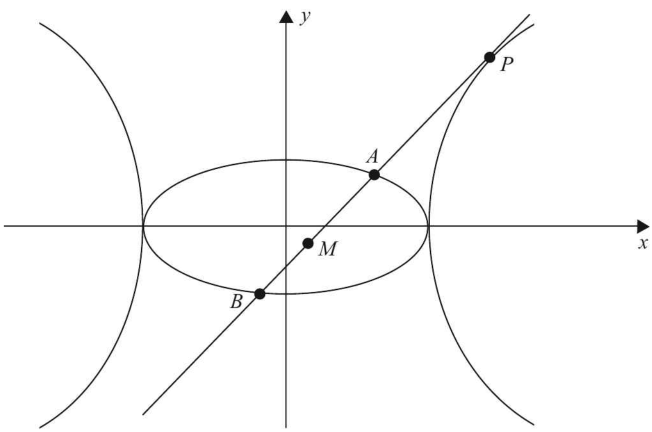
$$= 1 + \frac{1}{1+\dfrac{b^2}{a^2}-1} = 1 + \frac{1}{e^2-1} \qquad \left(\because\ e^2 = 1 + \frac{b^2}{a^2}\right)$$

$$= \frac{e^2}{e^2-1} \Rightarrow E = \frac{e}{\sqrt{e^2-1}}$$

As asserted, this is the eccentricity of the hyperbola described by P. ■

Example 47

A tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points A and B, as shown. Find the locus of the mid-point of AB.



Solution: We can assume $M(h,k)$ to be mid-point of the chord AB. Using the equation for the chord bisected at a given point in an ellipse, the equation of AB is

$$T(h,k) = S(h,k)$$

$$\Rightarrow \frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\Rightarrow y = \left(-\frac{b^2h}{a^2k}\right)x + \frac{b^2}{k}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

This is a tangent to the given hyperbola if the condition for tangency $(c^2 = a^2m^2 - b^2)$ is satisfied:

$$\frac{b^4}{k^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2 = a^2\left(-\frac{b^2h}{a^2k}\right)^2 - b^2$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2 = \frac{h^2}{a^2}-\frac{k^2}{b^2}$$

Thus, the locus of M is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2}-\frac{y^2}{b^2}$$ ■

Example 48

Show that an ellipse and a hyperbola having the same foci intersect orthogonally in four distinct points.

Solution: We can assume the equations of the ellipse and the hyperbola as follows:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(Ellipse)}$$

$$\frac{x^2}{c^2}-\frac{y^2}{d^2}=1 \quad \text{(Hyperbola)}$$

If we assume e_1 and e_2 to be their respective eccentricities, we have:

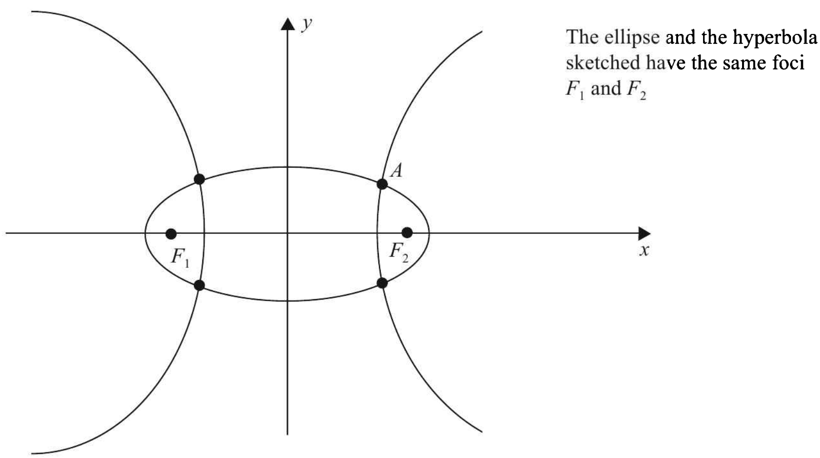
$$b^2 = a^2(1-e_1^2)$$

$$d^2 = c^2(e_2^2-1)$$

$$ae_1 = ce_2 \quad \text{(due to the foci being the same)}$$

$$\Rightarrow \quad d^2 = c^2\left(\frac{a^2e_1^2}{c^2}-1\right) = a^2e_1^2 - c^2$$

$$\Rightarrow \quad c^2+d^2 = a^2e_1^2 = a^2-b^2 \qquad (1)$$



To evaluate the angle(s) of intersection of the two curves, we must evaluate their respective derivatives at the points of intersection. By symmetry, it should be evident that the angles of intersection at all the four points will the same; thus we can focus our attention on any one point of intersection, say A. By solving the equations for the ellipse and the hyperbola simultaneously, A can be obtained:

$$A \equiv \left(\frac{ac\sqrt{b^2+d^2}}{\sqrt{b^2c^2+a^2d^2}}, \frac{bd\sqrt{a^2-c^2}}{\sqrt{b^2c^2+a^2d^2}} \right) = (\alpha, \beta) \left\{ \begin{array}{l} \alpha \text{ and } \beta \text{ are being} \\ \text{introduced just} \\ \text{for convenence} \end{array} \right\}$$

At A, the ellipse has a derivative given by

$$m_e = \left.\frac{dy}{dx}\right|_{A(\alpha,\beta)} = \left.\frac{-b^2x}{a^2y}\right|_{(\alpha,\beta)} = \frac{-b^2\alpha}{a^2\beta}$$

Similarly, at A, the hyperbola has a derivative given by

$$m_h = \left.\frac{dy}{dx}\right|_{A(\alpha,\beta)} = \left.\frac{d^2x}{c^2y}\right|_{(\alpha,\beta)} = \frac{d^2\alpha}{c^2\beta}$$

Now, we evaluate $m_e\, m_h$:

$$\begin{aligned} m_e\, m_h &= \frac{-b^2d^2}{a^2c^2} \times \frac{\alpha^2}{\beta^2} \\ &= \frac{-b^2d^2}{a^2c^2} \times \frac{a^2c^2}{b^2d^2} \times \frac{b^2+d^2}{a^2-c^2} \\ &= -1 \quad \text{(Using (1))} \end{aligned}$$

This implies that the two curves intersect orthogonally. ■

Example 49

For any triangle inscribed in a rectangular hyperbola, prove that its orthocentre lies on the hyperbola.

Solution: Assume the hyperbola to be $xy = c^2$ and assume any three points $A(t_1), B(t_2)$ and $C(t_3)$ on it. The equation of AB is

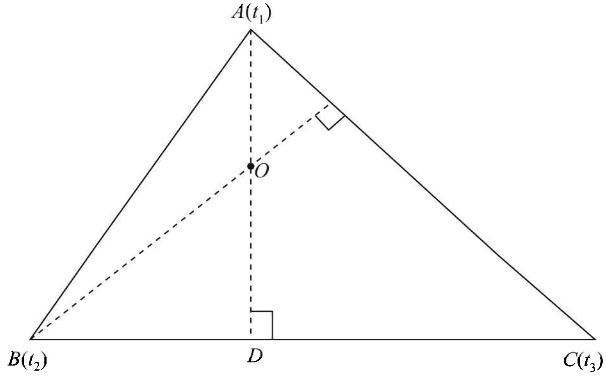
$$\frac{y - \dfrac{c}{t_2}}{x - ct_2} = \frac{\dfrac{c}{t_2} - \dfrac{c}{t_1}}{ct_2 - ct_1}$$

$$\Rightarrow \quad x + yt_1t_2 = c(t_1 + t_2)$$

Similarly, the equations of BC and CA are

$$BC : x + y\,t_2t_3 = c(t_2 + t_3)$$

$$CA : x + y\,t_3t_1 = c(t_3 + t_1)$$



We now find any two of the altitudes, say AD and BE, and find their point of intersection O:

$$AD : y - xt_2t_3 = \frac{c}{t_1} - ct_1t_2t_3 \tag{1}$$

$$BE : y - xt_1t_3 = \frac{c}{t_2} - ct_1t_2t_3 \tag{2}$$

The point of intersection O of (1) and (2) can easily be found :

$$O \equiv \left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3 \right)$$

From the coordinates of O, it should be clear that O lies on the hyperbola. ■

Conic Sections

PART-C: Advanced Problems

P1. Consider a parabola P given in parametric form:

$$x = ut\cos\alpha \qquad \{t \text{ is the parameter}\}$$

$$y = ut\sin\alpha - \frac{1}{2}gt^2$$

What is the distance of the directrix of P from the origin?

(A) $\frac{u^2}{2g}$ (B) $\frac{u^2}{g}$ (C) $\frac{2u^2}{g}$ (D) None of these

P2. An infinite rod is hinged at the focus of the parabola $y^2 = 4x$ so that it can rotate freely about the focus. The only impediment to its rotation is the parabola $x^2 = 4(y-1)$. What is the maximum angle through which it can rotate?

(A) 210° (B) 240° (C) 270° (D) 300° (E) None of these

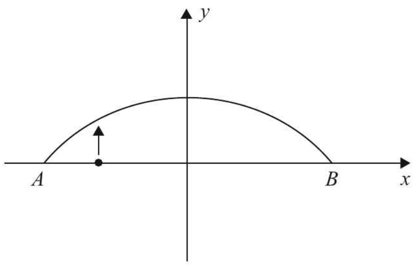
P3. Consider a parabola $y^2 = 4ax, a > 0$, and a variable chord $y = mx$ of the parabola. What is the value of m for which the area enclosed between the chord and the parabola lying left to the line $x = m$ is maximum?

(A) $(2a)^{1/3}$ (B) $(4a)^{1/3}$ (C) $2a^{1/3}$ (D) None of these

P4. Let O be the origin and A be the (non-origin) point of intersection of $y = 4x^2$ and $y^2 = ax$. Let S be the region bounded between these two curves. What are the possible values of a such that OA divides S into two parts of equal area?

(A) $a > 0$ (B) $a > 1$ (C) $a > 2$ (D) None of these

P5. A curve AB is given by $y = a - x^2$ in the region $y > 0$. From a point on the line segment AB, light is shone vertically onto the parabola.



What is the value of a for which the path of light resembles an M-shaped path after successive reflections between the parabola and the x-axes?

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) None of these

P6. A series of chords is drawn to the parabola $y^2 = 4ax$, so that their projections on a straight line which is inclined at an angle α to the axis of the parabola are all of constant length p. The locus of their mid-point(s) is specified as follows:

$$(y^2 - 4ax)\,(f(y))^2 + a^2p^2 = 0$$

The function $f(y)$ is given by

(A) $y\sin\alpha + 2a\cos\alpha$ (B) $y\sin\alpha + a\cos\alpha$

(C) $y\cos\alpha + 2a\sin\alpha$ (D) $y\cos\alpha + a\sin\alpha$

P7. (a) Given a circle $C: x^2 + y^2 = 4$ and a parabola $P: y^2 = 4ax$, what is the maximum value of a for which there exists a point from where perpendicular tangents can be drawn to C and P?

(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $3\sqrt{2}$ (D) None of these

(b) For that a, what is the angle between the common tangents to C and P?

(A) 30° (B) 60° (C) 90° (D) None of these

P8. A tangent is drawn at any point (x_1, y_1) on the parabola $y^2 = 4ax$. From any point on this tangent, tangents are now drawn to the circle $x^2 + y^2 = a^2$. The chord of contacts pass through the fixed point (x_2, y_2). It is further given that

$$\left(\frac{y_1}{y_2}\right)^2 = -\lambda\left(\frac{x_1}{x_2}\right), \quad \lambda > 0$$

The value of λ is

(A) 1 (B) 2 (C) 4 (D) 8

P9. Given a parabola $P_1: y^2 = 4ax$, another parabola P_2 is drawn having the same axis and the same latus rectum as P_1. At the point of intersection P of P_1 and P_2 in the first quadrant, tangents are drawn to P_1 and P_2, which meet the x-axis in Q and R. What is the ratio of the area of ΔPQR to the area enclosed between P_1 and P_2 for $y > 0$?

(A) $\sqrt{2}:1$ (B) 3:2 (C) 4:3 (D)2:1 (E) None of these

P10. What is the length of the normal chord of the parabola $y^2 = 4ax$ which subtends an angle of 90° at the origin?

(A) $4\sqrt{3}\,a$ (B) $5\sqrt{3}\,a$ (C) $6\sqrt{3}\,a$ (D) $8\sqrt{3}\,a$ (E) None of these

P11. If normals at two points P and Q on the parabola $y^2 = 4ax$ intersect on the parabola itself, the locus of the point of intersection of the tangents at P and Q is

(A) a straight line (B) a circle (C) a parabola, ellipse or a hyperbola (D) none of these

P12. (a) Prove that from the point $P(at^2, 2at)$, two normals can be drawn to the parabola $y^2 = 4ax$ which are normals to the parabola at points Q and R that are different from P.
(b) Which of the following equations will the parameters of the feet of these normals, *i.e.*, Q and R satisfy?

(A) $\lambda^2 + \lambda t + 2 = 0$ (B) $\lambda^2 + 2\lambda t + 3 = 0$ (C) $\lambda^2 - 2\lambda t + 3 = 0$ (D) $\lambda^2 - \lambda t + 2 = 0$

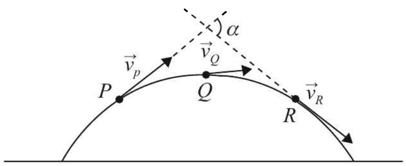
P13. (a) A parabola of latus rectum l touches a fixed parabola of the same size, the axes of the two curves being parallel. Prove that the locus of the vertex of the moving parabola is another parabola.
(b) What is the length of the latus rectum of this parabola?

(A) $\sqrt{2}\,l$ (B) $2\,l$ (C) $2\sqrt{2}\,l$ (D) None of these

P14. A gun can fire a shell with a (fixed) velocity u at a (variable) elevation α.
(a) Find the farthest distance at which an aeroplane at a height h can be hit by the gun
(b) The value of the angle of elevation in this case is

(A) $\dfrac{u}{\sqrt{u^2 - gh}}$ (B) $\dfrac{2u}{\sqrt{u^2 - gh}}$ (C) $\dfrac{u}{\sqrt{u^2 - 2gh}}$ (D) $\dfrac{2u}{\sqrt{u^2 - 2gh}}$

P15. Consider a particle in projectile motion, at three points P, Q and R in its flight:



Q is the highest point in flight.

The tangents at P and R intersect at angle α. The time of flight between P and R is

(A) $\dfrac{v_P v_R}{v_Q}\dfrac{\cos\alpha}{g}$ (B) $\dfrac{v_P v_R}{v_Q}\dfrac{\sin\alpha}{g}$ (C) $\dfrac{v_P v_R}{v_Q}\dfrac{\cos\alpha}{2g}$ (D) $\dfrac{v_P v_R}{v_Q}\dfrac{\sin\alpha}{2g}$

P16. If S_1 and S_2 are the foci of an ellipse and P a point on it, then what is the value of

$$\tan\left(\frac{1}{2}\angle PS_1S_2\right)\tan\left(\frac{1}{2}\angle PS_2S_1\right)$$

in terms of the eccentricity of the ellipse?

(A) $\sqrt{\dfrac{1-e^2}{1+e^2}}$ (B) $\dfrac{1-e}{1+e}$ (C) $\sqrt{\dfrac{1+e^2}{1-e^2}}$ (D) None of these

P17. Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a circle $x^2 + y^2 = r^2$ $(b < r < a)$, what is the minimum area of the quadrilateral formed by the common tangents to the circle and the ellipse?

(A) $2ab$ (B) $4ab$ (C) $8ab$ (D) None of these

P18. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. The angle between the tangents to the second ellipse at P and Q in degrees is

(A) 60 (B) 75 (C) 90 (D) 120

P19. What is the angle that the common tangent of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{c}=0 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}+\frac{2x}{c}=0$$

subtends at the origin?

(A) $\tan^{-1}\frac{a}{b}$ (B) $\tan^{-1}\frac{b}{a}$ (C) $\tan^{-1}\sqrt{ab}$ (D) None of these

P20. If the normals at $\alpha,\beta,\gamma,\delta$ on the ellipse are concurrent, the value of $(\Sigma\cos\alpha)(\Sigma\sec\alpha)$ is

(A) 2 (B) 4 (C) 8 (D) 16

P21. A point P moves such that the ratio of its distance from a fixed point to a fixed line is $\sqrt{a}\sin\frac{\pi}{a}$, where a is determined by the outcome of rolling a fair die. What is the probability that the locus of P is a hyperbola, given that the locus is not a parabola?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 1 (D) None of these

P22. The curve represented by the equation

$$\frac{x^2}{\sin\sqrt{2}-\sin\sqrt{3}}+\frac{y^2}{\cos\sqrt{2}-\cos\sqrt{3}}$$

is

(A) an ellipse with the foci on the x-axis (B) a hyperbola with the foci on the x-axis
(C) an ellipse with the foci on the y-axis (D) a hyperbola with the foci on the y-axis

P23. Consider the family of conic curves $x^2\cos\theta+y^2\sin\theta=1$, where $\theta=\frac{(k-1)\pi}{12}$ and k is determined by summing the output of two fair dice rolled simultaneously. What is the conditional probability that the curve is an ellipse, given that its eccentricity is in the range $\mathbb{R}^+\setminus\{1\}$?

(A) $\frac{1}{3}$ (B) $\frac{4}{9}$ (C) $\frac{5}{9}$ (D) $\frac{2}{3}$ (E) None of these

P24. The coordinate plane is constrained by the hyperbola $x^2-y^2=4$ so that only the region exterior of the hyperbola is accessible. What are the possible values of the ordinate of the center of a circular disc of radius 4 whose center lies on the y-axis?

(A) $(-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)$ (B) $(-\infty,-2\sqrt{3}]\cup[2\sqrt{3},\infty)$
(C) $(-\infty,-\sqrt{6}]\cup[\sqrt{6},\infty)$ (D) $(-\infty,-2\sqrt{6}]\cup[2\sqrt{6},\infty)$

P25. A variable straight line is such that the algebraic sum of perpendiculars from the points of intersection of the ellipse $2x^2+y^2=2$ and hyperbola $2x^2-4y^2=1$ is 0. What is the sum of the coordinates of the fixed point through which the straight line always passes?

(A) 0 (B) 1 (C) 2 (D) 3

P26. (a) If the lines $2x+y+5=0$ and $2px+y+1=0$ form the asymptotes of a hyperbola, the possible values of p are

(A) $\mathbb{R}\setminus\{1\}$ (B) $\mathbb{R}\setminus\{2\}$ (C) $\mathbb{R}\setminus\{3\}$ (D) $\mathbb{R}\setminus\{4\}$

(b) If such a hyperbola passes through (0, 1), find its equation.

P27. The locus of a point P from which tangents drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are inclined to each other at a constant angle θ is specified as

$$(x^2+y^2+b^2-a^2)^2=\mu(a^2y^2-b^2x^2+a^2b^2)$$

The value of μ is

(A) $2\tan^2\alpha$ (B) $4\tan^2\alpha a$ (C) $2\cot^2\alpha$ (D) $4\cot^2\alpha$

P28. The normal at P_1 to the hyperbola $xy=c^2$ meets the curve again at P_2; the normal at P_2 meets the curve again at P_3 and so on. We know that any point on this hyperbola can be specified in parametric form as $(ct,\frac{c}{t})$. The parameter of P_n in terms of the parameter of P_1 is

(A) $t_n=(-1)^n(t_1)^{(-3)^{n-1}}$ (B) $t_n=(-1)^{n-1}(t_1)^{(-3)^{n-1}}$

(C) $t_n=(-1)^n(t_1)^{(-3)^n}$ (D) $t_n=(-1)^{n-1}(t_1)^{(-3)^n}$

P29. From points on the circle $x^2+y^2=a^2$, tangents are drawn to the hyperbola $x^2-y^2=a^2$. The locus of the mid-points of the chords of contact is

(A) $(x^2-y^2)^2=2a^2(x^2+y^2)$ (B) $(x^2+y^2)^2=2a^2(x^2-y^2)$

(C) $(x^2-y^2)^2=a^2(x^2+y^2)$ (D) $(x^2+y^2)^2=a^2(x^2-y^2)$

P30. A variable circle S of radius r cuts a rectangular hyperbola H in four distinct points A, B, C and D. If O is the 'centre' of H, the value of

$$\lambda=\frac{OA^2+OB^2+OC^2+OD^2}{r^2} \text{ is}$$

(A) 4 (B) 8 (C) 16 (D) 32

SUBJECTIVE TYPE EXAMPLES

P31. Find the maximum and minimum distance of the conic $ax^2+2hxy+by^2=1$ from the origin.

P32. Consider the parabola $y^2=4ax$. From a point $P(h, k)$ in the plane, the line drawn parallel to the x-axis meets the y-axis in Q and the parabola in R, such that the area of ΔOPQ equals the area bounded between the parabola, the y-axis and QR. Find the locus of P.

P33. (a) Let C_1 and C_2 be, respectively, the parabolas $x^2=y-1$ and $y^2=x-1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line $y=x$. Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min\{PP_1, QQ_1\}$.
(b) Hence or otherwise, determine points P_0 and Q_0 on the parabolas C_1 and C_2 respectively such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2.

P34. Prove that on the axis of any parabola there is a point P which has the property that if a chord through P meets the parabola at A and B, then

$$\frac{1}{AP^2}+\frac{1}{BP^2}$$

is the same for all such chords. Find this point.

P35. A circle $x^2+y^2=r^2$ and a parabola $y^2=4ax$ have their radius and latus-rectum (respectively) varying sinusoidally with time. r varies from $\frac{1}{2}$ to $\frac{3}{2}$ and a from 1 to 2. Initially, both r and a have their minimum respective values. Also, the sinusoidal variations for both parameters have the same time period. Find the time after which the common tangents to the circle and parabola are perpendicular.

P36. Let P be any point on the parabola $y^2=4ax$ between its vertex and extremity of the latus rectum (with positive y coordinate). M is the foot of the perpendicular from the focus S to the tangent at P. What is the maximum value of the area of ΔSPM ?

P37. Find the length and angle of inclination of the shortest normal of the parabola $y^2=4ax$.

P38. Prove that the parabolas $y^2=4ax$ and $y^2=4c(x-b)$ cannot have a common normal other than the x-axis unless

$$\frac{b}{a-c}>2$$

P39. Prove that the normals at the extremities of each of a series of parallel chords of a parabola intersect on a fixed line, which itself is a normal to the parabola.

P40. PR and QR are chords of a parabola which are normals at P and Q. Prove that two of the common chords of the parabola and the circle circumscribing the triangle PQR meet on the directrix.

P41. Find the centre and radius of the smaller of the two circles that touch the parabola $75y^2=64(5x-3)$ at the point $P\left(\frac{6}{5}, \frac{8}{5}\right)$ and the x-axis.

P42. If v_1 and v_2 are the velocities of a particle at two points P and Q respectively on a parabolic trajectory, and PT and QT the lengths of the corresponding tangent-segments (T being the point of intersection of the two tangents), find the ratio $v_1 : v_2$ in terms of PT and QT.

P43. A particle is projected with a velocity $2\sqrt{ga}$ so that it just clears two walls of equal height a at a distance $2a$ apart. Find
(a) the length of the latus rectum of the flight path.
(b) the time taken by the particle to pass between the walls.

P44. If θ_1, θ_2 and θ_3 be the eccentric angles of three points on an ellipse, the normals at which are concurrent, then is the following statement always true?

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$$

P45. Tangents are drawn from a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the circle $x^2+y^2=r^2$. Prove that the chords are tangents to a fixed ellipse, and find the equation of this ellipse.

P46. An ellipse slides between two lines at right angles to one another. Find the locus of its center.

P47. Two tangents to a given ellipse intersect at right angles. Prove that the sum of the squares of the chords which the auxiliary circle intercepts on these tangents, is a constant.

P48. PS_1Q and PS_2R are two focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where S_1 and S_2 are the foci. The tangents at Q and R meet at T. Show that the locus of T as P moves around the ellipse, is another ellipse, and find its equation.

P49. Two sides of a triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are parallel to two given straight lines. Prove that its third side touches a certain ellipse.

P50. From any point P on the ellipse $a^2x^2+b^2y^2=(a^2+b^2)^2$, tangents are drawn to the ellipse $b^2x^2+a^2y^2=a^2b^2$. If the contact points are A, B, show that the orthocentre of ΔPAB lies on an ellipse, and find its equation.

P51. If the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are concurrent, then show that

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix}=0$$

P52. Chords at right angles are drawn through any point $P(\alpha)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The line joining the extremities of the chords meets the normal through P, at the point Q. Prove that Q is fixed, and find its coordinates.

P53. An ellipse intersects the parabola $x^2=4y$ orthogonally at the end points of the latus rectum of the parabola, and its axes are along the coordinate axes. Through one of the intersection points P, normals to the two curves at that point are drawn to intersect the x-axis at Q and R. Find the (a) centroid and (b) circumcenter of ΔPQR.

P54. A triangle with angles 90°, 15°, 75° is placed such that its right angled vertex lies on the positive y-axis and its two sides touch the ellipse $x^2+3y^2=3$, while the hypotenuse touches the auxiliary circle of that ellipse. Find the centroid and circumcenter of the triangle.

P55. A point moves such that the sum of the square of its distances from two fixed straight lines intersecting at angle 2α is a constant. Prove that the locus is an ellipse, and find its eccentricity.

P56. Given the base of a triangle and the sum of its sides, find the locus of the incentre of the triangle.

P57. With a given point and line as focus and directrix, a series of ellipses are described. Find the locus of the extremities of their minor axes.

P58. A perfectly rough plane is inclined at an angle α to the horizontal. Find the least eccentricity of an ellipse which can rest on the plane.

P59. Show that the tangents drawn from any point on the circle $x^2 + y^2 = a^2 + b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equally inclined to the asymptotes of the hyperbola $x^2 - y^2 = a^2 - b^2$

P60. If the chord joining points the $A(\alpha)$ and $B(\beta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a focal chord, then prove that

(a) $\pm e \cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

(b) $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) + \frac{ke-1}{ke+1} = 0$ where $k = \pm 1$

P61. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid point of the chord of contact.

P62. Find the equations and the length of the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

P63. What are the possible values of λ if two tangents can be drawn to the different branches of the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (λ, λ^2)?

P64. A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola and one of its diagonals is a chord of the hyperbola. Show that the other diagonal passes through the centre of the hyperbola.

P65. (a) Consider a rectangular hyperbola H and a circle C intersecting H at four points P, Q, R and S. Let O_H and O_C denote the centers of H and C respectively. If X is the mean-point of P, Q, R, S, and Y is the mid-point of O_H and O_C, find the length XY in terms of the appropriate parameters.

(b) Five points are selected on a circle of radius r. Consider the centers of the five rectangular hyperbolas, each passing through four of these five points.

(i) Show that these centers lie on a circle.

(ii) Find the radius of this circle.

P66. A variable circle through the origin and touching the hyperbola $xy = c^2$ at some point, cuts the hyperbola again at two distinct points A and B. Find the locus of the foot of the perpendicular drawn from the origin to the line AB.

P67. From a point P on a rectangular hyperbola, a tangent is drawn which intersects the asymptotes of the hyperbola at A and B. Show that the area of the triangle OAB (O is the centre of the hyperbola) is a constant, independent of the position P.

P68. A triangle is inscribed in a rectangular hyperbola such that the tangent at one of the vertices is perpendicular to the opposite side. Prove that the triangle is right-angled at the same vertex.

P69. The axis of a parabola and a rectangular hyperbola is the same and the vertex of the parabola is the same as the centre of the hyperbola. Prove that the locus of the point whose chord of contact with respect to the parabola touches the rectangular hyperbola, is an ellipse with the same centre and axis.

Conic Sections

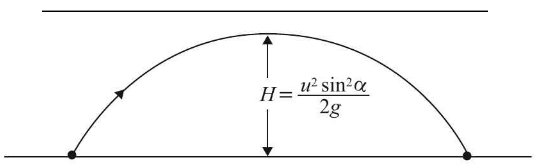
PART-D: Important Ideas and Tips

S1. The equation of the 'trajectory', after eliminating the variable t is

$$y = x\tan\alpha - \frac{g x^2}{2u^2\cos^2\alpha}$$

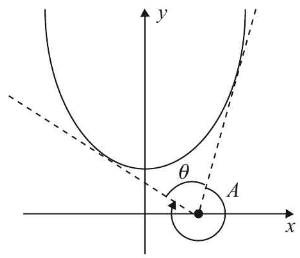
$$\Rightarrow \quad 4\lambda y = (x-\mu)^2$$

where $\quad \lambda = \dfrac{u^2\cos^2\alpha}{2g}$.



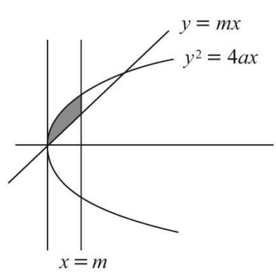
Thus, the directrix is at a distance of $\frac{u^2\sin^2\alpha}{2g} + \frac{u^2\cos^2\alpha}{2g} = \frac{u^2}{2g}$ from the origin. The correct option is (A).

S2. The focus of $y^2 = 4x$ is (1,0).



We only need to find θ, the angle between the tangents at the two sides. But note that since the x-axis is the directrix for the parabola $x^2 = 4(y-1)$,the tangents must be perpendicular. Thus, $\theta = 90°$,which means that the rod can rotate through a maximum angle of 270°. The correct option is (C).

S3. The required area is $A(m) = \int\limits_0^m (\sqrt{4ax} - mx)dx$



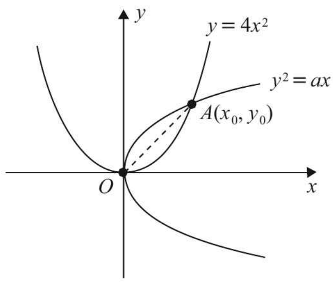
Using the Leibnitz rule,

$$\frac{dA(m)}{dm} = \sqrt{4am} - m^2 = 0$$

$$\Rightarrow \quad m = (4a)^{1/3}$$

The correct option is (B).

S4. A is given by $(x_0, y_0) = (a^{1/3}2^{-4/3}, a^{2/3}\ 2^{-2/3})$. Thus,



$$\int_0^{x_0} (\sqrt{ax} - \lambda x)dx = \int_0^{x_0} (\lambda x - 4x^2)dx$$

where $\lambda = a^{1/3}2^{2/3}$ is the slope of OA.

$$\Rightarrow \int_0^{x_0} (\sqrt{ax} + 4x^2 - 2\lambda x)dx = 0$$

$$\Rightarrow \left(\frac{2}{3}\sqrt{a}x^{3/2} + \frac{4}{3}x^3 - \lambda x^2\right)\Bigg|_0^{x_0} = 0$$

$$\Rightarrow \frac{a}{6} + \frac{a}{12} = \frac{a}{4}, \text{ which always holds.}$$

Thus, any value of a greater than 0 will do. The correct option is (A).

S5. The origin (0, 0) must be the focus of the parabola $y = a - x^2$, or

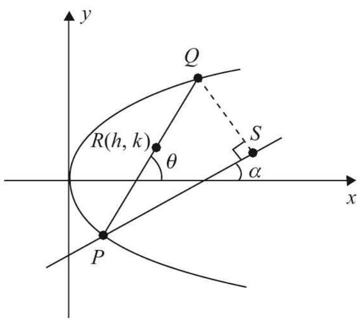
$$x^2 = -4\left(\frac{1}{4}\right)(y - a).$$

The distance from the vertex $(0, a)$ to the focus (0, 0) must be $\frac{1}{4}$, and so

$$a = \frac{1}{4}$$

The correct option is (B).

S6. You may find this problem somewhat complicated in terms of the calculations involved, but the solution broadly consists of the following sequence of simple steps.



Step 1: Assuming that the mid-point of PQ is $R(h, k)$, the equation of PQ is $'T = S_1'$ *i.e.*,

$$y = \left(\frac{2a}{k}\right)x + \left(\frac{k^2 - 2ah}{k}\right) = mx + c \text{ (say)}$$

where we have used m and c for convenience.

Step 2: Evaluate the length $PQ = l$ in terms of m and c, by solving the equation of the parabola with that of the line PQ. This comes out to

$$l^2 = \frac{16a(1+m^2)\,(a-mc)}{m^4} \tag{1}$$

Step 3: Since $PS = p$, while the angle between PQ and PS is $\theta - \alpha$ (where $\tan\theta = m$), we have

$$l^2 = p^2\sec^2(\theta-\alpha) = \frac{p^2\sec^2\alpha(1+m^2)}{(1+m\tan\alpha)^2} \tag{2}$$

Step 4: Compare (1) and (2), and substitute $m = \frac{2a}{k}$, $c = \frac{k^2-2ah}{k}$ to obtain an equation in h and k, which after some simplification, is

$$(k^2-4ah)\,(k\cos\alpha + 2a\sin\alpha)^2 + a^2p^2 = 0$$

Thus, the required locus is

$$(y^2-4ax)\,(y\cos\alpha + 2a\sin\alpha)^2 + a^2p^2 = 0$$

Comparing with the expression given in the problem, we see that

$$f(y) = y\cos\alpha + 2a\sin\alpha$$

Thus, the correct option is (C).

S7. (a) A point from where perpendicular tangents can be drawn to C must lie on the director circle of C, *i.e.*, on the circle $x^2+y^2=8$. Thus, we can assume such a point to be $(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$. If it is possible to also draw perpendicular tangents from this point to the parabola P, it must lie on the directrix of P, which means that

$$-a = 2\sqrt{2}\cos\theta \quad\Rightarrow\quad a_{\max} = 2\sqrt{2}$$

The correct option is (B).

(b) For this a, the angle between the common tangents to C and P is 90° (verify). The correct option is therefore (C).

S8. The equation of the tangent will be $yy_1 = 2a(x+x_1)$,on which any point can be taken as $(z, \frac{2a(z+x_1)}{y_1})$.From this point, the chord of contact for $x^2+y^2=a^2$ will be $T=0$, *i.e.*,

$$zx + \frac{2a(z+x_1)}{y_1}y = a^2$$

$$\Rightarrow\quad (xy_1+2ay)z + (2ax_1y - a^2y_1) = 0$$

This is a family of lines passing through (x_2, y_2), which is the intersection point of the lines

$$xy_1 + 2ay = 0,\ 2ax_1y - a^2y_1 = 0$$

Thus,

$$x_2 = -\frac{a^2}{x_1},\quad y_2 = \frac{ay_1}{2x_1}$$

$$\Rightarrow\quad \left(\frac{y_1}{y_2}\right)^2 = -4\left(\frac{x_1}{x_2}\right)$$

$$\Rightarrow\quad \lambda = 4$$

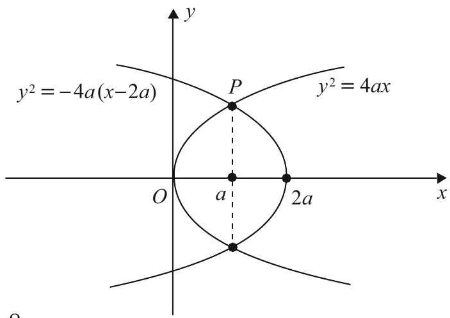
The correct option is (C).

S9. Note that P is the point $(a, 2a)$. The tangents at P to the two curves have the following equations:

$$2ay = 2a(x+a), \text{ and } 2ay = -2a(x-3a)$$

$$\Rightarrow \quad y = x + a \text{ and } y = 3a - x$$

These intersect the x-axis in $Q(-a, 0)$ and $R(3a, 0)$ (not shown here). The area of ΔPQR is $\frac{1}{2} \times 4a \times 2a = 4a^2$, while



the area enclosed between P_1 and P_2 for $y > 0$ is $2\int_0^a \sqrt{4ax}\, dx = \frac{8}{3}a^2$. The required ratio is 3:2. The correct option is (B).

S10. The equation of a normal (chord) can be written as

$$y + tx = 2at + at^3$$

where t is the (normal) contact point. The joint equation of the lines joining the origin to the points of contact is:

$$y^2 = 4ax\left(\frac{y+tx}{2at+at^3}\right)$$

$$\Rightarrow \quad (4at)x^2 - (2at + at^3)y^2 + 4axy = 0$$

Since this represents perpendicular lines,

$$4at = 2at + at^3 \Rightarrow \quad t = \pm\sqrt{2}$$

The other end point becomes $\mp 2\sqrt{2}$. The required length l is

$$l^2 = (a(\sqrt{2})^2 - a(-2\sqrt{2})^2)^2 + (2a(\sqrt{2}) - 2a(-2\sqrt{2}))^2 = 108a^2$$

$$\Rightarrow \quad l = 6\sqrt{3}\, a$$

The correct option is (C).

S11. Let $P \equiv t_1$ and $Q \equiv t_2$:

Step I: Use the fact that the normals intersect on the parabola itself to prove that $t_1 t_2 = 2$.
Step II: The tangents at P and Q intersect at $(h, k) \equiv (at_1 t_2, a(t_1 + t_2))$:

$$\Rightarrow \quad h = 2a$$

The locus is the straight line $x = 2a$. The correct option is (A).

S12. This question is quite straightforward. We make $P(at^2, 2at)$ satisfy the general equation of a parabola's normal, *i.e.*, $y = mx - 2am - am^3$, and obtain the equation

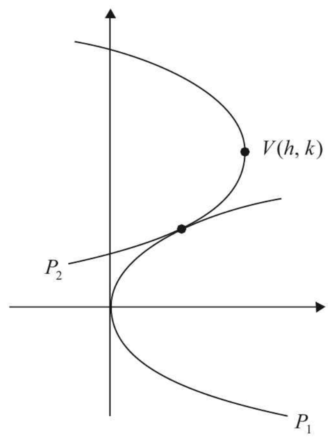
$$(m+t)(m^2 - mt + 2) = 0$$

One value $m = -t$ corresponds to the point P itself. For the other two points, $m^2 - mt + 2$. If the parameters of the other points are represented by λ, then (since $m = -\lambda$ for a normal) $\lambda^2 + \lambda t + 2 = 0$.

S13. We take P_1 as the fixed parabola and P_2 as the variable parabola:

$$P_1 : y^2 = lx \quad (1)$$

$$P_2 : (y-k)^2 = -l(x-h) \quad (2)$$



Since P_1 and P_2 just touch each other, solving (1) and (2) simultaneously should give just one root:

$$\left.\begin{aligned} y^2 &= \ell x \\ (y-k)^2 &= -l(x-h) \end{aligned}\right\} \xrightarrow[x]{\text{Eliminating}} 2y^2 - 2ky + k^2 - lh = 0$$

$$\Rightarrow \quad 4k^2 = 8(k^2 - lh)$$

$$\Rightarrow \quad k^2 - 2lh = 0$$

The required locus of V is thus $y^2 = 2lx$, which is a parabola of latus rectum $2l$. The correct option for part-(b) is therefore (B).

S14. (a) The trajectory of the shell is described by the following equation:

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

If it passes through (d, h), we have

$$gd^2\tan^2\alpha - 2u^2 d\tan\alpha + gd^2 + 2u^2 h = 0 \quad (1)$$

Since $\tan\alpha$ must be real, we have

$$4u^4d^2 - 4gd^2(gd^2 + 2u^2h) \geq 0$$

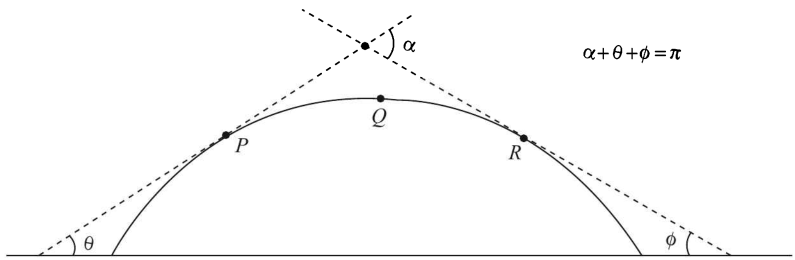
$$\Rightarrow \quad d \leq \frac{u}{g}\sqrt{u^2 - 2gh} \Rightarrow \quad d_{\max} = \frac{u}{g}\sqrt{u^2 - 2gh}$$

(b) Substituting the value of $d_{\max}$ in (1) yields

$$\tan\alpha = \frac{u}{\sqrt{u^2 - 2gh}}$$

The correct option is (C).

S15. Consider the following diagram carefully:



Let t be the required time. Since the horizontal component of the velocity stays the same throughout the flight,

we have

$$v_P \cos\theta = v_Q = v_R \cos\phi \tag{1}$$

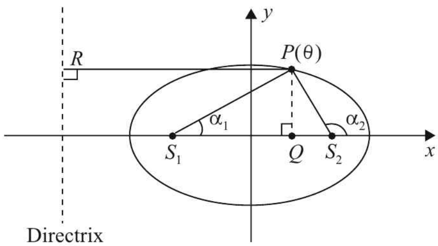
Also, $v_R \sin\phi - (-v_P \sin\theta) = gt$:

$$\Rightarrow \quad t = \frac{v_R \sin\phi + v_P \sin\theta}{g} \times \frac{v_Q}{v_Q}$$

$$= \frac{v_P v_R\,(\sin\phi\cos\theta + \sin\theta\cos\phi)}{g v_Q} \qquad \text{(using (1))}$$

$$= \frac{v_P v_R}{v_Q}\,\frac{\sin(\theta+\phi)}{g} = \frac{v_P v_R}{v_Q}\,\frac{\sin\alpha}{g}$$

S16. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that $S_1 \equiv (-ae, 0)$, $S_2 \equiv (ae, 0)$, and $P \equiv (a\cos\theta, b\sin\theta)$.



Now,

$$\cos\alpha_1 = \frac{QS_1}{PS_1} = \frac{QS_1}{e(PR)} = \frac{a(\cos\theta + e)}{a(1 + e\cos\theta)} \Rightarrow \tan^2\frac{\alpha_1}{2} = \left(\frac{1-e}{1+e}\right)\tan^2\frac{\theta}{2}$$

Similarly,

$$\tan^2\frac{\alpha_2}{2} = \left(\frac{1+e}{1-e}\right)\tan^2\frac{\theta}{2}$$

But

$$\frac{\alpha_2}{2} = \frac{\pi}{2} - \frac{1}{2}\angle PS_2S_1 \Rightarrow \tan\frac{\alpha_2}{2} = \cot\left(\frac{1}{2}\angle PS_2S_1\right).$$

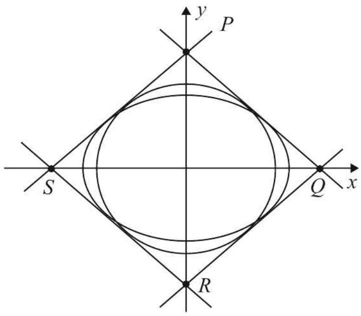
Thus,

$$\tan\left(\frac{1}{2}\angle PS_1S_2\right)\tan\left(\frac{1}{2}\angle PS_2S_1\right) = \frac{1-e}{1+e}$$

The correct option is (B).

S17. We have

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Circle: $x^2 + y^2 = r^2$



Since PQ is a tangent to the ellipse, its equation can be written as

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

for some θ. Thus, $P\equiv\left(0,\dfrac{a}{\cos\theta}\right)$ and $Q=\left(\dfrac{b}{\sin\theta},0\right)$, so that

$$\text{Area }(PQRS)=4\times\frac{1}{2}\times\frac{a}{\cos\theta}\times\frac{b}{\sin\theta}=\frac{2ab}{\sin\theta\cos\theta}$$

Since PQ is also a tangent to the circle, its distance from the origin must equal r, which leads to

$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{r^2}\quad\Rightarrow\quad\cos^2\theta=\frac{a^2}{r^2}\left(\frac{r^2-b^2}{a^2-b^2}\right)$$

and

$$\sin^2\theta=\frac{b^2}{r^2}\left(\frac{a^2-r^2}{a^2-b^2}\right)$$

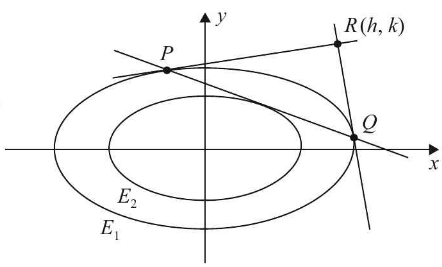
Thus,

$$\text{Area }(PQRS)=\frac{2ab}{\dfrac{ab}{a^2-b^2}\sqrt{\left(1-\dfrac{b^2}{r^2}\right)\left(1-\dfrac{a^2}{r^2}\right)}}=\frac{2\,(a^2-b^2)}{\sqrt{\left(1-\dfrac{b^2}{r^2}\right)\left(1-\dfrac{a^2}{r^2}\right)}}$$

which has a maximum value of $4ab$ when $r^2=\frac{2a^2b^2}{a^2+b^2}$. The correct option is (B).

S18. The problem involves two simple steps. First, since PQ is the chord of contact from R to E_1, we have

$$\frac{xh}{6}+\frac{yk}{3}=1 \qquad (1)$$



Now, since PQ is a tangent to $E_2:\frac{x^2}{4}+y^2=1$, the equation of PQ must be of the form

$$y=mx+\sqrt{4m^2+1} \qquad (2)$$

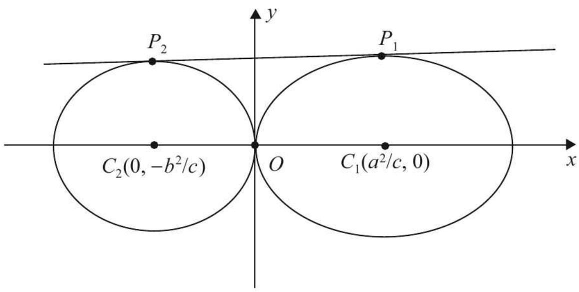
Since (1) and (2) are identical, we obtain $m = -\frac{h}{2k}$, $\sqrt{4m^2+1} = \frac{3}{k}$.

Eliminating m, we obtain $h^2 + k^2 = 9$, which implies that $R(h, k)$ lies on the director circle of E_1. Thus, RP and RQ must be at right angles to each other.

S19. The equations of the ellipses can be rewritten as

$$\frac{(x-a^2/c)^2}{a^4/c^2} + \frac{y^2}{a^2b^2/c^2} = 1; \quad \frac{(x+b^2/c)^2}{b^4/c^2} + \frac{y^2}{a^2b^2/c^2} = 1$$

From these equations, it is easy to observe that the centers of these ellipses are at $C_1(a^2/c, 0)$ and $C_2(-b^2/c, 0)$, so they'll be positioned in the plane as follows:



Note that the common tangent is parallel to the x-axis, so that P_1 and P_2 are immediately obvious:

$$P_1 \equiv \left(\frac{a^2}{c}, \frac{ab}{c}\right), \quad P_2 \equiv \left(-\frac{b^2}{c}, \frac{ab}{c}\right)$$

It is now easy to deduce that OP_1 and OP_2 are perpendicular, *i.e.*, P_1P_2 subtends a right angle at O. None of the first three options is therefore correct, so the correct choice is (D).

S20. A normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the general equation

$$(a\sec\theta)x - (b\operatorname{cosec}\theta)y = a^2e^2$$

If this passes through (h, k), then

$$ah\sec\theta - bk\operatorname{cosec}\theta = a^2e^2$$

This has four roots in general, which suggests that we should rewrite this equation as a fourth degree polynomial equation:

$$(ah\sec\theta - a^2e^2)^2 = b^2k^2\operatorname{cosec}^2\theta$$

Using $\operatorname{cosec}^2\theta = \frac{\sec^2\theta}{\sec^2\theta - 1}$, this reduces to a fourth degree equation in $\sec\theta$:

$$a^2h^2\sec^4\theta - 2a^3e^2h\sec^3\theta + (a^4e^4 - a^2h^2 - b^2k^2)\sec^2\theta + 2a^3e^2h\sec\theta - a^4e^4 = 0$$

Thus, $\sum\sec\alpha = \frac{2ae^2}{h}$. On the other hand, we could multiply this equation by $\cos^4\theta$, so that the equation changes to a fourth degree equation in $\cos\theta$, from which we can infer that $\sum\cos\alpha = \frac{2h}{ae^2}$.

Therefore,

$$\left(\sum\cos\alpha\right)\left(\sum\sec\alpha\right) = 4$$

The correct option is (B).

S21. The 6 possible values of the eccentricity e are given by $e_i = \sqrt{i}\sin\frac{\pi}{i},\ i = 1, 2, ..., 6$:

$$e_1 = 0, \quad e_2 = \sqrt{2}, \quad e_3 = \frac{3}{2}$$

$$e_4 = \sqrt{2}, \quad e_5 = \sqrt{5}\sin\frac{\pi}{5} > (2)\times\left(\frac{1}{2}\right) = 1 \Rightarrow e_5 > 1$$

$$e_6 = \sqrt{\frac{3}{2}} > 1$$

Therefore, 5 of the possible e-values correspond to hyperbolas, while one corresponds to a circle. The required probability is $\frac{5}{6}$. The correct option is (C). Note that the information that the locus is not a parabola did not turn out to be useful in any way, as none of the six loci is a parabola anyway.

S22. We need to find there lative magnitudes of $\sin\sqrt{2}, \sin\sqrt{3}, \cos\sqrt{2}, \cos\sqrt{3}$, and $\lambda = \sin\sqrt{2} - \sin\sqrt{3}$, $\mu = \cos\sqrt{2} - \cos\sqrt{3}$. We note that $\sqrt{2} \in (0, \frac{\pi}{2})$ and $\sqrt{3} \in (\frac{\pi}{2}, \pi)$.

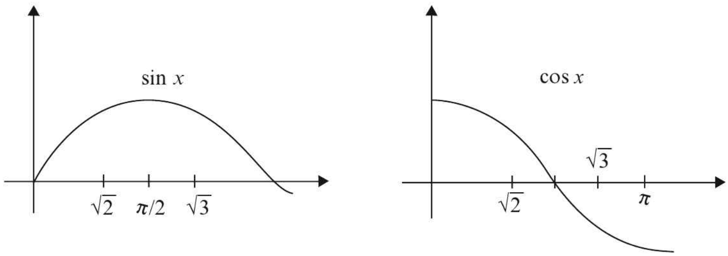


Figure not to scale

We note from the figures that both $\sin\sqrt{2}$ and $\sin\sqrt{3}$ are positive. Also, upto three decimal places, $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, so that

$$\sqrt{2} + \sqrt{3} \approx 1.414 + 1.732 \approx 3.146$$

In fact, the actual value will be slightly larger than this. On the other hand, $\pi \approx 3.141$. Thus, the 'mid-point' of $\sqrt{2}$ and $\sqrt{3}$ lies slightly to the right of $\frac{\pi}{2}$. From the graph of $\sin x$, this means that $\sqrt{2}$ is closer to $\frac{\pi}{2}$ than $\sqrt{3}$, so that $\sin\sqrt{2} > \sin\sqrt{3}$, *i.e.*, $\lambda > 0$.

Now, $\cos\sqrt{2} > 0$ and $\cos\sqrt{3} < 0$, which means that $\mu = \cos\sqrt{2} - \cos\sqrt{3} > 0$.

Thus, we conclude that $\lambda, \mu > 0$ which implies that the curve is an ellipse. Now we need to compare the relative magnitudes of λ and μ. Although to some readers, it may be evident that $\lambda < \mu$, we still need to prove this rigorously. We note that

$$\lambda - \mu = (\sin\sqrt{2} - \sin\sqrt{3}) - (\cos\sqrt{2} - \cos\sqrt{3})$$

Denoting $\sqrt{3} - \sqrt{2}$ by ψ, we construct a function $f(x)$ gives by

$$f(x) = \sin x - \cos x - \sin(x+\psi) + \cos(x+\psi)$$

$$\Rightarrow \quad f'(x) = \cos x + \sin x - \cos(x+\psi) - \sin(x+\psi)$$

$$= \sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right) - \sin\left(x + \psi + \frac{\pi}{4}\right)\right)$$

$$= -2\sqrt{2}\cos\left(x + \frac{\pi}{4} + \frac{\psi}{2}\right)\sin\frac{\psi}{2}$$

We note that

$$f'(x)\begin{cases} <0 & \text{for } x\in\left(0,\dfrac{\pi}{4}-\dfrac{\psi}{2}\right) \\ >0 & \text{for } x\in\left(\dfrac{\pi}{4}-\dfrac{\psi}{2},\dfrac{5\pi}{4}-\dfrac{\psi}{2}\right) \end{cases}$$

Also, the zeroes of $f(x)$ are given by $f(x)=0$, *i.e.*,

$$\sin x-\cos x=\sin(x+\psi)-\cos(x+\psi)$$

$$\Rightarrow \quad \sin(x-\pi/4)=\sin(x+\psi-\pi/4)$$

This can only happen if

$$\frac{\left(x-\dfrac{\pi}{4}\right)+\left(x+\psi-\dfrac{\pi}{4}\right)}{2}=-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\text{ etc.}$$

$$\Rightarrow \quad x=-\frac{\pi}{4}-\frac{\psi}{2},\frac{3\pi}{4}-\frac{\psi}{2},\frac{7\pi}{4},-\frac{\psi}{2},\text{ etc.}$$

Thus, $f(x)$ has one zero in $(0,\pi)$, namely$(\frac{3\pi}{4}-\frac{\psi}{2})$. Also, $f(0)<0$. We conclude that $f(x)$ is negative throughout the interval $(0,\frac{3\pi}{4}-\frac{\psi}{2})$, which means that $f(\sqrt{2})<0$:

$$\sin\sqrt{2}-\cos\sqrt{2}-\sin(\sqrt{2}+(\sqrt{3}-\sqrt{2}))+\cos(\sqrt{2}+(\sqrt{3}-\sqrt{2}))<0$$

$$\Rightarrow \quad \sin\sqrt{2}-\sin\sqrt{3}<\cos\sqrt{2}-\cos\sqrt{3}$$

$$\Rightarrow \quad \lambda<\mu$$

Thus, the given curve represents an ellipse with its foci on the y-axis. The correct option is (C).

S23. Given the range of the eccentricity, the curve must be either an ellipse or a hyperbola (it cannot be a parabola, a circle or a line or pair of lines). Consider the following table which tabulates all the cases corresponding to the different values of k.

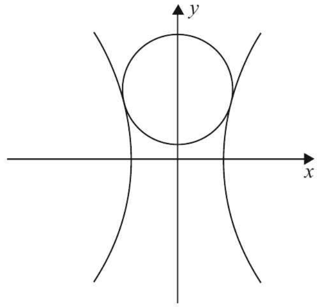
Value of k	**No. of outcomes (Total outcomes = 36)**	**Type of conic**
2	1	Ellipse
3	2	Ellipse
4	3	Circle
5	4	Ellipse
6	5	Ellipse
7	6	Pair of lines
8	5	Hyperbola
9	4	Hyperbola
10	3	Hyperbola
11	2	Hyperbola
12	1	Hyperbola

Check the entries in the last column carefully. The required probability as $\frac{4}{9}$ (how?). The correct option is (B).

S24. In the figure shown, we have drawn the circle just touching the hyperbola.
Let the equation of this circle be $x^2 + (y - \lambda)^2 = 16$.
Solving this with the hyperbola's equation must yield real roots, which leads to $\lambda = \pm 2\sqrt{6}$.

Thus, the possible values of λ are

$$(-\infty, -2\sqrt{6}] \cup [2\sqrt{6}, \infty).$$



Consider the circle touching the hyperbola.

S25. The ellipse and the hyperbola will intersect in four points, and it can be easily deduced that the coordinates of these points will be

$$x = \pm\frac{3}{\sqrt{10}},\ y = \pm\frac{1}{\sqrt{5}}$$

If the four points are represented by (x_i, y_i), $i = 1, 2, 3, 4$, we conclude that

$$\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i = 0$$

Now, if the variable line is represented by $ax + by + c = 0$, the (algebraic) length of the perpendicular p_i from any one of the four points of intersection is

$$p_i = \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}$$

If $\sum_{i=1}^{4} p_i = 0$, we have

$$a\sum_{i=1}^{4} x_i + b\sum_{i=1}^{4} y_i + c = 0$$

$$\Rightarrow \quad c = 0$$

Thus, the variable line always passes through (0, 0). We have 0 + 0 = 0, and so the correct option is (A).

S26. Recall that if the equation of the asymptotes are known ($L_1 = 0$ and $L_2 = 0$), the equation of the hyperbola can be written as $L_1L_2 = \lambda$, where λ is some constant.

(a) The equation of the hyperbola can be written as

$$(2x + y + 5)(2px + y + 1) = \lambda$$

$$\Rightarrow \quad 4px^2 + 2(1+p)xy + y^2 + (2 + 10p)x + 6y + 5 - \lambda = 0 \tag{1}$$

This equation will be a valid representation for a hyperbola if $h^2 > ab$, *i.e.*,

$$(1+p)^2 > 4p \quad \Rightarrow \quad (1-p)^2 > 0 \quad \Rightarrow \quad p \neq 1$$

Thus, $\quad p \in \mathbb{R} \setminus \{1\}$

The correct option is (A).

(b) If the hyperbola passes through (0, 1), we have from (1):

$$(6)(2) = \lambda \quad \Rightarrow \quad \lambda = 12$$

The required equation will be

$$(2x + y + 5)(2px + y + 1) = 12, \quad p \in \mathbb{R} \setminus \{1\}$$

S27. Assuming P to be (h, k), and making these coordinates satisfy the general tangent equation applicable in this case, which is

$$y = mx + \sqrt{a^2m^2 - b^2},$$

we have

$$k = mh + \sqrt{a^2m^2 - b^2} \quad \Rightarrow \quad (k - mh)^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad (h^2 - a^2)m^2 - 2hkm + k^2 + b^2 = 0$$

This quadratic in m will have two roots m_1 and m_2, which will satisfy

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}, \; m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2} \tag{1}$$

Since, the tangents are inclined to other at angle θ, we have

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \quad \Rightarrow \quad \tan^2\theta\,(1 + m_1m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 \tag{2}$$

Using the values of (1) in (2), simplifying, and finally using (x, y) instead of (h, k), we'll obtain the desired locus as

$$(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2\alpha(a^2y^2 - b^2x^2 + a^2b^2)$$

Comparing this with the expression given in the question, we have $\mu = 4\cot^2\alpha$. Thus, the correct option is (D). Note that we can now deduce that for the director circle of this hyperbola, $\alpha = 90^\circ$, so $\cot\alpha = 0$, and its equation will be $x^2 + y^2 = a^2 - b^2$.

S28. In the sequence of points: $P_1, P_2, \ldots, P_n$, we will evaluate the relation between the coordinates of $P_k(ct_k, \frac{c}{t_k})$ and $P_{k+1}(ct_{k+1}, \frac{c}{t_{k+1}})$. The normal at P_k has the slope t_k^2 (why), *i.e.*, the line joining P_k and P_{k+1} has the slope t_k^2:

$$\frac{\dfrac{c}{t_{k+1}} - \dfrac{c}{t_k}}{ct_{k+1} - ct_k} = t_k^2 \quad \Rightarrow \quad t_{k+1} = -\frac{1}{t_k^3}$$

Thus,

$$t_2 = -\frac{1}{t_1^3}, \; t_3 = -\frac{1}{t_2^3} = t_1^9, \; t_4 = -\frac{1}{t_3^3} = -\frac{1}{t_1^{27}}$$

and so on, and we will eventually obtain

$$t_n = (-1)^{n-1}(t_1)^{(-3)^{n-1}}$$

The correct option is (B).

S29. If $P(h, k)$ is the mid-point of the chord(s) of contact, the equation of the chord will be given by $T(h, k) = S(h, k)$, or

$$xh - yk - a^2 = h^2 - k^2 - a^2$$

$$\Rightarrow \quad xh - yk = h^2 - k^2 \tag{1}$$

Assuming a (variable) point on the circle as $Q(a\cos\theta, a\sin\theta)$, the chord of contact from Q to the hyperbola will have the equation $T(h, k) = 0$, or

$$\begin{aligned} &ax\cos\theta - ay\sin\theta = a^2 \\ \Rightarrow \quad &x\cos\theta - y\sin\theta = a \end{aligned} \tag{2}$$

(1) and (2) essentially represent the same line, and thus we have

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{h^2 - k^2}$$

We can easily eliminate θ, and replace (h, k) with (x, y) to obtain the desired locus as

$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$

The correct option is (C).

S30. Let $H \equiv xy = c^2$ and $S \equiv (x-\lambda)^2 + (y-\beta)^2 = r^2$, so that $O \equiv (0,0)$. Any point on H can be taken as $P(ct, \frac{c}{t})$. If P lies on S as well, we have

$$(ct - \alpha)^2 + \left(\frac{c}{t} - \beta\right)^2 = r^2$$

$$\Rightarrow \quad c^2t^4 - 2\alpha ct^3 + (\alpha^2 + \beta^2 - r^2)t^2 - 2\beta ct + c^2 = 0 \tag{1}$$

This will have four roots t_1, t_2, t_3 and t_4, corresponding to A, B, C and D. Now,

$$\lambda = \frac{\sum_{i=1}^{4}\left\{(ct_i - 0)^2 + \left(\frac{c}{t_i} - 0\right)^2\right\}}{r^2}$$

$$= \frac{c^2}{r^2}\sum_{i=1}^{4}\left(t_i^2 + \frac{1}{t_i^2}\right) = \frac{c^2}{r^2}\left\{\underbrace{\sum_{i=1}^{4} t_i^2}_{S_1} + \underbrace{\sum_{i=1}^{4}\frac{1}{t_i^2}}_{S_2}\right\} \tag{2}$$

We will now evaluate S_1 and S_2 separately.

$$S_1 = t_1^2 + t_2^2 + t_3^2 + t_4^2 = (t_1 + t_2 + t_3 + t_4)^2 - 2\sum_{\substack{1\le i,\, j\le 4 \\ i\ne j}} t_i t_j$$

$$= \left(\frac{2\alpha c}{c^2}\right)^2 - 2\left(\frac{\alpha^2 + \beta^2 - r^2}{c^2}\right) \qquad \text{(From (1))}$$

$$= \frac{2(\alpha^2 - \beta^2 + r^2)}{c^2} \tag{3}$$

$$S_2 = \frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} + \frac{1}{t_4^2} = \frac{t_1^2t_2^2t_3^2 + t_1^2t_2^2t_4^2 + t_1^2t_3^2t_4^2 + t_2^2t_3^2t_4^2}{(t_1t_2t_3t_4)^2}$$

$$= \frac{(t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)^2 - 2t_1t_2t_3t_4\left(\sum_{\substack{1\le i,\, j\le 4 \\ i\ne j}} t_it_j\right)}{(t_1t_2t_3t_4)^2}$$

You are urged to carefully verify the last step. From (1) again, we have

$$S_2 = \frac{\left(\frac{2\beta c}{c^2}\right)^2 - 2\left(\frac{c^2}{c^2}\right)\left(\frac{\alpha^2+\beta^2-r^2}{c^2}\right)}{\left(\frac{c^2}{c^2}\right)^2} = \frac{2(\beta^2-\alpha^2+r^2)}{c^2} \quad (4)$$

From (2), (3) and (4),

$$\lambda = \frac{c^2}{r^2}(S_1+S_2) = \frac{c^2}{r^2}\cdot\frac{4r^2}{c^2}$$

$$\Rightarrow \quad \lambda = 4$$

The correct option is (A).

SUBJECTIVE TYPE EXAMPLES

S31. Using $x = r\cos\theta$, $y = r\sin\theta$, we have

$$ar^2\cos^2\theta + hr^2\sin 2\theta + br^2\sin^2\theta = 1$$

$$\Rightarrow \quad r^2 = \frac{1}{a\cos^2\theta + h\sin 2\theta + b\sin^2\theta} = \frac{1}{D} \text{ (say)}$$

To maximize/minimize r, we have to minimize/maximize

$$D(\theta) = a\cos^2\theta + h\sin 2\theta + b\sin^2\theta \tag{1}$$

$$\Rightarrow \quad D'(\theta) = -2a\sin 2\theta + 2h\cos 2\theta + 2b\sin 2\theta$$

$$= 2\{(b-a)\sin 2\theta + h\cos 2\theta\}$$

This is *0* when $\tan 2\theta = \frac{h}{a-b}$. Note that $\tan 2\theta = \frac{h}{a-b}$ implies that

$$\sin 2\theta = \frac{\pm h}{\sqrt{h^2+(a-b)^2}}, \quad \cos^2\theta = \frac{1}{2} \pm \frac{|a-b|}{2\sqrt{h^2+(a-b)^2}}, \quad \sin^2\theta = \frac{1}{2} \mp \frac{|a-b|}{2\sqrt{h^2+(a-b)^2}}$$

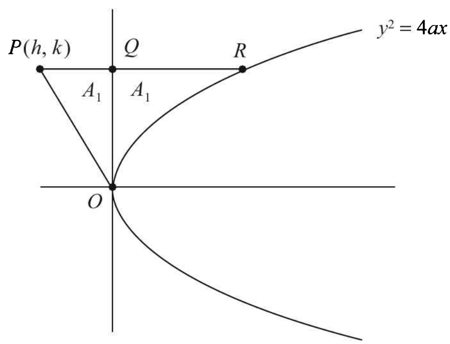
Therefore, using (1), we have

$$D_{\text{extreme}}(\theta) = \left(\frac{a+b}{2}\right) \pm \left(\frac{a|a-b| - b|a-b| + h^2}{2\sqrt{h^2+(a-b)^2}}\right)$$

$$= \left(\frac{a+b}{2}\right) \pm \frac{1}{2}\sqrt{h^2+(a-b)^2}$$

These are the required extreme values of $D(\theta)$, from which the extreme values of r are immediately obvious.

S32. Since $A_1 = A_2$, we have

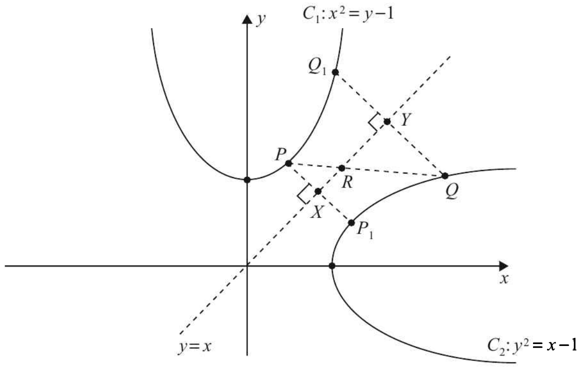


$$\left|\frac{1}{2}hk\right| = \left|\int_0^k \frac{y^2}{4a}\,dy\right|$$

$$\Rightarrow \quad k = 0 \text{ or } k^2 = 6a|h|$$

Since $h < 0$, the required locus is $y^2 = -6ax$.

S33. We note that since C_1 and C_2 are mirror reflections of each other in the line $y = x$ (C_2 can be obtained from C_1 by $x \to y, y \to x$), P_1 obviously lies on C_2 and Q_1 lies on C_1.



Also, $PR \ge PX$, $QR \ge QY$, so that

$$PQ = PR + QR \ge PX + QY = \frac{1}{2}(PP_1 + QQ_1) \ge \min\{PP_1, QQ_1\}$$

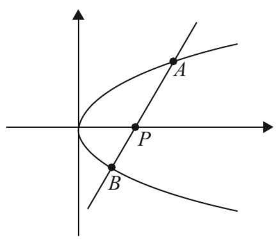
We note that P_0 and Q_0 represent minimum-distance points for C_1 and C_2. The required value of P_0Q_0 is the absolute minimum value of PP_1(or QQ_1), which will be the same since P_0 and Q_0 must be reflections of each other. If $P \equiv (t, t^2+1)$, then $P_1 \equiv (t^2+1, t)$, so that

$$PP_1 = \sqrt{2}(t^2 - t + 1)$$

The minimum value of this expression is achieved for $t = \frac{1}{2}$, so that

$$P_0Q_0 = PP_{1(\min)} = PP_{1\left(t=\frac{1}{2}\right)} = \frac{3}{2\sqrt{2}}$$

S34. Consider the following figure:



Since we are dealing with distances here, it would be best if we use the polar form of a line's equation to represent the points A and B. Thus, we use $(h + r\cos\theta, r\sin\theta)$ as any point lying on AB. If this lies on the parabola, then

$$(r\sin\theta)^2 = 4a(h + r\cos\theta) \quad \Rightarrow \quad (\sin^2\theta)r^2 - (4a\cos\theta)r - 4ah = 0$$

Let the two roots of this equation be r_1 and r_2 (corresponding to AP and BP). Thus,

$$\frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{(r_1 + r_2)^2 - 2(r_1 r_2)}{(r_1 r_2)^2} = \frac{\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^2 + \frac{8ah}{\sin^2\theta}}{\left(\frac{-4ah}{\sin^2\theta}\right)^2}$$

$$= \frac{2a\cos^2\theta + h\sin^2\theta}{2ah^2}$$

This is fixed if $h = 2a$. Thus, P is the point $(2a, 0)$.

S35. The variations in r and a with time t are described by the following equations:

$$r = \frac{2-\cos t}{2}, \quad a = \frac{3-\cos t}{2}$$

The common tangents are perpendicular when $a = \sqrt{2}r$, a result that is not very difficult to deduce. Thus,

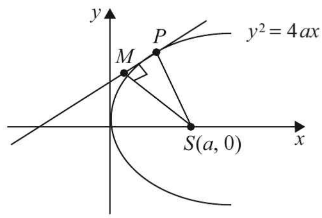
$$\frac{3-\cos t}{2} = \sqrt{2}\left(\frac{2-\cos t}{2}\right)$$

$$\Rightarrow \quad t = \cos^{-1}\left|\frac{2\sqrt{2}-3}{\sqrt{2}-1}\right|$$

S36. If P is the point $(at^2, 2at)$, then the equation of MP is $x - yt + at^2 = 0$.

Thus,

$$SM = \frac{a(1+t^2)}{\sqrt{1+t^2}} = a\sqrt{1+t^2}$$



Also, $SP = at^2 + a = a(1+t^2)$, which implies that

$$PM = \sqrt{SP^2 - SM^2} = at\sqrt{1+t^2}$$

Finally,

$$A = \text{area }(\Delta SPM) = \frac{1}{2} \times PM \times SM = \frac{1}{2}a^2 t(1+t^2)$$

Since $t_{\max} = 1$ (from the problem), $A_{\max} = a^2$.

S37. This is a standard maxima-minima problem with its basis in coordinate geometry. We are going to broadly outline the steps involved; the reader is expected to fill in the details.

Step-1: Assuming that the normal is of slope m, its equation will be $y = mx - 2am - am^3$, while the two contact points can be shown to be

$$P:(am^2, -2am), \quad Q:\left(a\left(m^2 + \frac{4}{m^2} + 4\right), 2a\left(m + \frac{2}{m}\right)\right)$$

Step 2: We write an expression for the length l of the normal chord PQ in terms of m:

$$l^2 = PQ^2 = \frac{16a^2(1+m^2)^3}{m^4}$$

Step 3: We minimize l^2. This happens when $m^2 = 2$, so that

$$l_{\min} = 6\sqrt{3}\,a$$

The slope of the corresponding normal chord is $\pm\sqrt{2}$.

S38. The obvious approach is to write the equations for the normals to the two curves separately, and then find the values of the parameters for which the two equations are identical.

$$\text{Normal to } y^2 = 4ax: \quad y = m_1 x - 2am_1 - am_1^3$$

$$\text{Normal to } y^2 = 4c(x-b): \quad y = m_2(x-b) - 2cm_2 - cm_2^3$$

If these are identical, then

$$m_1 = m_2, \quad 2am_1 + am_1^3 = (b+2c)m_2 + cm_2^3$$

From these, we deduce the possible values of m_1. Apart from the obvious values of $m_1 = m_2 = 0$,we get

$$m_1 = m_2 = \pm\sqrt{\frac{b}{a-c} - 2}$$

Thus, we must have $\frac{b}{a-c} > 2$.

S39. Let the parabola be $y^2 = 4ax$, and the family of parallel chords have a slope m. For any chord of this family, let the end-points be $P(t_1)$ and $Q(t_2)$. Since the slope of PQ is m, we can prove from this that $t_1 + t_2 = \frac{2}{m}$.

Now, the normals at P and Q have the equations

$$y = -t_1x + 2at_1 + at_1^3$$

$$y = -t_2x + 2at_2 + at_2^3$$

If these intersect at $R(h, k)$, then it is easy to show that

$$h = 2a + a\left(\frac{4}{m^2} - t_1 t_2\right), \quad k = -\left(\frac{2}{m}\right)a\, t_1 t_2$$

Eliminating $t_1 t_2$ from these, we obtain

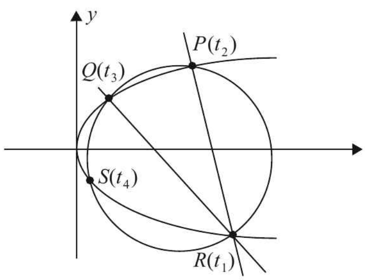
$$k = \frac{2}{m}(h - 2a) - \frac{8a}{m^3}$$

Using (x, y) instead of (h, k), and letting $\frac{2}{m} = m'$, the locus of R is

$$y = m'x - 2am' - am'^3$$

This itself is a normal to $y^2 = 4ax$.

S40. We make use of the following result which we have obtained elsewhere. If R is the point t_1, then the parameters t_2 and t_3 of P and Q are the roots of $t^2 + t_1 t + 2 = 0$ *i.e.*, $t_2 + t_3 = -t_1, t_2 t_3 = 2$.



Also, it is straightforward to show that $t_1 + t_2 + t_3 + t_4 = 0$, by considering the equation in x formed by simultaneously solving the equations of the circle and parabola, and observing that the sum of roots of this equation is 0. From this, we obtain $t_4 = 0$. Thus, the point S is actually the origin (the representation of S in the figure above is not entirely correct). Now, we write down the equations of PQ and RS:

$$PQ: y - 2at_3 = \frac{2}{t_2 + t_3}(x - at_3^2)$$

$$\Rightarrow \quad y = -\frac{2x}{t_1} + 2at_3\left(1 + \frac{t_3}{t_1}\right)$$

$$RS :\Rightarrow \quad y = \left(\frac{2}{t_1}\right)x$$

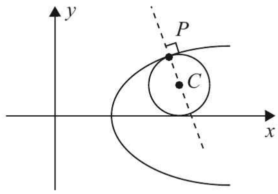
Equating these, we have

$$-\frac{2x}{t_1} + 2at_3\left(1 + \frac{t_3}{t_1}\right) = \frac{2x}{t_1} \Rightarrow \quad \frac{2x}{t_1} = \frac{at_3}{t_1} \times (t_1 + t_3) = -\frac{at_2 t_3}{t_1}$$

$$\Rightarrow \quad x = -a$$

Thus, PQ and RS intersect on the directrix $x + a = 0$.

S41. The tangent at P has the equation $3y - 4x = 0$,which further implies that the normal's equation at P is $3x + 4y - 10 = 0$. If r is the radius of the circle, the coordinates of C, the circle's center, can be assumed as (α, r).



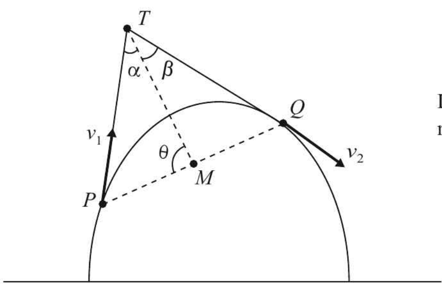
Since C must lie on the normal PC, we have $3\alpha + 4r - 10 = 0$. Another constraint on α and r can be obtained using the fact that the distance PC must equal r.

$$\left(\alpha - \frac{6}{5}\right)^2 + \left(r - \frac{8}{5}\right)^2 = r^2$$

Solving these two constraints gives $r^2 - 5r + 4$, that is $r = 1, 4$. This further gives $\alpha = \pm\, 2$. The values we were looking for are $\alpha = 2, r = 1$, and the required equation is

$$(x-2)^2 + (y-1)^2 = 1$$

S42. TM must be vertical, since it joins T (point of intersection of tangents) to M (mid-point of the corresponding chord), so it must be parallel to the axis of the parabola (why?), which is vertical. Thus, since the horizontal velocity of the particle is unchanged, we have $v_1 \sin\alpha = v_2 \sin\beta$.



Let M be the mid point of PQ.

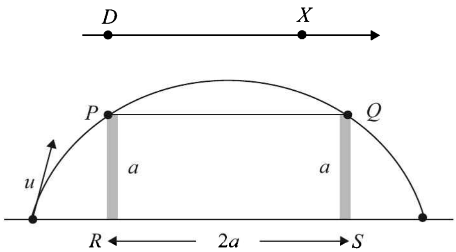
Also, by the sine rule,

$$\frac{PT}{\sin\theta} = \frac{PM}{\sin\alpha} \quad \text{and} \quad \frac{QT}{\sin(\pi - \theta)} = \frac{MQ}{\sin\beta}$$

$$\Rightarrow \quad \frac{PT}{QT} = \frac{\sin\beta}{\sin\alpha} = \frac{v_1}{v_2}$$

The required value is thus $PT : QT$.

S43. Let DX be the directrix of the parabola, and F (not shown) be the focus.



We can easily show that the distance of DX from RS is

$$\frac{u^2}{2g} = 2a$$

Therefore, $PD = a \Rightarrow PF = a$. Also, $QF = a$. Thus F must be the mid-point of PQ. This implies that the latus rectum is of length $2a$. Now, from the trajectory equation, the length of the latus rectum can be deduced to be $\frac{2}{g}(u\cos\alpha)^2$, so that

$$\frac{2}{g}(u\cos\alpha)^2 = 2a \quad \Rightarrow \quad u\cos\alpha = \sqrt{ag}$$

Thus,

$$T = \frac{2a}{u\cos\alpha} = 2\sqrt{\frac{a}{g}}$$

S44. The general equation of a normal to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$(a\sec\theta)x - (b\operatorname{cosec}\theta)y = a^2 - b^2 = a^2e^2 \tag{1}$$

Let this normal pass through (h, k). Using $\tan\frac{\theta}{2} = t$, so that $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$, (1) becomes

$$bkt^4 + 2(ah + a^2e^2)t^3 + 2(ah - a^2e^2)t - bk = 0 \tag{2}$$

This has in general four roots, which means that (in general) four normals can be drawn from any point to a given ellipse. Note that $\sum_{i\neq j} t_i\, t_j = 0$, while $\prod t_i = -1$. Now, in the sum $\sum_{i\neq j} t_i\, t_j$, we have a total of six terms, so we'll have three pairs of the form (each pair has a term involving t_4 which we want to get rid of)

$$S_1 = t_1 t_2 + t_3 t_4 = t_1 t_2 - \frac{t_3}{t_1 t_2 t_3}$$

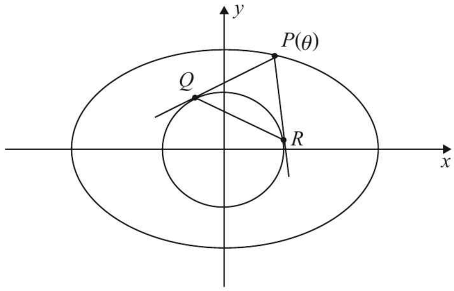
$$= \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} - \cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} \quad \text{(how?)}$$

$$= \frac{(\cos\theta_1 + \cos\theta_2)\sin\theta_3}{\sin\theta_1\,\sin\theta_2\sin\theta_3} \quad \text{(how?)}$$

Similarly, we'll have two other pairs S_2 and S_3 where $S_2 = t_2 t_3 + t_4 t_1$ and $S_3 = t_3 t_1 + t_2 t_4$. The sum $S_1 + S_2 + S_3$ must be 0, which means that

$$(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) + (\cos\theta_2 \sin\theta_3 + \cos\theta_3 \sin\theta_2) + (\cos\theta_3 \sin\theta_1 + \cos\theta_1 \sin\theta_3)$$
$$= \sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0.$$

S45. The equation of the chord of contact QR is



$$xa\cos\theta + yb\sin\theta = r^2 \qquad (1)$$

We have to show that this equation is of the form

$$y = mx \pm \sqrt{m^2A^2 + B^2}$$

for the appropriate A and B.
To do that, we write (1) as $y = (-\frac{a}{b}\cot\theta)x + \frac{r^2}{b}\operatorname{cosec}\theta$.

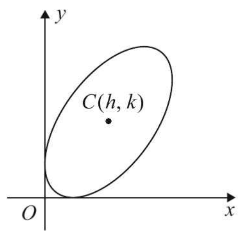
If $-\frac{a}{b}\cot\theta$ is written as m, then

$$\cot\theta = -\frac{bm}{a} \Rightarrow \quad \operatorname{cosec}\theta = \pm\frac{1}{a}\sqrt{m^2b^2 + a^2}$$

$$\Rightarrow \quad y = mx \pm \frac{r^2}{ab}\sqrt{m^2b^2 + a^2} = mx \pm \sqrt{\frac{m^2r^4}{a^2} + \frac{r^4}{b^2}}$$

Thus, $A = \frac{r^4}{a^2}$, $B = \frac{r^4}{b^2}$,which means that QR is tangent to the ellipse $a^2x^2 + b^2y^2 = r^4$.

S46. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We need to find the locus of the center C. We make the simple observation that in the configuration shown, O must lie on the director circle of the ellipse. Thus, OC must equal the radius of the director circle:



$$h^2 + k^2 = a^2 + b^2$$

The required locus of C is a circle with the equation

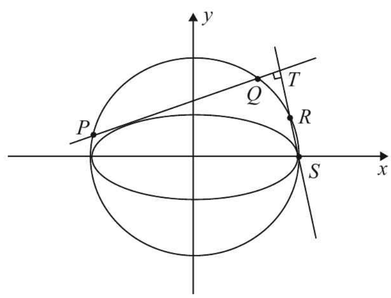
$$x^2 + y^2 = a^2 + b^2$$

A very interesting solution indeed!

S47. We need to prove that

$$PQ^2 + RS^2 = \text{constant}$$

This problem involves slightly detailed calculations, so we are going to outline the steps involved broadly, with the details left to the reader to fill in.



Step 1: Any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will have the form $y = mx + \sqrt{a^2m^2 + b^2}$.
By solving this equation with that of the auxiliary circle $x^2 + y^2 = a^2$, we can determine the length of the intercept:

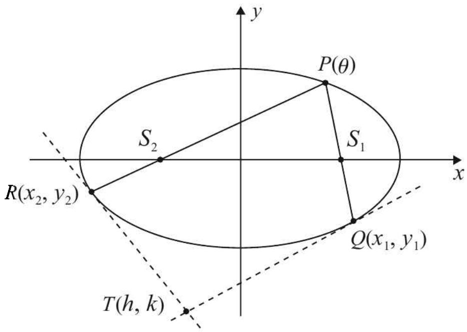
$$l = \frac{4(a^2 - b^2)}{1 + m^2} \qquad (1)$$

Step 2: If PQ has slope m, then RS has slope $-\frac{1}{m}$, so that using (1), we have

$$PQ = \frac{4(a^2 - b^2)}{1+m^2}, \quad RS = \frac{4m^2(a^2 - b^2)}{1+m^2} \tag{2}$$

Step 3: $PQ^2 + RS^2$, using (2), comes out to $4(a^2 - b^2)$, which is independent of m.

S48. Let P be the point θ. Note that $S_1 \equiv (ae, 0)$, $S_2 \equiv (-ae, 0)$. The solution broadly consists of the following steps.



Step 1:Find the coordinates of Q and R.
This is done by first writing the equations of PS_1 and PS_2. For example, from the two-point form,

$$PS_1 : y = \frac{b\sin\theta}{(\cos\theta - e)}(x - ae)$$

We now find the intersection of PS_1 with the ellipse, *i.e.*, the point $Q(x_1, y_1)$:

$$x_1 = \frac{a(2e - (1+e^2)\cos\theta)}{1+e^2 - 2e\cos\theta}, \quad y_1 = \frac{b(e^2-1)\sin\theta}{1+e^2-2e\cos\theta}$$

To obtain $R(x_2, y_2)$, use $e \to -e$ in the expressions above.

Step 2: Write the equations of the tangents at Q and R.

These are $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$, and $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$, where (x_1, y_1) and (x_2, y_2) are known from Step-1.

Step 3: Find relations between θ and (h, k).
Since $T(h, k)$ must satisfy both the tangent equations, we can use this to show that

$$\cos\theta = -\frac{h}{a}, \quad \sin\theta = \frac{k}{b}\left(\frac{e^2-1}{e^2+1}\right)$$

Step 4: Eliminate θ and use $(h, k) \to (x, y)$.
The required locus is

$$\frac{x^2}{a^2} + \frac{(1-e^2)^2 y^2}{(1+e^2)^2 b^2} = 1$$

This is an ellipse.

S49. If $P(\theta_1)$, $Q(\theta_2)$, and $R(\theta_3)$ are the vertices of the variable triangle, and if we assume that PQ and PR have fixed slopes, then it is straightforward to show that

$$\cot\left(\frac{\theta_1+\theta_2}{2}\right)=\lambda_1,\quad \cot\left(\frac{\theta_1+\theta_3}{2}\right)=\lambda_2,\text{where } \lambda_1, \lambda_2 \text{ are fixed}$$

$$\Rightarrow\quad \theta_1+\theta_2=2\mu_1, \theta_1+\theta_3=2\mu_2, \text{ where } \mu_1 \text{ and } \mu_2 \text{ are fixed.}$$

The equation of the chord QR is

$$\frac{x}{a}\cos\left(\frac{\theta_2+\theta_3}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_2+\theta_3}{2}\right)=\cos\left(\frac{\theta_2-\theta_3}{2}\right)$$

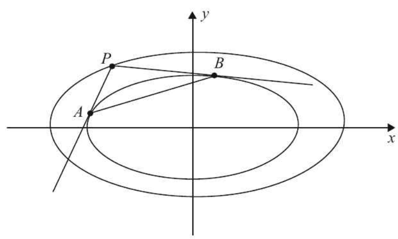
Using $\frac{\theta_2-\theta_3}{2}\to\mu_1-\mu_2$ and $\frac{\theta_2+\theta_3}{2}\to\theta$, the equation of QR becomes

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\cos(\mu_1-\mu_2)$$

This is a tangent to the ellipse

$$\frac{x^2}{a^2\cos^2(\mu_1-\mu_2)}+\frac{y^2}{b^2\cos^2(\mu_1-\mu_2)}=1$$

S50. We denote the two ellipses by E_1 and E_2. To evaluate the coordinates of H, we will actually need to use the coordinates of the vertices of triangle PAB. This suggests that we start by assuming A and B, say α and β. Thus, the equations of PA and PB become



$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1,\quad \frac{x\cos\beta}{a}+\frac{y\sin\beta}{b}=1$$

These intersect at P, whose coordinates can be evaluated by solving the two equations above, to obtain:

$$P\equiv\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)},\ \frac{b\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

This must satisfy the equation of E_1, from which we obtain a constraint on α and β:

$$a^4\cos^2\left(\frac{\alpha+\beta}{2}\right)+b^4\sin^2\left(\frac{\alpha+\beta}{2}\right)=(a^2+b^2)^2\cos^2\left(\frac{\alpha-\beta}{2}\right) \tag{1}$$

The next step is to write the equations of the altitudes from A to PB and from B to PA:

Altitude from A to PB: $(a\sin\beta)x-(b\cos\beta)y=a^2\sin\beta\cos\alpha-b^2\cos\beta\sin\alpha$

Altitude from B to PA: $(a\sin\alpha)x-(b\cos\alpha)y=a^2\sin\alpha\cos\beta-b^2\cos\alpha\sin\beta$

These two intersect at $H(h, k)$ given by

$$h=\frac{a^2}{a^2+b^2}\frac{\cos(\frac{\alpha+\beta}{2})}{\cos(\frac{\alpha-\beta}{2})},\quad k=\frac{b^2}{a^2+b^2}\frac{\sin(\frac{\alpha+\beta}{2})}{\cos(\frac{\alpha-\beta}{2})}$$

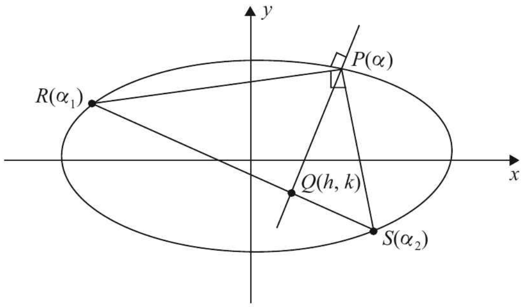
By virtue of the relation in (1), it can be observed that (h, k) satisfies the equation of E_2. Thus, H lies on E_2.

S51. Use the fact that that the equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_0, y_0) is

$$(b^2x_0)y-(a^2y_0)x+(a^2-b^2)x_0y_0=0$$

and apply the standard condition for concurrency for three lines.

S52. All the relevant parameters have been assumed as shown in the figure. We present the outline of the solution, with the reader expected to fill in the details.



Step 1: Since $PR \perp PS$, their slopes have a product of -1. Thus,

$$\frac{-b\cos(\frac{\alpha+\alpha_1}{2})}{a\sin(\frac{\alpha+\alpha_1}{2})}\times\frac{-b\cos(\frac{\alpha+\alpha_2}{2})}{a\sin(\frac{\alpha+\alpha_2}{2})}=-1$$

This when simplified, leads to

$$\cos\left(\frac{\alpha_1-\alpha_2}{2}\right)=\frac{a^2e^2}{a^2+b^2}\cos\left(\alpha+\frac{\alpha_1+\alpha_2}{2}\right) \tag{1}$$

Step 2: We write the equations of the normal at P and the chord RS:

$$\text{Normal at } P:\ (a\sec\alpha)x-(b\ \text{cosec}\ \alpha)y=a^2e^2 \tag{2}$$

$$RS\text{: } \frac{x}{a}\cos\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)=\cos\left(\frac{\alpha_1-\alpha_2}{2}\right) \tag{3}$$

Step 3: Use (1) in (3) to obtain the equation of RS as

$$b\cos\left(\frac{\alpha_1+\alpha_2}{2}\right)x+a\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)y=\frac{a^3be^2}{a^2+b^2}\cos\left(\alpha+\frac{\alpha_1+\alpha_2}{2}\right) \tag{4}$$

Step 4: Solve (2) and (4) to obtain $Q(h, k)$. To make things simpler, rewrite (4) using $\frac{\alpha_1+\alpha_2}{2}=\theta$. Thus,

$$(a\sec\alpha)x-(b\text{cosec}\ \alpha)y=a^2e^2 \tag{2}$$

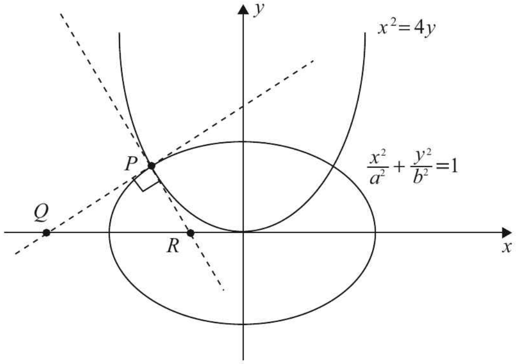
$$(b\cos\theta)x+(a\sin\theta)y=\frac{a^3be^2}{a^2+b^2}\cos(\alpha+\theta) \tag{4}$$

Solving these equations, we find that θ gets eliminated.

$$h = \frac{a^3e^2\cos\alpha}{a^2+b^2},\quad k = -\frac{a^2be^2\sin\alpha}{a^2+b^2}$$

The point Q is therefore $\left(\frac{a^3e^2\cos\alpha}{a^2+b^2}, \frac{-a^2be^2\sin\alpha}{a^2+b^2}\right)$.

S53. Note that $P \equiv (-2, 1)$, since it is one-end of the latus rectum of the parabola.

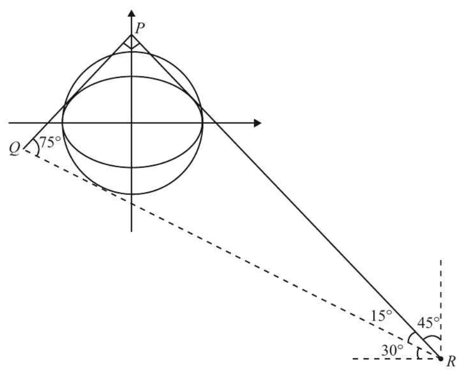


(a) At (–2, 1), the slope of the parabola is –1, while the slope of the ellipse is $\left.\frac{-b^2x}{a^2y}\right|_{(-2,1)} = \frac{2b^2}{a^2}$. Since the two curves are orthogonal at this point, we have $a^2 = 2b^2$. However, since (–2, 1) lies on the ellipse, we also have $\frac{4}{a^2}+\frac{1}{b^2}=1$. From these two conditions, $a^2 = 6$ and $b^2 = 3$, Thus, the equation of the ellipse is $\frac{x^2}{6}+\frac{y^2}{3}=1$.

(b) We have $Q \equiv (-3, 0)$ and $R \equiv (-1, 0)$. The circumcenter is the mid-point of QR, *i.e.*, (–2, 0) while the centroid is $(-2, \frac{1}{3})$.

S54. Note that P lies on the director circle $x^2 + y^2 = 4$. The equation of PR can be written as $\frac{x}{\sqrt{3}}\cos + y\sin\theta = 1$ for some θ. Since the slope of PR is –1, $\sin\theta = \frac{1}{2}$ and so the equation of PR is $x + y = 2$. This implies that the equation of PQ is $y - x = 2$. The slope of QR is $-\frac{1}{\sqrt{3}}$, while using the fact that QR is tangent to the auxiliary circle $x^2 + y^2 = 3$, its equation can be written as $x\cos\theta_1 + y\sin\theta_1 = \sqrt{3}$ for some θ_1. Thus, $-\cot\theta_1 = -\frac{1}{\sqrt{3}}$ or $\theta_1 = 60°$, and so the equation of QR is $x+\sqrt{3}y = -2\sqrt{3}$.

We now have the equations of the three sides of ΔPQR, from which we can obtain all the points:

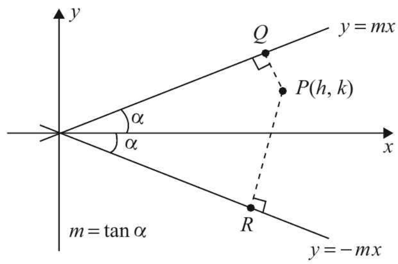


$$P \equiv (0, 2),\quad Q \equiv (-6+2\sqrt{3}, -4+2\sqrt{3}),\quad R \equiv (6+2\sqrt{3}, -4-2\sqrt{3})$$

The centroid is $(\frac{4}{\sqrt{3}}, -2)$ while the circumcenter is the mid-point of QR, *i.e.*, $(2\sqrt{3}, -4)$.

S55. As in many coordinate geometry problems, the ease with which you can solve this problem depends on your choice of the reference axes. Since no coordinates are mentioned anywhere in the problem, it is up to us to choose the appropriate reference axis. We choose our axes so that the two fixed lines are placed as shown in the figure.

Since $PQ^2 + PR^2 = \lambda$ (a constant), we have



$$\frac{(k-mh)^2}{1+m^2} + \frac{(k+mh)^2}{1+m^2} = \lambda$$

$$\Rightarrow \quad k^2 + m^2h^2 = \frac{\lambda}{2}(1+m^2)$$

Using (x, y) instead of (h, k), the locus of P is

$$\frac{x^2}{\frac{\lambda}{2}(1+\frac{1}{m^2})} + \frac{y^2}{\frac{\lambda}{2}(1+m^2)} = 1$$

To determine the eccentricity, we note that

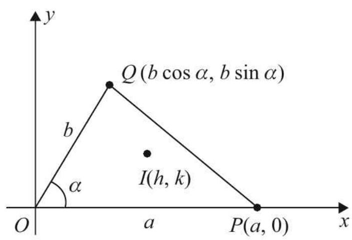
$$\alpha < \frac{\pi}{4} \Rightarrow \quad m < 1 \Rightarrow \quad 1 + \frac{1}{m^2} > 1 + m^2$$

$$\Rightarrow \quad e = \sqrt{1 - \frac{(1+m^2)}{\left(1+\frac{1}{m^2}\right)}} = \sqrt{1-m^2} = \frac{\sqrt{\cos 2\alpha}}{\cos\alpha}$$

Similarly, $\alpha > \frac{\pi}{4} \Rightarrow \quad m > 1 \Rightarrow \quad e = \sqrt{1 - \frac{1}{m^2}} = \frac{\sqrt{-\cos 2\alpha}}{\sin\alpha}$

Thus, the eccentricity can take two values depending on what value α takes.

S56. Perhaps the most appropriate choice of the reference axes is the one presented below. Let s be the sum of the sides. We are given that a and s are fixed. Therefore,



$$PQ^2 = (s-(a+b))^2 = a^2 + b^2 - 2ab\cos\alpha \qquad (1)$$

Now, if $I(h, k)$ is the incenter, then

$$h = \frac{ab\cos\alpha + ab}{s}, \quad k = \frac{ab\sin\alpha}{s} \qquad (2)$$

From (1) and (2), it is straightforward to show that $b = \frac{s}{2} + h - a$. Also, if we eliminate α from (2), we have

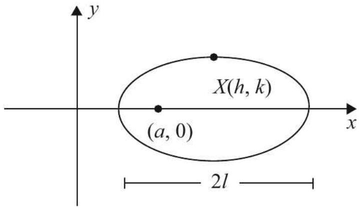
$$s^2k^2 + (sh - ab)^2 = a^2b^2 \qquad (3)$$

We substitute $b = \frac{s}{2} + h - a$ in (3), and thus a relation in h and k is obtained, which can be rearranged to

$$\left(h - \frac{a(s-2a)}{2s}\right)^2 + \left(\frac{s}{s-2a}\right)k^2 = \frac{a^2(s-2a)^2}{s^2}$$

This is an ellipse with its center at $(\frac{a(s-2a)}{2s}, 0)$

S57. The choice of the reference axes should be more or less obvious. The directrix should be taken to be the y-axis, while the focus can be taken as $(a, 0)$:



Let the (variable) eccentricity of the ellipse be denoted by e. Thus, for a point $P(x, y)$ on the ellipse,

$$(x-a)^2 + y^2 = e^2x^2 \quad \text{(how?)} \tag{1}$$

Let $X(h, k)$ be the extremity of the minor axis. To obtain h, k, we need to use the major-axis length $2l$ of the ellipse. We can easily show that $l = \frac{ae}{1-e^2}$. Thus,

$$h = \frac{l}{e} = \frac{a}{1-e^2} \quad \Rightarrow \quad k = \frac{\pm ae}{\sqrt{1-e^2}} \qquad \text{(Using (1))}$$

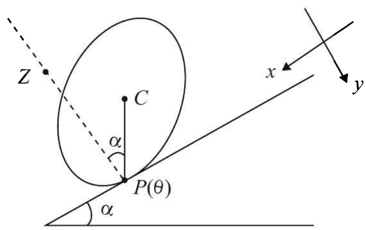
$$\Rightarrow \quad k^2 = a(h-a)$$

The locus of the extremities is therefore the parabola $y^2 = a(x-a)$.

S58. Carefully observe how we have chosen the reference axes, as shown in the figure. Also, the origin is the point C. With respect to this axes, let $P \equiv (a\cos\theta, b\sin\theta)$:

$\Rightarrow$ Slope of $CP = \frac{b}{a}\tan\theta$

$\Rightarrow$ Slope of normal $PZ = \frac{a}{b}\tan\theta$



$$\Rightarrow \quad \tan\alpha = \frac{a^2-b^2}{2ab}\sin 2\theta \quad \Rightarrow \quad \sin 2\theta = \frac{2ab}{a^2-b^2}\cdot\tan\alpha$$

Using $\sin 2\theta \le 1$ and $e^2 = 1 - \frac{b^2}{a^2}$, we proceed as follows:

$$\frac{2\left(\frac{b}{a}\right)}{1-\frac{b^2}{a^2}}\tan\alpha \le 1 \quad \Rightarrow \quad \frac{2\sqrt{1-e^2}}{e^2}\tan\alpha \le 1$$

$$\Rightarrow \quad e^4 + 4e^2\tan^2\alpha - 4\tan^2\alpha \ge 0$$

If we consider the expression on the left hand side as a quadratic in e^2, then the positive zero of that quadratic will be $2\tan\alpha(\sec\alpha - \tan\alpha)$. Thus,

$$e^2 \geq 2\tan\alpha(\sec\alpha - \tan\alpha) = \frac{2\sin\alpha}{1+\sin\alpha}$$

$$\Rightarrow \quad e \geq \sqrt{\frac{2\sin\alpha}{1+\sin\alpha}}$$

The minimum possible value of e is $\sqrt{\frac{2\sin\alpha}{1+\sin\alpha}}$.

S59. We simply observe that since C is the director circle of E, the tangents drawn from any point on C to E must be perpendicular. On the other hand, the asymptotes of H are $y = \pm x$. The tangents are therefore equally inclined to the asymptotes.

S60. The general equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ through the points α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right) \qquad (1)$$

(a) This result is proved directly by making the coordinates $(\pm ae, 0)$ of the foci satisfy (1).
(b) From the first part,

$$\pm e = \frac{\cos(\frac{\alpha+\beta}{2})}{\cos\frac{\alpha-\beta}{2}} \quad \Rightarrow \quad \frac{\pm e - 1}{\pm e + 1} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

Hence, the assertion of the problem holds.

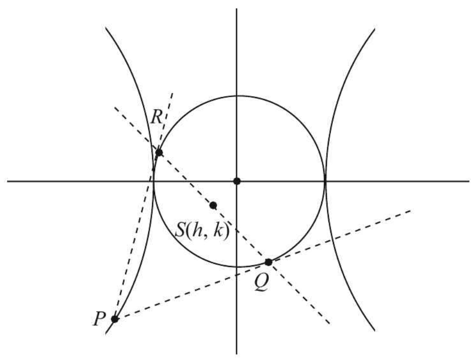
S61. We assume $P \equiv (3\sec\theta, 2\tan\theta)$.
The equation of the chord of contact from P to the given circle will be

$$(3\sec\theta)x + (2\tan\theta)y = 9 \qquad (1)$$

The chord of the circle with mid-point $S(h, k)$ will be

$$hx + ky = h^2 + k^2 \qquad (2)$$

Since (1) and (2) are essentially representing the same line (QR), we have



$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2} \quad \Rightarrow \quad \begin{cases} \sec\theta = \dfrac{3h}{h^2+k^2} \\ \tan\theta = \dfrac{(9/2)k}{h^2+k^2} \end{cases}$$

Eliminating θ, we have

$$\left(\frac{3h}{h^2+k^2}\right)^2 - \left(\frac{9/2\,k}{h^2+k^2}\right)^2 = 1 \quad \Rightarrow \quad 36h^2 - 81k^2 = 4(h^2+k^2)^2$$

The required locus is thus

$$36x^2 - 81y^2 = 4(x^2 + y^2)^2$$

S62. The general equations of tangents to the two hyperbolas can be respectively written as

$$y = m_1 x \pm \sqrt{a^2 m_1^2 - b^2}, \quad x = m_2 y \pm \sqrt{a^2 m_2^2 - b^2}$$

For the two be identical,

$$m_2 = \frac{1}{m_1} \quad \Rightarrow \quad a^2 m_1^2 - b^2 = a^2 - \frac{b^2}{m_2^2} \quad \Rightarrow \quad m_1 = \pm 1$$

Thus, there are four common tangents, with the equations

$$y = \pm\, x \pm \sqrt{a^2 - b^2}$$

We pick one of these common tangents, say $y = x + \sqrt{a^2 - b^2}$, and find its points of contact $A(x_1, y_1)$ and $B(x_2, y_2)$ with the two hyperbolas. This is straightforward and left to the reader to complete:

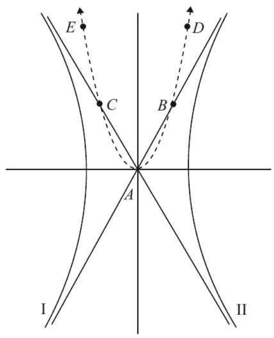
$$A(x_1, y_1) \equiv \left(\frac{-a^2}{\sqrt{a^2 - b^2}}, \frac{-b^2}{\sqrt{a^2 - b^2}} \right); \quad B(x_2, y_2) = \left(\frac{b^2}{\sqrt{a^2 - b^2}}, \frac{a^2}{\sqrt{a^2 - b^2}} \right)$$

Using the distance formula,

$$AB = \frac{\sqrt{2}(a^2 + b^2)}{\sqrt{a^2 - b^2}}$$

Note that all the common tangents will be of equal lengths, due to the symmetry of the situation.

S63. The asymptotes of the given hyperbola are $y = \pm 2x$, as shown below. The dotted curve represents $y = x^2$, the curve on which any point will be of be the form (λ, λ^2).



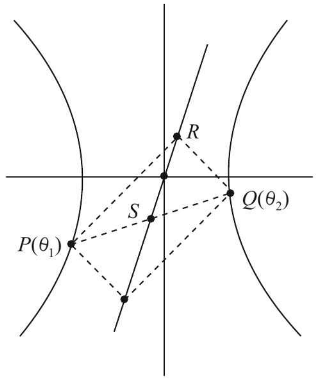
The crucial step towards the solution is that two tangents can be drawn to the two different branches of the hyperbola from any point on the curve $y = x^2$ *only if* that point lines on the portions BD (upto ∞) and CE (upto ∞), and *not on* the portions AB or AC.

To clarify this further, think of a point on the portion AC of $y = x^2$. From this point, it is possible to drawn two tangents to the *same* branch of the hyperbola, namely I, but none to branch II. On the other hand, from any point on the portion AB, it is possible to draw two tangents but again to the *same* branch, namely II, but none to branch I. However, from any point on either of CE or BD (the reader

should consider these extending to ∞), it is possible to draw exactly one tangent to I and exactly one tangent to II. Thus, we require $\lambda^2 > 2|\lambda|$ or

$$\lambda \in (-\infty, -2) \cup (2, \infty)$$

S64. We assume the hyperbola as



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

so that its asymptotes will be

$$y = \pm\frac{b}{a}x$$

In addition, we assume P and Q as two opposite vertices of the parallelogram as well as lying on the hyperbola, with eccentric angles θ_1 and θ_2.

Since we have the coordinates of P and Q and the slopes of both PR and QR (they are parallel to the asymptotes), we have

$$\begin{aligned}&\text{Equation of} \quad y - b\tan\theta_1 = -\frac{b}{a}(x - a\sec\theta_1)\\ &PR{:}(L_1 = 0){:} \quad \Rightarrow \quad bx - ay - ab(\sec\theta_1 + \tan\theta_1) = 0\end{aligned} \tag{1}$$

$$\begin{aligned}&\text{Equation of} \quad y - b\tan\theta_2 = -\frac{b}{a}(x - a\sec\theta_2)\\ &QR{:}\,(L_2 = 0){:} \quad \Rightarrow \quad bx + ay - ab(\sec\theta_2 + \tan\theta_2) = 0\end{aligned} \tag{2}$$

Using a 'family of lines' approach, the equation of a line through R can be written as $L_1 + \lambda L_2 = 0$, or

$$\{bx - ay - ab(\sec\theta_1 - \tan\theta_1)\} + \lambda\{bx + ay - ab(\sec\theta_2 + \tan\theta_2)\} = 0 \tag{3}$$

We can easily find that value of λ for which this line passes through the origin. The final step is to shown that the coordinates of S, *i.e.*, $(\frac{a}{2}(\sec\theta_1 + \sec\theta_2), \frac{b}{2}(\tan\theta_1 + \tan\theta_2))$, satisfy (3) for this value of λ. This is straightforward and left to the reader as an exercise.

S65. (a) We take $H \equiv xy = c^2$ and $C \equiv (x-\alpha)^2 + (y-\beta)^2 = r^2$. If any point on H, say $(ct, \frac{c}{t})$, lies on C, we have

$$(ct - \alpha)^2 + \left(\frac{c}{t} - \beta\right)^2 = r^2$$

$$\Rightarrow \quad c^2t^4 - 2\alpha ct^3 + (\alpha^2 + \beta^2 - r^2)t^2 - 2\beta ct + c^2 = 0 \tag{1}$$

The mean of the four points P, Q, R, S will be

$$X \equiv \left(\frac{\sum_{i=1}^{4} ct_i}{4}, \frac{\sum_{i=1}^{4} \frac{c}{t_i}}{4} \right) = \left(\frac{c}{4}\sum_{i=1}^{4} t_i, \frac{c}{4}\sum_{i=1}^{4} \frac{1}{t_i} \right)$$

From (1), we have $\sum_{i=1}^{4} t_i = \frac{2\alpha}{c}$, while

$$\sum_{i=1}^{4} \frac{1}{t_i} = \frac{t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4}{t_1t_2t_3t_4} = \frac{\frac{2\beta c}{c^2}}{\frac{c^2}{c^2}} = \frac{2\beta}{c}$$

Thus,

$$X \equiv \left(\frac{\alpha}{2}, \frac{\beta}{2} \right)$$

On the other hand, the mid-point of O_H and O_C is

$$Y \equiv \left(\frac{0+\alpha}{2}, \frac{0+\beta}{2} \right) = \left(\frac{\alpha}{2}, \frac{\beta}{2} \right)$$

This means that X and Y are the same points! Hence, $XY = 0$.

(b) Let the circle be $x^2 + y^2 = r^2$, and the five points be $P_i \equiv (r\cos\theta_i, r\sin\theta_i)$, $i = 1, 2, 3, 4, 5$. Let (h_i, k_i) represent the center of that rectangular hyperbola which passes through the four points P_j where $j \neq i$. For example, (h_1, k_1) will be the center of the hyperbola passing through P_2, P_3, P_4, P_5. Now, from the result of part - (a), we have

$$\frac{h_1}{2} = \frac{r}{4}(\cos\theta_2 + \cos\theta_3 + \cos\theta_4 + \cos\theta_5), \frac{k_1}{2} = \frac{r}{4}(\sin\theta_2 + \sin\theta_3 + \sin\theta_4 + \sin\theta_5)$$

$$\Rightarrow \quad 2h_1 = r\sum_{i=1}^{5} \cos\theta_i - r\cos\theta_1, 2k_1 = r\sum_{i=1}^{5} \sin\theta_i - r\sin\theta_1 \qquad (2)$$

However, r and the quantities $\sum_{i=1}^{5} \cos\theta_i$ and $\sum_{i=1}^{5} \sin\theta_i$ are fixed. If we let $r\sum_{i=1}^{5} \cos\theta_i = \lambda$ and $r\sum_{i=1}^{5} \sin\theta_i = \mu$ where λ and μ are constants, we have from (2),

$$2h_1 = \lambda - r\cos\theta_1, \quad 2k_1 = \mu - r\sin\theta$$

$$\Rightarrow \quad \left(h_1 - \frac{\lambda}{2} \right)^2 + \left(k_1 - \frac{\mu}{2} \right)^2 = \left(\frac{r}{2} \right)^2 \qquad (3)$$

(2) will be satisfied not just by (h_1, k_1) but every (h_i, k_i) for i = 1, 2, 3, 4, 5. Thus, the center of the hyperbolas lie on the cirlce

$$\left(x - \frac{\lambda}{2} \right)^2 + \left(y - \frac{\mu}{2} \right)^2 = \left(\frac{r}{2} \right)^2$$

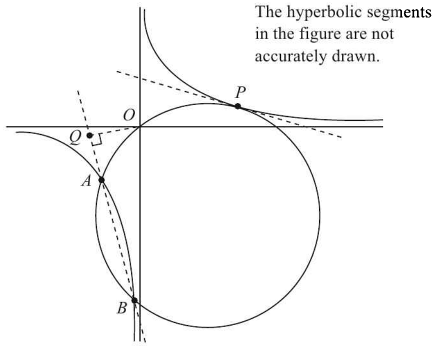
which has its radius as $\frac{r}{2}$.

S66. We assume the hyperbola to be $xy = c^2$, and P to be the point $(ct, \frac{c}{t})$, so that the equation of the tangent at P is

$$x + yt^2 = 2ct \tag{1}$$

A neat artifice for this problem will be to assume the equation of the line AB in perpendicular form, because OQ is the perpendicular to this line from the origin, and we have to find the locus of Q.

Thus, if assume OQ is of length p, we have



$$\left.\begin{aligned} Q &\equiv (h,k) \equiv (p\cos\theta,\ p\sin\theta) \\ AB &\equiv x\cos\theta + y\sin\theta = p \end{aligned}\right\} \quad (\theta \text{ corresponds to the inclination of } OQ.) \tag{2}$$

The equation of a second degree curve (which passes through A and B, and touches the hyperbola at P) can now be written from (1) and (2) as follows.

$$\underbrace{(x + yt^2 - 2ct)}_{AB}\ \underbrace{(x\cos\theta + y\sin\theta - p)}_{\text{Tangent at } P} + \lambda\ \underbrace{(xy - c^2)}_{\text{Hyperbola}} = 0 \tag{3}$$

The next step is to impose the necessary constraints for (3) to represent a circle passing through the origin. This will give.

$$\cos\theta = t^2\sin\theta, \quad 2pt = \lambda c, \quad \lambda + \sin\theta + t^2\cos\theta = 0 \tag{4}$$

We can finally eliminate t and θ to obtain a relation satisfied by h and k, which will come out to be $4hk = c^2$. Thus, the required locus is

$$4xy = c^2$$

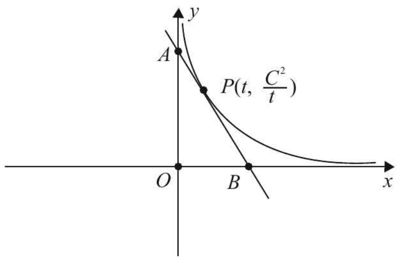
S67. Let the hyperbola be $xy = c^2$. If P is the point t, the tangent at P has the equation

$$y = \left(-\frac{c^2}{t^2}\right)x + \frac{2c^2}{t}$$

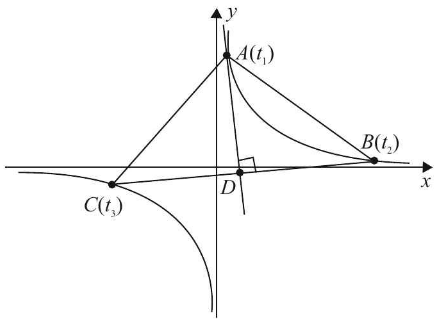
This intersects the axes at

$$A \equiv \left(0, \frac{2c^2}{t}\right), \quad B \equiv (2t, 0)$$

The area of ΔOAB is $\frac{1}{2} \times OB \times OA = \frac{1}{2} \times 2t \times \frac{2c^2}{t} = 2c^2$, which is independent of the position of P.



S68. Let the rectangular hyperbola be $xy = c^2$, and the vertices of the triangle be $A(t_1)$, $B(t_2)$ and $C(t_3)$.



We are given that $AD \perp BC$. The slope of AD, which is the tangent at A, is $-c^2/t_1^2$, while the slope of BC is

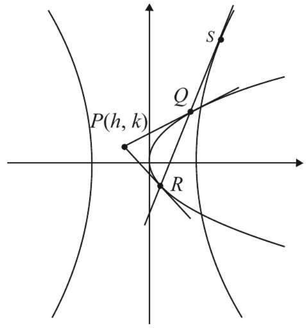
$$m_{BC} = \frac{\frac{c^2}{t_2} - \frac{c^2}{t_3}}{t_2 - t_3} = -\frac{c^2}{t_2 t_3}$$

Thus, $\dfrac{c^4}{t_1^2 t_2 t_3} = -1$

Now, we consider the product of the slopes of AB and AC:

$$m_{AB} \times m_{AC} = \frac{\frac{c^2}{t_2} - \frac{c^2}{t_1}}{t_2 - t_1} \times \frac{\frac{c^2}{t_3} - \frac{c^2}{t_1}}{t_3 - t_1} = \frac{c^4}{t_1^2\, t_2\, t_3} = -1$$

Therefore, ΔABC is right angled at A.



S69. Let the hyperbola be $x^2 - y^2 = c^2$, while the parabola be $y^2 = 4ax$. We wish to find the locus of $P(h, k)$. The equation of the chord of contact QR from P is $ky = 2a(x+h)$, *i.e.*,

$$y = \frac{2a}{k}x + \left(\frac{2ah}{k}\right)$$

Since this touches the hyperbola (at S), it must be of the form

$$y = mx + c\sqrt{m^2 - 1}$$

Thus,

$$\left.\begin{aligned} m &= \frac{2a}{k} \\ c\sqrt{m^2-1} &= \frac{2ah}{k} \end{aligned}\right\} \xrightarrow[m]{\text{Eliminate}} 4a^2h^2 + c^2k^2 = 4a^2c^2$$

Using $(h, k) \to (x, y)$, the required locus is $\frac{x^2}{c^2} + \frac{y^2}{4a^2} = 1$, which is an ellipse, with the same center and axis as the hyperbola.

3-D Geometry

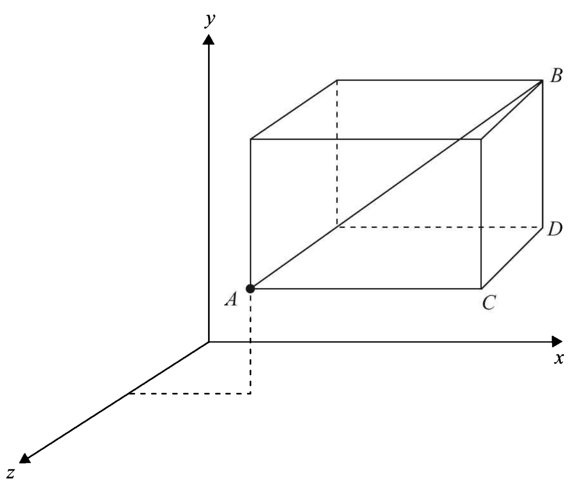
PART-A: Summary of Important Concepts

Most of the results in 3-D geometry follow in a straight forward manner from the various results in Vectors; it is thus strongly suggested that as a prelude to this chapter, you should have covered the chapter on Vectors as thoroughly as possible.

1. Important Results and Formula

1.1 Distance Formula

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two arbitrary points:



Note that since
$A \equiv (x_1, y_1, z_1)$
$B \equiv (x_2, y_2, z_2)$

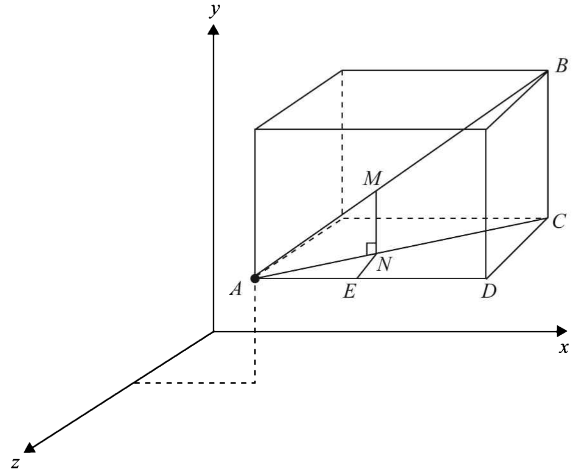
we have,
$AC = |x_2 - x_1|$
$BD = |y_2 - y_1|$
$CD = |z_2 - z_1|$

We have $AB^2 = AC^2 + BD^2 + CD^2$, and so

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

1.2 Section Formula

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two arbitrary points. Let M be the point dividing AB internally in the ratio $m : n$, as shown later.



Let M be the point on AB such that

$$\frac{AM}{MB} = \frac{m}{n}.$$

Drop a perpendicular (MN) from M onto AC.

We note that $\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$.

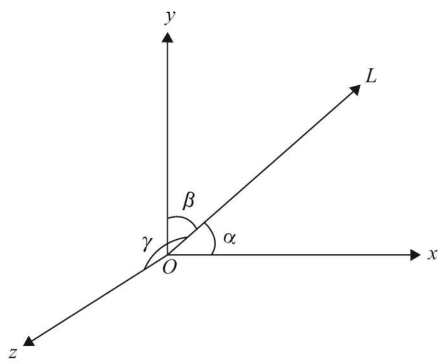
The coordinates of M can consequently be obtained as

$$M \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

The coordinates of the point M which divides AB externally in the ratio $m:n$ can be obtained by substituting $-n$ for n in the expressions for the coordinates of M.

1.3 Direction Cosines and Direction Ratios

The direction cosines of a (directed) line are the cosines of the angles which the line makes with the positive directions of the coordinate axes. Consider a line OL as shown, passing through the origin O. Let OL be inclined at angles α, β, γ to the coordinate axes.



Thus, the direction cosines are given by

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

Note that for the line LO (*i.e.*, the directed line segment in the direction opposite to OL), the direction cosines will be $-l, -m, -n$. The direction cosines for a directed line L *not* passing through the origin are the same as the direction cosines of the directed line parallel to L and passing through the origin.

Note that for any point, P lying on the line OL with direction cosines l, m, n, such that $OP = r$, the coordinates of P will be

$$x = lr, \quad y = mr, \quad z = nr \quad \Rightarrow \quad P \equiv (lr, mr, nr)$$

Now, since $OP = r$, we have

$$\Rightarrow \quad \sqrt{l^2r^2 + m^2r^2 + n^2r^2} = r \quad \Rightarrow \quad \boxed{l^2 + m^2 + n^2 = 1}$$

The direction cosines of *any* line will satisfy this relation.
The direction ratios are simply a set of three real numbers a, b, c proportional to l, m, n, *i.e.*,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

From this relation, we can write

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \pm\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}} = \pm\sqrt{a^2+b^2+c^2}$$

$$\Rightarrow \quad \boxed{l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}}$$

These relations tell us how to find the direction cosines from direction ratios. Note that the direction cosines for any line must be unique. However, there are infinitely many sets of direction ratios since direction ratios are just a set of any three numbers proportional to the direction cosines.

(a) We note the very important fact that for two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ such that $AB = d$, the direction cosines of AB are given by

$$l = \frac{x_2 - x_1}{d}, \; m = \frac{y_2 - y_1}{d}, \; n = \frac{z_2 - z_1}{d}$$

(b) Let the lines L_1 and L_2 have direction cosines $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$.

(i) The angle θ between L_1 and L_2 is given by

$$\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$$

(ii) L_1 is parallel to L_2 if

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

(iii) L_1 is perpendicular to L_2 if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

(c) The direction cosines of a line L_3 perpendicular to both L_1 and L_2 will be

$$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$$

2. Planes

In the chapter on Vectors, we learn how to write the equations for a plane in different forms. In this section, we will extend that discussion and learn how to write the equation of a plane in three dimensional coordinates form.

The general vector equation of a plane is of the form

$$\vec{r}\cdot\vec{n} = l; \qquad l \text{ is a constant}$$

where $\vec{r}$ is the variable vector $x\hat{i} + y\hat{j} + z\hat{k}$ representing any point on the plane, while $\vec{n}$ is a fixed vector, say $a\hat{i}+b\hat{j}+c\hat{k}$ which is perpendicular to the plane. Thus, the equation of the plane can be written as

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(a\hat{i} + b\hat{j} + c\hat{k}) = l$$

$$\Rightarrow \quad ax + by + cz = l$$

$$\Rightarrow \quad \boxed{ax + by + cz + d = 0}; \qquad d = -l$$

This is the most general equation of a plane in coordinate form. Note that this equation of the plane contains only three arbitrary constants, for, it can be written as

$$\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0$$

$$\Rightarrow \quad \lambda_1 x + \lambda_2 y + \lambda_3 z + 1 = 0$$

Thus, three independent constraints are sufficient to uniquely determine a plane. For example, three non collinear points are sufficient to uniquely determine the plane passing through them. Note the following important results:

(a) The equation of an arbitrary plane passing through the point $A(x_1, y, z_1)$ can be written as

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

(b) The equation of the plane intercepting lengths a, b and c on the $x-$, $y-$ and $z-$axis is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(c) The acute angle of intersection θ of the two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

is the acute angle between their normals, and is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) The two planes are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
(ii) The two planes are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(d) The distance of the point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by

$$l = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(e) The distance between the two parallel planes

$$ax + by + cz + d_1 = 0$$
$$ax + by + cz + d_2 = 0$$

is given by

$$l = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

(f) The equations of the planes bisecting the angles between two given planes

$$P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$$
$$P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$$

are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

As expected, we get two angle bisector planes, one corresponding to the "+" and one to the "–" sign. As in the case of straight line angle bisectors, we can prove that the equation of the angle bisector containing the origin will be given by the "+" sign if d_1 and d_2 are of the same sign.

(g) For two planes with the equations

$$P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0 \equiv \vec{r} \cdot \vec{n}_1 + d_1 = 0$$

$$P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0 \equiv \vec{r} \cdot \vec{n}_2 + d_2 = 0,$$

we have already proved in the chapter on vectors that *any* plane passing through the intersection line of P_1 and P_2 can be written as

$$P_1 + \lambda P_2 = 0, \quad \lambda \in \mathbb{R}$$

3. Straight Lines

In this section, we will discuss how to write the equation for a straight line in coordinate form. There are essentially two different ways of doing so:

(a) Unsymmetrical form of the equation of a line

A line can be defined as the intersection of two planes. Thus, the equations of two planes considered together represents a straight line. For example, the set of equations

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

represents the straight line formed by the intersection of these two planes. Recall that the planes will intersect only if they are non-parallel, *i.e.*, only if

$$a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$$

(b) Symmetrical form of the equation of a line

Consider a line with direction cosines *l*, *m*, *n* and passing through the point $A(x_1, y_1, z_1)$. For any point $P(x, y, z)$ on this line, the set of numbers $\{(x - x_1), (y - y_1), (z - z_1)\}$ must be proportional to the direction cosines, as has already been discussed. Thus, the equation of this line can be written as

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Extending this, we can write the equation of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Note that for any point $P(x, y, z)$ at a distance *r* from $A(x_1, y_1, z_1)$ along the line with direction cosines *l*, *m*, *n*, we have

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

Thus, the coordinates of *P* can be written as

$$\boxed{x = x_1 + lr, \quad y = y_1 + mr, \quad z = z_1 + nr}$$

This is a useful fact and we'll be using it frequently.

IMPORTANT IDEAS AND TIPS

1. ***2-D and 3-D Geometry:*** It is helpful to remember the different formulae in 3-D geometry if you are able to see how they are more general cases of the analogous results in 2-D geometry. Here are two simple examples:

	2-D Result	**3-D Result**
Intercept form of line/plane	$\frac{x}{a} + \frac{y}{b} = 1$	$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
Distance between parallel lines/planes	$\frac{\lvert c_1 - c_2 \rvert}{\sqrt{a^2 + b^2}}$	$\frac{\lvert d_1 - d_2 \rvert}{\sqrt{a^2 + b^2 + c^2}}$

2. ***Vectors and 3-D Geometry:*** The most important idea you can learn from this chapter is that there is a very close connection between vectors and 3-D geometry. For example, the results which we have encountered for lines and planes in 3-D geometry are analogous to the relevant results in vectors. Making this connection between the two subjects, in our opinion, is really important. In the problems, we will use concepts of vectors in many cases to highlight this connection.

3. ***DCs and DRs:*** We have discussed this in the chapter on Vectors, and repeating it here for the sake of emphasis. Keep in mind the difference between the direction cosines (DCs) and direction ratios (DRs) of a directed straight line (or a vector):

(a) DCs are unique and represented conventionally by l, m, n, such that $l^2 + m^2 + n^2 = 1$.
(b) DRs are any three numbers proportional to the DCs. They are not unique.

Many times, the following question may bother you: If DCs are sufficient to specify the direction of a vector, then what is the need of DRs? The answer is that DRs are a matter of convenience. Suppose that the DCs of a line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$. If I were to tell you the direction of the line, I could say that the direction is given by (1, 2, 3) instead of saying that the direction is given by $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$. The former is easier to say, and using the former (the non-unique DRs), I can anyway deduce the latter (the unique DCs). I could even have said that the direction is given by (2,4,6) or (200,400,600), etc. That is all there is to the concept of DRs, a set of numbers which help us easily to specify the direction of a directed straight line (or segment).

4. ***Visualization:*** As in the case of studying vectors, visualization is extremely important for effective problem solving. For example, the concept of skew lines, the concept of specifying a line through the intersection of two planes, etc, can be understood much better if you take the help of your imagination (or surroundings) to actually visualize them.

3-D Geometry

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

The number of lines which we can draw that are equally inclined to each of the three coordinate axes is

(A) 3 (B) 4 (C) 6 (D) 8

Solution: Intuitively, we can expect the answer to be 8, one for each of the 8 octants. Lets try to derive this answer rigorously. Assume the direction cosines of the lines to be l, m, n. Thus,

$$l^2 + m^2 + n^2 = 1 \qquad (1)$$

But since the lines are equally inclined to the three axes, we have $|l| = |m| = |n|$. This gives using (1),

$$|l| = |m| = |n| = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad l = \pm\frac{1}{\sqrt{3}},\ m = \pm\frac{1}{\sqrt{3}},\ n = \pm\frac{1}{\sqrt{3}}$$

It is obvious that 8 combinations of l, m, n are possible. Hence, 8 lines can be drawn which are equally inclined to the axes. The correct option is (D). ■

Example 2

What is the angle between the lines whose direction cosines are given by the following equations?

$$3l + m + 5n = 0, \quad 6mn - 2nl + 5lm = 0$$

(A) $\sin^{-1}\frac{1}{6}$ (B) $\cos^{-1}\frac{1}{6}$ (C) $\sin^{-1}\frac{1}{3}$ (D) $\cos^{-1}\frac{1}{3}$ (E) None of these

Solution: Using the value of m from the first equation in the second, we have

$$-6(3l + 5n)n - 2nl - 5l(3l + 5n) = 0$$

$$\Rightarrow \quad 45ln + 30n^2 + 15l^2 = 0 \quad \Rightarrow \quad 2n^2 + 3ln + l^2 = 0$$

$$\Rightarrow \quad (2n + l)(n + l) = 0 \quad \Rightarrow \quad 2n = -l \text{ or } n = -l$$

For $l = -2n$, we obtain $m = n$. A set of direction ratios of one line is therefore $\{-2n, n, n\}$.
For $l = -n$, we obtain $m = -2n$. A set of direction ratios of the other line is therefore $\{-n, -2n, n\}$.

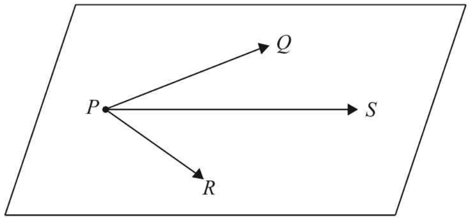
The angle between the two lines can now be easily evaluated to be $\cos^{-1}(\frac{1}{6})$. The correct option is (B). ■

Example 3

Let the equation of the plane passing through the points $P(1, 1, 0)$, $Q(1, 2, 1)$ and $R(-2, 2, -1)$ be $ax + by + cz = 5$. The value of $a + b + c$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Let $S(x, y, z)$ be any arbitrary point in the plane whose equation we wish to determine:



Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ will be perpendicular to this plane, we must have

$$\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0 \quad \Rightarrow \quad \{(x-1)\hat{i} + (y-1)\hat{j} + z\hat{k}\} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \{(x-1)\hat{i} + (y-1)\hat{j} + z\hat{k}\} \cdot (-2\hat{i} - 3\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow \quad 2(x-1) + 3(y-1) - 3z = 0 \quad \Rightarrow \quad 2x + 3y - 3z = 5$$

We could have proceeded alternatively as follows: any arbitrary plane through $P(1, 1, 0)$ will be of the form

$$a(x-1) + b(y-1) + cz = 0$$

$$\Rightarrow \quad \lambda_1(x-1) + \lambda_2(y-1) + z = 0; \quad \lambda_1 = \frac{a}{c}, \lambda_2 = \frac{b}{c}$$

If this passes through $Q(1, 2, 1)$ and $R(-2, 2, -1)$, we have

$$\lambda_2 + 1 = 0 \quad \text{and} \quad -3\lambda_1 + \lambda_2 - 1 = 0$$

$$\Rightarrow \quad \lambda_1 = -\frac{2}{3}, \lambda_2 = -1$$

Thus, the equation of the plane is

$$-\frac{2}{3}(x-1) - (y-1) + z = 0$$

$$\Rightarrow \quad 2(x-1) + 3(y-1) - 3z = 0$$

$$\Rightarrow \quad 2x + 3y - 3z = 5$$

We see that that the value of $a + b + c$ is 2. The correct option is (B). ■

Example 4

What is the distance of the point A (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) None of these

Solution: The direction cosines of the line parallel to whom we wish to measure the distance, can be evaluated to be $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$. Thus, any point on the line through A with these direction cosines, at a distance r from A, will have the coordinates

$$\left(1+\frac{2r}{7},\ -2+\frac{3r}{7},\ 3-\frac{6r}{7}\right)$$

If this point lies on the given plane, we have

$$\left(1+\frac{2r}{7}\right)-\left(-2+\frac{3r}{7}\right)+\left(3-\frac{6r}{7}\right)=5 \quad \Rightarrow \quad r=1$$

Thus, the required distance is 1 unit. ■

SUBJECTIVE TYPE EXAMPLES

Example 5

Find the locus of a point P which moves so that its distances from the points A (0, 2, 3) and B (2, –2, 1) are always equal.

Solution: P will obviously lie on the perpendicular bisector of AB. Let the coordinates of P be (x, y, z). Therefore,

$$PA^2 = PB^2$$

$$\Rightarrow \quad x^2 + (y-2)^2 + (z-3)^2 = (x-2)^2 + (y+2)^2 + (z-1)^2$$

$$\Rightarrow \quad x - 2y - z + 1 = 0$$

This is the required locus of P (it is a plane, *i.e.*, the perpendicular bisector is a plane). ■

Example 6

Find the projection of the line segment joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ onto a line with direction cosines l, m, n.

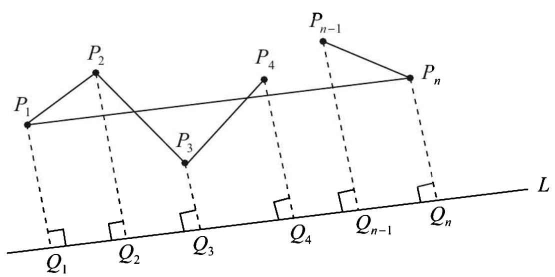
Solution: Let us first consider a vector approach to this problem. The vector $\overrightarrow{AB}$ can be written as

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

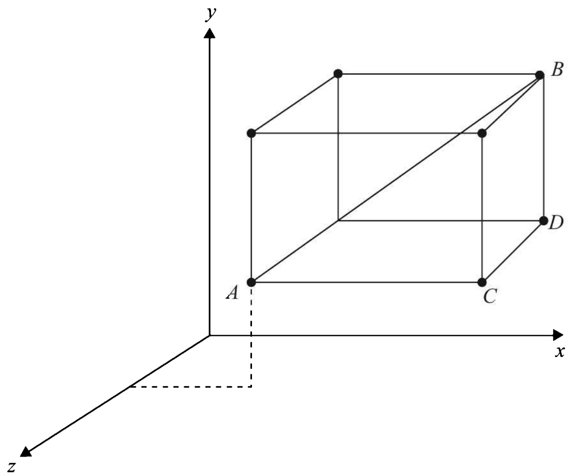
A unit vector $\hat{u}$ along the line with direction cosines l, m, n will be $\hat{u} = l\hat{i} + m\hat{j} + n\hat{k}$. Therefore, the projected length of $\overrightarrow{AB}$ upon this line will be.

$$d = |\overrightarrow{AB} \cdot \hat{u}| = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

Let us discuss an alternative approach without using vectors. For this, we first understand the projection of a sequence of line segments on a given line. Assume $P_1, P_2, P_3, \ldots, P_n$ to be n points in space. The sum of projections of the sequence of segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$ onto a fixed line L will be the same as the projection of P_1P_n onto L. This should be obvious from the following diagram:



The projection of the segment P_1P_n onto L is Q_1Q_n. The sum of projections of segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$ onto L is $Q_1Q_2 + Q_2Q_3 + \cdots + Q_{n-1}Q_n = Q_1Q_n$. We use this fact in our original problem as shown later.



The projection d of AB onto any line L (with direction cosines, say l, m, n) will be sum of projections of AC, BD, CD onto L are $l(x_2 - x_1), m(y_2 - y_1)$ and $n(z_2 - z_1)$ respectively. Thus, we get the total projection of AB onto L as:

$$d = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

■

Example 7

Find the direction cosines of the line $6x - 2 = 3y + 1 = 2z - 2$.

Solution: We have

$$6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z - 1)$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z - 1}{3}$$

Comparing this with the symmetrical form of the equation of a line, we can say that the direction ratios of this line are proportional to 1, 2, 3. Thus, the direction cosines are:

$$l = \frac{1}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}, \quad m = \frac{2}{\sqrt{14}}, \quad n = \frac{3}{\sqrt{14}}$$

$\Rightarrow$ The direction cosines are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$. ■

Example 8

Find a set of direction ratios of the line

$$a_1x + b_1y + c_1z + d_1 = 0; \quad a_2x + b_2y + c_2z + d_2 = 0$$

$$a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$$

Solution: The equation of the line has been specified in unsymmetric form, *i.e.*, as the intersection of two non-parallel planes. Visualize in your mind that when two planes intersect, the line of intersection will be perpendicular to normals to both the planes. Normal vectors to the two planes can be taken to be

$$\vec{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$
$$\vec{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

Thus, the line of intersection will be parallel to $\vec{n}_1 \times \vec{n}_2$, *i.e.*, to

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \hat{i}(b_1c_2 - b_2c_1) + \hat{j}(c_1a_2 - a_1c_2) + \hat{k}(a_1b_2 - a_2b_1)$$

A set of direction ratios of the line of intersection can be taken to be

$$(b_1c_2 - b_2c_1), (c_1a_2 - a_1c_2), (a_1b_2 - a_2b_1)$$ ■

Example 9

Find the angle of intersection of the two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

Solution: Though we already know the result of this problem (it is mentioned in the theory), it will be helpful if you know the proof of the result. From the equations of the planes, it is evident that the following vectors are perpendicular to these planes respectively:

$$\vec{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$
$$\vec{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

Since the acute angle θ, between the two planes, will be the acute angle between their normals, we have

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Incidentally, we can now derive the conditions for these planes to be parallel or perpendicular.

Planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2 \quad \Rightarrow \quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Planes are perpendicular if $\vec{n}_1 \times \vec{n}_2 = 0 \quad \Rightarrow \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

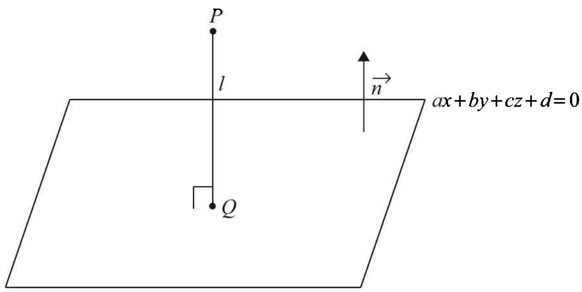
It should be obvious that for two parallel planes, their equations can be written so that they differ only in the constant term. Thus, any plane parallel to $ax + by + cz + d = 0$ can be written as $ax + by + cz + d' = 0$ where $d' \in \mathbb{R}$ (and $d' \neq d$). ■

Example 10

(a) Find the distance of the point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$.
(b) Find the distance between the two parallel planes

$$ax + by + cz + d_1 = 0$$
$$ax + by + cz + d_2 = 0$$

Solution: **(a)** The distance l of P from the given plane will obviously be measured along the normal to the plane passing through P:



We write the equation of the plane as $\vec{r}\cdot\vec{n} = -d$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is any point on the plane and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ is the normal to the plane. Let O be the origin. Since Q lies on the plane, its position vector $\overrightarrow{OQ}$ must satisfy the equation of the plane. But $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$. Thus,

$$(\overrightarrow{OP} + \overrightarrow{PQ})\cdot\vec{n} = -d$$

Note that $\overrightarrow{PQ} = \dfrac{\lambda\vec{n}}{|\vec{n}|}$ where $\lambda = \pm l$ (which sign to take depends on which direction $\vec{n}$ points in). Thus,

$$\left(\overrightarrow{OP} + \frac{\lambda\vec{n}}{|\vec{n}|}\right)\cdot\vec{n} = -d \quad \Rightarrow \quad \overrightarrow{OP}\cdot\vec{n} + \frac{\lambda\vec{n}\cdot\vec{n}}{|\vec{n}|} = -d$$

$$\Rightarrow \ (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})\cdot\ (a\hat{i} + b\hat{j} + c\hat{k}) + \lambda\ |\vec{n}| = -d$$

$$\Rightarrow \ ax_1 + by_1 + cz_1 + d = -\lambda\,|\vec{n}|$$

$$\Rightarrow \ |\lambda| = l = \frac{|ax_1 + by_1 + cz_1 + d|}{|\vec{n}|}$$

$$\Rightarrow \ l = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(b) Assume any point $P(x_1, y_1, z_1)$ on the first plane. We have

$$ax_1 + by_1 + cz_1 + d_1 = 0$$

$$\Rightarrow \ d_1 = -\ (ax_1 + by_1 + cz_1) \qquad (1)$$

The distance of P from the second plane, say l, can be evaluated as described in part (a):

$$l = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} \tag{2}$$

Using (1) in (2), we have

$$l = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

This is the required distance between the two planes. ■

Example 11

Find the equation of the plane(s) bisecting the angle(s) between two given planes

$$P_1 \equiv a_1x + b_1x + c_1z + d_1 = 0$$

$$P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$$

Solution: Note that as in the case of the intersection of straight lines, there will be two (supplementary) angles formed when two planes intersect: one will be acute and the other obtuse (or both could be right). The angle bisector plane of two planes has essentially the same property as the angle bisector of two lines: any point on the angle bisector plane of the planes P_1 and P_2 will be equidistant from P_1 and P_2 (visualize this fact with as much detail as possible). If we assume an arbitrary point $S(x, y, z)$ on the angle bisector plane(s) of P_1 and P_2, we have:

Distance of S from P_1 = Distance of S from P_2:

$$\Rightarrow \quad \frac{|a_1x + b_1y + c_1z + d_1|}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{|a_2x + b_2y + c_2z + d_2|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \quad \boxed{\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}}$$

As expected, we get two angle bisector planes, one corresponding to the "+" and one to the "–" sign. As in the case of straight line angle bisectors, we can prove that the equation of the angle bisector containing the origin will be given by the "+" sign if d_1 and d_2 are of the same sign. You are urged to prove this as an exercise. ■

Example 12

Find the equation of the plane passing through the line of intersection of

$$P_1 \equiv x + 3y - 6 = 0$$

$$P_2 \equiv 3x - y + 4z = 0$$

and at a unit distance from the origin.

Solution: Any plane through the intersection line of P_1 and P_2 can be written as

$$P_1 + \lambda P_2 = 0$$

$$\Rightarrow \quad (1 + 3\lambda)x + (3 - \lambda)y + 4\lambda z - 6 = 0 \tag{1}$$

The distance of this plane from the origin (0, 0, 0) is 1. We thus have, using the formula for the distance of a point from a plane,

$$\frac{|(1+3\lambda)0+(3-\lambda)0+4\lambda(0)-6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2}}=0$$

$$\Rightarrow \quad (1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2=36$$

$$\Rightarrow \quad \lambda=\pm 1$$

Thus, in fact two such planes will exist. Using the values of λ obtained in (1), the equations of these two planes will be $2x+y+2z+3=0$ and $-x+2y-2z+3=0$. ■

Example 13

Find the equation of the plane passing through the line

$$2x+y-z-3=0=5x-3y+4z+9$$

and parallel to the line $\dfrac{x-1}{2}=\dfrac{y-3}{4}=\dfrac{z-5}{5}$.

Solution: In terms of a parameter λ, the equation of the plane that we require can be written as

$$(2x+y-z-3)+\lambda(5x-3y+4z+9)=0$$

$$\Rightarrow \quad (2+5\lambda)x+(1-3\lambda)y+(4\lambda-1)z+(9\lambda-3)=0 \tag{1}$$

For this plane to be parallel to the given line, its normal must be perpendicular to the given line. Using the condition for perpendicularity, we thus have

$$2(2+5\lambda)+4(1-3\lambda)+5(4\lambda-1)=0$$

$$\Rightarrow \quad 3+18\lambda=0 \quad \Rightarrow \quad \lambda=-\frac{1}{6}$$

Using this value of λ in (1), we get the required equation of the plane as $7x+9y-10z=27$. ■

3-D Geometry

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. A variable line passing through $(\lambda, \lambda, \lambda)$ intersects the following 3 lines:

$$y = mx,\ z = c$$
$$y = -mx,\ z = -c$$
$$y = z,\ mx = -c$$

What are the possible values of λ?

(A) $\frac{c}{m}$ (B) $-\frac{c}{m}$ (C) $\frac{2c}{m}$ (D) $-\frac{2c}{m}$ (E) None of these

P2. The point $P(\lambda, \lambda^2, -\lambda)$ is equidistant from the following two lines:

$$y - mx = 0 = z - c; \qquad y + mx = 0 = z + c$$

The possible values of λ are

(A) 0 (B) $\sqrt{c\left(m + \frac{1}{m}\right)}$ (C) $-\sqrt{c\left(m + \frac{1}{m}\right)}$ (D) $\pm\sqrt{cm}$ (E) None of these

P3. Let P be the plane $6x + 3y + 2z = 6$. If the coordinate axes are 'rotated' so that the origin stays the same, but P cuts the new axes at distances $\lambda, \lambda, \lambda$ from the origin, the value of λ is

(A) $\frac{3\sqrt{2}}{7}$ (B) $\frac{4\sqrt{2}}{7}$ (C) $\frac{5\sqrt{3}}{7}$ (D) $\frac{6\sqrt{3}}{7}$ (E) None of these

P4. A plane P makes intercepts with the axes, the sum of whose squares is a constant equal to k^2. The foot of the perpendicular from the origin to P is $(\lambda, \lambda, \lambda)$. The value of λ in terms of k is

(A) $\pm\frac{k}{3\sqrt{3}}$ (B) $\pm\frac{k}{2\sqrt{3}}$ (C) $\pm\frac{k}{\sqrt{3}}$ (D) $\pm\frac{2k}{\sqrt{3}}$ (E) None of these

P5. Through the point $P(1,2,3)$, a plane is drawn perpendicular to OP, O being the origin, such that the plane meets the axes in A, B, C. The area of ΔABC is

(A) $\frac{34}{3}\sqrt{14}$ (B) $\frac{49}{3}\sqrt{14}$ (C) $\frac{55}{3}\sqrt{14}$ (D) $\frac{67}{3}\sqrt{14}$ (E) None of these

P6. Consider a square $ABCD$ of diagonal length $2a$. The square is folded along the diagonal AC so that the plane of ΔABC is perpendicular to the plane of ΔADC. The shortest distance between AB and CD is

(A) $\frac{a}{2\sqrt{3}}$ (B) $\frac{a}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}a$ (D) None of these

SUBJECTIVE TYPE EXAMPLES

P7. Find the condition(s) for two lines L_1 and L_2 to be (i) parallel and (ii) perpendicular when their direction cosines (l, m, n) satisfy the following systems of equations (the two parts below are two separate problems):

(a) $ul + vm + wn + 0, \quad al^2 + bm^2 + cn^2 = 0$

(b) $al + bm + cn = 0, \quad fmn + gnl + hlm = 0$

P8. Let there be two lines L_1 and L_2 inclined at a small angle $\delta\theta$ to one another, such that their direction cosines are (l, m, n) and $(l + \delta l, m + \delta m + n + \delta n)$ respectively. Show that (the sign'$\approx$'should be read as *approximately equal to*)

$$(\delta\theta)^2 \approx (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

P9. Consider two lines L_1 and L_2. You have to test whether L_1 and L_2 are coplanar, and if they are, you have to find the plane containing them, in the following cases:

(a) $L_1 : \dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z+5}{1}, \quad L_2 : \dfrac{x-2}{1} = \dfrac{y}{2} = \dfrac{z+3}{-1}$

(b) $L_1 : 7x - 4y + 7z + 16 = 0 = 4x + 3y - 2z + 3$

$L_2 : x - 3y + 4z + 6 = 0 = x - y + z + 1$

P10. Find the shortest distance between the lines L_1 and L_2 given by

$$L_1 : \frac{y}{b} + \frac{z}{c} = 1, \quad x = 0$$

$$L_2 : \frac{x}{a} - \frac{z}{c} = 1, \quad y = 0$$

P11. Find the volume of the tetrahedron formed by the following planes:

$$my + nz = 0$$

$$nz + lx = 0$$

$$lx + my = 0$$

$$lx + my + nz = p$$

P12. Find the equation of the plane containing the line $2x - y + z - 3 = 0, 3x + y + z = 5$ and at a distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, –1).

P13. A point P moves on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP (O is the origin) meets the coordinate axes in A, B and C. If the planes through A, B and C parallel to the coordinate planes meet at Z, find the locus of Z.

P14. Consider the following three planes:

$$P_1: x = y\sin\alpha + z\sin\beta$$

$$P_2: y = z\sin\theta + x\sin\alpha$$

$$P_3: z = x\sin\beta + y\sin\theta$$

If P_1, P_2, P_3 intersect in the line $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\beta} = \dfrac{z}{\cos\alpha}$, find the value of $\alpha + \beta + \theta$.

P15. Find the reflection of the plane $P_1 \equiv ax + by + cz + d = 0$ in the plane $P_2 \equiv px + qy + rz + s = 0$.

P16. A ball is dropped from the point $A(1, 1, 21)$ onto the plane $P \equiv x + y + z = 3$, where the z-axis is along the vertical direction. The ball hits the plane the first time at the point B, and the second time at the point C. Find the coordinates of C. All length units are in meters. Assume that the collision is elastic.

P17. A, B, C, D are coplanar points. A', B', C', D' are their projections respectively on the $y-z$ plane. Prove that

$$Vol\,|AB'C'D'| = Vol\,|A'BCD|$$

3-D Geometry

PART-D: Solutions to Advanced Problems

S1. The (families of) planes passing through the first two lines can be represented as:

$$y - mx + \mu_1(z - c) = 0$$

$$y + mx + \mu_2(z + c) = 0$$

Since the plane mentioned in the problem also passes through the third line $y = z$, $mx = -c$, we have

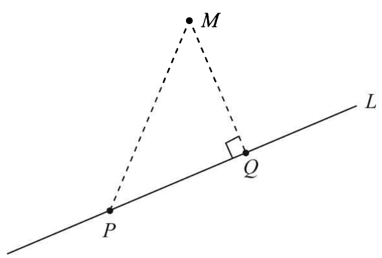
$$\mu_1 = \frac{c+y}{c-y}, \quad \mu_2 = \frac{c-y}{c+y} \quad \Rightarrow \quad \mu_1 \, \mu_2 = 1$$

$$\Rightarrow \quad \left(\frac{y-mx}{z-c}\right)\left(\frac{y+mx}{z+c}\right) = 1$$

$$\Rightarrow \quad y^2 - m^2 x^2 = z^2 - c^2$$

Since $(\lambda, \lambda, \lambda)$ must satisfy this relation, we obtain the required values of λ as $\pm\frac{c}{m}$. The correct options are (A) and (B).

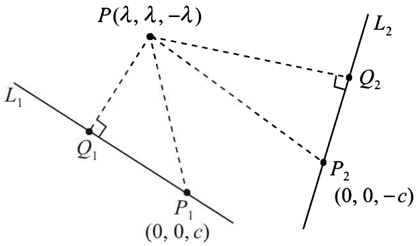
S2. Consider a line L and a point P on L, and a point M not on L (visualize this in three dimensions):



To evaluate MQ (the perpendicular distance of M from L), we can find the projection PQ of MP on L, and use

$$MQ^2 = MP^2 - PQ^2$$

This is what we will do for the current problem. The reader can easily deduce that a point on L_1 is $P_1(0,0,c)$ and a unit vector along L_1 is $\hat{u}_1 = \frac{\hat{i}+m\hat{j}}{\sqrt{1+m^2}}$, while a point on L_2 is $P_2 = (0, 0, -c)$, and a unit vector along L_2 is $\hat{u}_2 = \frac{\hat{i}-m\hat{j}}{\sqrt{1+m^2}}$:



Also,

$PP_1^2 = \lambda^2 + \lambda^4 + (\lambda + c)^2$

$PP_2^2 = \lambda^2 + \lambda^4 + (\lambda - c)^2$

$P_1Q_1 = |\overrightarrow{P_1P} \cdot \hat{u}_1|$

$P_2Q_2 = |\overrightarrow{P_2P} \cdot \hat{u}_2|$

From the figure and accompanying observations, we can deduce that

$$P_1Q_1 = \frac{\lambda + \lambda^2 m}{\sqrt{1+m^2}}, \quad P_2Q_2 = \frac{\lambda - \lambda^2 m}{\sqrt{1+m^2}}$$

If $PQ_1 = PQ_2$ as given in the problem, we have

$$PP_1^2 - P_1Q_1^2 = PP_2^2 - P_2Q_2^2$$

Using the values for these terms and simplifying, we will obtain

$$\lambda = 0, \pm\sqrt{c\left(m + \frac{1}{m}\right)}$$

The correct options are (A), (B) and (C).

S3. Note that the distance of the origin from the plane remains the same. Referred to the new axes, the equation of the plane becomes

$$x + y + z = \lambda$$

Thus,

$$\frac{6}{7} = \frac{\lambda}{\sqrt{3}} \Rightarrow \lambda = \frac{6\sqrt{3}}{7}$$

The correct option is (D).

S4. The normal to the plane will have direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (why?), and the distance of this plane from the origin will be $OP = \sqrt{3}\,\lambda$. Thus, the equation of the plane will be

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = \sqrt{3}\lambda \quad \Rightarrow \quad x + y + z = 3\lambda$$

The intercepts of this plane are of lengths $3\lambda, 3\lambda, 3\lambda$. Thus,

$$(3\lambda)^2 + (3\lambda)^2 + (3\lambda)^2 = k^2 \quad \Rightarrow \quad \lambda = \pm\frac{k}{3\sqrt{3}}$$

The correct option is (A).

S5. The equation of this plane can be written as

$$x + 2y + 3z + d = 0$$

Since (1, 2, 3) must satisfy this equation, we have $d = -14$, and the equation of the plane as $x + 2y + 3z = 14$.

The intersection points of this plane with the axes are:

$$A(14, 0, 0),\ B(0, 7, 0),\ C\left(0, 0, \frac{14}{3}\right)$$

Thus,

$$\overrightarrow{AB} = -14\hat{i} + 7\hat{j},\ \overrightarrow{AC} = -14\hat{i} + \frac{14}{3}\hat{k}$$

so that the area of ΔABC equals

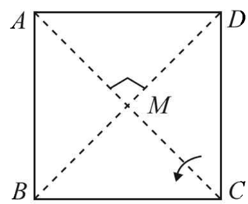
$$\left|\frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -14 & 7 & 0 \\ -14 & 0 & \frac{14}{3} \end{matrix}\right\|\right| = 49\left|\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \hat{k}\right| = \frac{49}{3}\sqrt{14}$$

The correct option is (B).

S6. Let us define our axes as follows: B as the origin, BC and BA as x and y axis and z-axis perpendicular to the plane of ΔABC. Now, note the following coordinates carefully:

$$A \equiv (0, \sqrt{2}a, 0), \quad C \equiv (\sqrt{2}a, 0, 0), \quad D \equiv \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, a\right)$$

Therefore,



$$\overrightarrow{BA} = \sqrt{2}a\,\hat{j},\ \overrightarrow{CD} = \left(\left(\frac{a}{\sqrt{2}} - \sqrt{2}a\right)\hat{i} + \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}\right)$$

$$\Rightarrow \quad \vec{n} = \overrightarrow{BA} \times \overrightarrow{CD} = a^2\hat{k} + \sqrt{2}a^2\hat{i} = a^2(\sqrt{2}\hat{i} + \hat{k})$$

The minimum distance d is

$$d = \left|\overrightarrow{BC} \cdot \frac{\vec{n}}{|\vec{n}|}\right| = \frac{2}{\sqrt{3}}a$$

The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

S7. **(a)** The reader is urged to carefully understand that L_1 and L_2 will have two *different* sets of direction cosines, given by the system of equations provided to us. We proceed by eliminating l from the system:

$$l = -\frac{(vm+wn)}{u} \quad \Rightarrow \quad a\frac{(vm+wn)^2}{u^2} + bm^2 + cn^2 = 0$$

Rearraning this, we will obtain

$$(av^2+bu^2)m^2 + (aw^2+cu^2)n^2 + 2avwmn = 0$$

$$\Rightarrow \quad (av^2+bu^2)\left(\frac{m}{n}\right)^2 + 2avw\left(\frac{m}{n}\right) + (aw^2+cu^2) = 0 \qquad (1)$$

This will give two roots, namely $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$

(i) For L_1 and L_2 to be perpendicular, we must have

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \quad \Rightarrow \quad \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

Thus, (1) must have equal roots, *i.e.*, its $D = 0$:

$$(2avw)^2 = 4(av^2+bu^2)(aw^2+cu^2)$$

Simplifying, we will obtain the required condition as

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0$$

(ii) For L_1 and L_2 to be perpendicular, we must have

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad (2)$$

From (1), we have

$$\frac{m_1m_2}{n_1n_2} = \frac{aw^2+cu^2}{av^2+bu^2} \quad \Rightarrow \quad \frac{m_1m_2}{aw^2+cu^2} = \frac{n_1n_2}{av^2+bu^2} \qquad (3)$$

By symmetry, we can conclude from (3) that

$$\frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{aw^2+cu^2} = \frac{n_1n_2}{av^2+bu^2} = \lambda \text{ (say)}$$

$$\Rightarrow \begin{cases} l_1l_2 = \lambda(bw^2+cv^2) \\ m_1m_2 = \lambda(aw^2+cu^2) \\ n_1n_2 = \lambda(av^2+bu^2) \end{cases} \qquad (4)$$

Using these values in (2), and simplifying, we will obtain the required condition as

$$u^2(b+c) + v^2(c+a) + w^2(a+b) = 0$$

(b) We follow an exactly analogous approach and the conditions will be obtained as follows:

(i) $L_1 \parallel L_2 : \sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$ (ii) $L_1 \perp L_2 : \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

S8. We have

$$l^2 + m^2 + n^2 = 1$$

$$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$$

From these, we can immediately obtain

$$(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l(\delta l) + m(\delta m) + (\delta n))$$

Also,

$$\cos(\delta\theta) = 1 - 2\sin^2\left(\frac{\delta\theta}{2}\right) = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$= (l^2 + m^2 + n^2) + \{l(\delta l) + m(\delta m) + n(\delta n)\}$$

$$\Rightarrow \quad 1 - 2\sin^2\left(\frac{\delta\theta}{2}\right) = 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$$

$$\Rightarrow \quad \sin^2\left(\frac{\delta\theta}{2}\right) = \frac{1}{4}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$$

If $\delta\theta$ is very small, we can approximate $\sin(\frac{\delta\theta}{2}) \sim (\frac{\delta\theta}{2})$, and thus obtain

$$(\delta\theta)^2 \approx (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

S9. **(a)** The equation of any plane containing L_1 can be written as

$$\alpha(x-1) + \beta(y+1) + \gamma(z+5) = 0 \tag{1}$$

where

$$2\alpha + 3\beta + \gamma = 0 \tag{2}$$

For this plane to contain L_2, the point (2, 0, –3) must lie on it and its normal must be perpendicular to L_2. Thus,

$$\alpha + \beta + 2\gamma = 0 \tag{3}$$

$$\alpha + 2\beta - \gamma = 0 \tag{4}$$

If we consider (2), (3) and (4) as a system of equations in α, β, γ, we have

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 0 \quad \Rightarrow \quad \text{The system is consistent in } \alpha, \beta, \gamma$$

From this, we can conclude that there will exist values of α, β and γ (not all zero) for which the plane in (1) will contain L_2. To determine that plane, we eliminate α, β, γ from the system formed by (1), (2) and (3). Thus, we obtain

$$\begin{vmatrix} x-1 & y+1 & z+5 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \quad \Rightarrow \quad 5x - 3y - z = 13$$

(b) We take a 'family of planes' approach. The equation of any plane P_1 containing L_1 and any plane P_2 containing L_2 can be written as

$$P_1: (7x - 4y + 7z + 16) + \lambda(4x + 3y - 2z + 3) = 0$$

$$\Rightarrow \quad (7 + 4\lambda)x + (-4 + 3\lambda)y + (7 - 2\lambda)z + 16 + 3\lambda = 0 \tag{1}$$

$$P_2: (x - 3y + 4z + 6) + \mu(x - y + z + 1) = 0$$

$$\Rightarrow \quad (1 + \mu)x + (-3 - \mu)y + (4 + \mu)z + 6 + \mu = 0 \tag{2}$$

For some values of λ and μ, (1) and (2) must be identical, and thus,

$$\frac{7+4\lambda}{1+\mu}=\frac{-4+3\lambda}{-3-\mu}=\frac{7-2\lambda}{4+\mu}=\frac{16+3\lambda}{6+\mu} \tag{3}$$

It can be easily shown (exercise for the reader) that $\lambda=-1$ and $\mu=\frac{1}{2}$ satisfy the relations in (3) consistently. Thus, the two lines are coplanar, and putting $\lambda=-1$ in (1), we get the equation of the containing plane as

$$3x-7y+9z+13=0$$

S10. We take two specific points on L_1 and L_2, namely $\vec{r}_1$ and $\vec{r}_2$ respectively, where

$$\vec{r}_1=(0,0,c),\quad \vec{r}_2=(a,0,0)$$

$$\Rightarrow\quad \vec{r}_2-\vec{r}_1=a\hat{i}-c\hat{k} \tag{1}$$

Now, the direction ratios of L_1 (say p_1, q_1, r_1), will be given by the cross product of the normal vectors of the planes $\frac{y}{b}+\frac{z}{c}=1$ and $x=0$. Thus,

$$(p_1,q_1,r_1)\equiv\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & \frac{1}{b} & \frac{1}{c}\\ 1 & 0 & 0\end{vmatrix}=\frac{1}{c}\hat{j}-\frac{1}{b}\hat{k}\equiv\left(0,\frac{1}{c},-\frac{1}{b}\right)$$

Similarly, the direction ratios of L_2 will be

$$(p_2,q_2,r_2)\equiv\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{1}{a} & 0 & -\frac{1}{c}\\ 0 & 1 & 0\end{vmatrix}=\frac{1}{c}\hat{i}+\frac{1}{a}\hat{k}=\left(\frac{1}{c},0,\frac{1}{a}\right)$$

The direction ratios of the line L perpendicular to both L_1 and L_2 will be

$$(p,q,r)\equiv(p_1,q_1,r_1)\times(p_2,q_2,r_2)$$

$$\equiv\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & \frac{1}{c} & -\frac{1}{b}\\ \frac{1}{c} & 0 & \frac{1}{a}\end{vmatrix}=\frac{1}{ac}\hat{i}-\frac{1}{bc}\hat{j}-\frac{1}{c^2}\hat{k}\equiv\left(\frac{1}{ac},-\frac{1}{bc},-\frac{1}{c^2}\right)$$

Thus, a unit vector along L, say $\hat{u}$, can be written as

$$\hat{u}=\frac{\frac{1}{ac}\hat{i}-\frac{1}{bc}\hat{j}-\frac{1}{c^2}\hat{k}}{\sqrt{(\frac{1}{ac})^2+(-\frac{1}{bc})^2+(-\frac{1}{c^2})^2}}=\frac{\frac{1}{a}\hat{i}-\frac{1}{b}\hat{j}-\frac{1}{c}\hat{k}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \tag{2}$$

From (1) and (2), the shortest distance (denote it by d) will be

$$d=|(\vec{r}_2-\vec{r}_1)\cdot\hat{u}|=\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

S11. Denote the 4 planes by P_1, P_2, P_3 and P_4. The first three planes intersect at $(0, 0, 0)$. P_1, P_2 and P_4 intersect at $\left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)$. The other two points of intersection are evident by symmetry. Now, the relevant STP (scalar triple product) can be used to evaluate the volume V of the tetrahedron.

$$V = \frac{1}{6}\begin{vmatrix} \frac{p}{l} & \frac{-p}{l} & \frac{p}{l} \\ \frac{p}{m} & \frac{p}{m} & \frac{-p}{m} \\ \frac{-p}{n} & \frac{p}{n} & \frac{p}{n} \end{vmatrix} = \frac{p^3}{3lmn}$$

S12. The line given to us has been specified as the intersection between two planes. Thus, any plane passing through the given line can be specified as

$$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$$

$$\Rightarrow \quad (2 + 3\lambda)x + (\lambda - 1)y + (1 + \lambda)z - (3 + 5\lambda) = 0$$

If this is at a distance of $\frac{1}{\sqrt{6}}$ from (2, 1, −1), we have

$$\left| \frac{(2+3\lambda)(2) + (\lambda - 1)(1) + (1+\lambda)(-1) - (3+5\lambda)}{\sqrt{(2+3\lambda)^2 + (\lambda-1)^2 + (1+\lambda)^2}} \right| = \frac{1}{\sqrt{6}}$$

Squaring and simplifying gives

$$\lambda(5\lambda + 24) = 0 \quad \Rightarrow \quad \lambda = 0, -\frac{24}{5}$$

The two planes are thus

$$P_1(\text{for } \lambda = 0): \quad 2x - y + z = 3$$

$$P_2\left(\text{for } \lambda = -\frac{24}{5}\right): \quad 62x + 29y + 19z = 105$$

S13. If P is assumed to be the (variable) point (h, k, l), we have

$$\frac{h}{a} + \frac{k}{b} + \frac{l}{c} = 1 \tag{1}$$

Also, it can be easily shown that the equation of the plane passing through P and perpendicular to OP will be

$$hx + ky + lz = h^2 + k^2 + l^2 \quad \text{(verify)} \tag{2}$$

This meets the coordinate axes at (say) X_1, X_2 and X_3. From the statement of the problem, it is obvious that if Z is the point (α, β, γ), then α, β, γ are respectively the $x-$, $y-$ and $z-$coordinates of X_1, X_2 and X_3. Thus,

$$\alpha = \frac{h^2 + k^2 + l^2}{h}, \quad \beta = \frac{h^2 + k^2 + l^2}{k}, \quad \gamma = \frac{h^2 + k^2 + l^2}{l} \tag{3}$$

$$\Rightarrow \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{h^2 + k^2 + l^2} \tag{4}$$

From (1) and (3)

$$\frac{h}{a}+\frac{k}{b}+\frac{l}{c}=1=\frac{h^2+k^2+l^2}{a\alpha}+\frac{h^2+k^2+l^2}{b\beta}+\frac{h^2+k^2+l^2}{c\gamma}$$

$$\Rightarrow \quad \frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+l^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

Using $(\alpha, \beta, \gamma) \to (x, y, z)$, the required locus is

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$$

S14. We will outline the steps involved in the solution; the reader can fill in the details herself. The first thing to understand is that if three planes pass through the same line, their normal vectors will be coplanar. In this case, applying the observation made, we will have

$$\begin{vmatrix} -1 & \sin\alpha & \sin\beta \\ \sin\alpha & -1 & \sin\theta \\ \sin\beta & \sin\theta & -1 \end{vmatrix}=0 \quad \Rightarrow \quad \begin{array}{c} -1(1-\sin^2\theta) \\ + \\ \sin\alpha(\sin\beta\sin\theta+\sin\alpha)=0 \\ + \\ \sin\beta(\sin\alpha\sin\theta+\sin\beta) \end{array}$$

$$\Rightarrow \quad \sin^2\alpha+\sin^2\beta+\sin^2\theta+2\sin\alpha\sin\beta\sin\theta=1 \qquad (1)$$

Now, the cross product $\hat{r}$ of the normals of P_1 and P_2 will be parallel to the given line. The direction ratios of $\hat{r}$ are given by

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \sin\alpha & \sin\beta \\ \sin\alpha & -1 & \sin\theta \end{vmatrix} \equiv \{(\sin\alpha\sin\theta+\sin\beta), (\sin\alpha\sin\beta+\sin\theta), (1-\sin^2\alpha)\}$$

Since $\hat{r}$ is parallel to the given line, we have

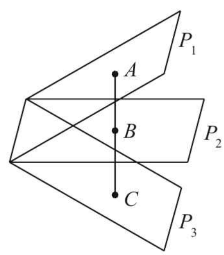
$$\frac{\sin\alpha\sin\theta+\sin\beta}{\cos\theta}=\frac{\sin\alpha\sin\beta+\sin\theta}{\cos\beta}=\frac{1-\sin^2\alpha}{\cos\alpha}=\lambda(\text{say}) \qquad (2)$$

Using (1) and (2), and some manipulations, we will arrive at

$$\lambda\cos\alpha=\lambda\sin(\beta+\theta) \quad \Rightarrow \quad \sin(\beta+\theta)=\sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\Rightarrow \quad \alpha+\beta+\theta=2n\pi+\frac{\pi}{2}, \quad n\in\mathbb{Z}$$

S 15: For any point A on the plane P_1, we need to find the reflection C of A in the plane P_2, and then deduce the locus of C, which will obviously be another plane.



Let $A \equiv (\alpha_1, \beta_1, \gamma_1)$ and let C be $(\alpha_2, \beta_2, \gamma_2)$. We use the following two facts:

Fact-1: The mid-point of AC, *i.e.*, B, **lies on P_2**.

$$\Rightarrow \quad p\left(\frac{\alpha_1+\alpha_2}{2}\right)+q\left(\frac{\beta_1+\beta_2}{2}\right)+r\left(\frac{\gamma_1+\gamma_2}{2}\right)+s=0 \qquad (1)$$

Fact-2: AC **is perpendicular to P_2**.

$$\Rightarrow \quad \frac{\alpha_1-\alpha_2}{p}=\frac{\beta_1-\beta_2}{q}=\frac{\gamma_1-\gamma_2}{r}=t \qquad \text{(say)}$$

$$\Rightarrow \quad \alpha_1=\alpha_2+pt,\ \beta_1=\beta_2+qt,\ \ \gamma_1=\gamma_2+rt \qquad (2)$$

Using the values of (2) in (1), and simplifying, we have

$$p(2\alpha_2+pt)+q(2\beta_2+qt)+r(2\gamma_2+rt)+2s=0$$

$$\Rightarrow \quad t=-2\frac{(p\alpha_1+q\beta_2+r\gamma_2+s)}{p^2+q^2+r^2} \qquad (3)$$

Also, since A lies on P_1, we have

$$a\alpha_1+b\beta_1+c\gamma_1+d=0$$

$$\Rightarrow \quad a(\alpha_2+pt)+b(\beta_2+qt)+c(\gamma_2+rt)+d=0$$

$$\Rightarrow \quad t=-\frac{(a\alpha_2+b\beta_2+c\gamma_2+d)}{ap+bq+cr} \qquad (4)$$

From (3) and (4), we have

$$(p^2+q^2+r^2)(a\alpha_2+b\beta_2+c\gamma_2+d)=2(ap+bq+cr)(p\alpha_2+q\beta_2+r\gamma_2+s)$$

The required locus can be obtained by using $(\alpha_2,\beta_2,\gamma_2)\to(x,y,z)$, and we get

$$(p^2+q^2+r^2)(ax+by+cz+d)=2(ap+bq+cr)(px+qy+rz+s)$$

S16. Since the ball falls vertically at first, the x and y coordinates of B will be those of A only, and thus $B\equiv(1,1,1)$. Since $AB = 20$ m, the ball will have achieved a velocity of

$$v=\sqrt{2g(AB)}=20 \text{ m/s} \qquad \text{(just before impacting the plane at } A\text{)}$$

Now, if θ is the angle between AB and the normal to the plane, we have

$$\cos\theta=\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \cos 2\theta=-\frac{1}{3},\ \sin 2\theta=\frac{2\sqrt{2}}{3}$$

After impacting the plane at A, the ball will have the z-component of its velocity as

$$v_z=20\sin\left(\frac{\pi}{2}-2\theta\right)=-\frac{20}{3}\text{ m/s}$$

The component of the ball's velocity in the $x-y$ plane will be

$$v_{xy}=20\cos\left(\frac{\pi}{2}-2\theta\right)=\frac{40\sqrt{2}}{3}\text{ m/s}$$

By symmetry, the x and y components will separately be

$$v_x = v_y = \frac{40}{3}\,\text{m/s}$$

Thus, after impacting the plane at A, the coordinates of the ball's position as a function of time t can be written as

$$x = 1 + \frac{40}{3}t,\ y = 1 + \frac{40t}{3},\ z = 1 - \frac{20}{3}t - 5t^2,$$

where we have used the fact that gravity acts in the vertical direction. For some value of $t > 0$, these coordinates will satisfy the equation of the plane, and that value of t will correspond to the point C. Thus, we have

$$\left(1 + \frac{40}{3}t\right) + \left(1 + \frac{40}{3}t\right) + \left(1 - \frac{20}{3}t - 5t^2\right) = 3$$

$$\Rightarrow \quad 5t^2 - 20t = 0 \quad \Rightarrow \quad t = 0, 4$$

$t = 0$ corresponds to B, and $t = 4$ corresponds to C. Therefore the coordinates of C are

$$C \equiv \left(\frac{163}{3}, \frac{163}{3}, -\frac{317}{3}\right)$$

S17. Let A, B, C, D be the points $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ and (d_1, d_2, d_3). We have

$$Vol[AB'C'D'] = \left| \frac{1}{6} \begin{vmatrix} a_1 & a_2 - b_2 & a_3 - b_3 \\ a_1 & a_2 - c_2 & a_3 - c_3 \\ a_1 & a_2 - d_2 & a_3 - d_3 \end{vmatrix} \right|$$

$$Vol[A'BCD] = \left| \frac{1}{6} \begin{vmatrix} -b_1 & a_2 - b_2 & a_3 - b_3 \\ -c_1 & a_2 - c_2 & a_3 - c_3 \\ -d_1 & a_2 - d_2 & a_3 - d_3 \end{vmatrix} \right|$$

The trick now is to evaluate the sum of the volumes of these two tetrahedrons in the algebraically correct way:

$$Vol[AB'C'D'] + Vol[A'BCD] = \frac{1}{6} \begin{vmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ a_1 - d_1 & a_2 - d_2 & a_3 - d_3 \end{vmatrix}$$

But the value of this determinant is 0, since A, B, C, D are coplanar. Hence, the result follows.

Functions

PART-A. Summary of Important Concepts

1. Basic Facts about Functions

It is most helpful to think of a function as a machine, or a black box if you like, which takes as input a certain independent variable, say x, and outputs according to some specific rule a dependent variable, say y.

$$x \longrightarrow \boxed{f} \longrightarrow y$$

Since y depends on x, while x varies independently, we say that y is the dependent variable while x is the independent variable, and we write this relationship as $y = f(x)$, which is to be interpreted as follows: f *acts* on the independent variable x to produce the dependent variable y.

Now, it may be possible that there is only a particular set of values which f can take as its input in a given case, to be able to produce a valid output. Or, we might wish to restrict the input set of values based on some reasons of our own. In either case, the possible set of input values for a function is called the *domain* of the function. When f acts on each element in the domain, it will produce a set of output values. That set, the set of values which f outputs, is called the *range* of the function. As of now, we are concerned only with real-valued functions, that is, whenever we are studying a function (at this stage), we will always assume that it will produce a real output. Here are two quick examples:

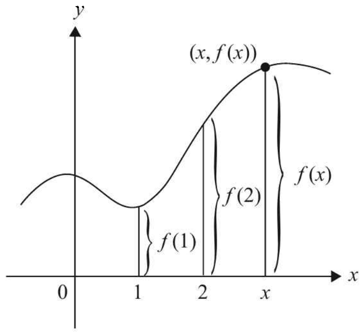
	Domain	**Range**
$f(x) = \sqrt{x}$	The domain is $[0, \infty)$, because if $x < 0$, it will not produce a real output. In this case, the domain has been restricted by the nature of the function (square-root). If we were studying complex functions, we could have allowed x to take on negative values as well (or even complex values), but since we are not, we have had to restrict the domain accordingly.	The range is $[0, \infty)$, since $\sqrt{x}$ is always non-negative. If f was a complex function, the range would have been a much larger (complex) set, but we are restricted to real functions, where the output must be real, so the input (the domain) must be restricted accordingly.
$f(x) = x^2, 0 \le x \le 2$	The domain is $[0, 2]$ because we have ourselves specified it to be so, even though x can take real values outside this set and still produce its output as real values.	Since the domain is $[0, 2]$, the range is $[0, 4]$.

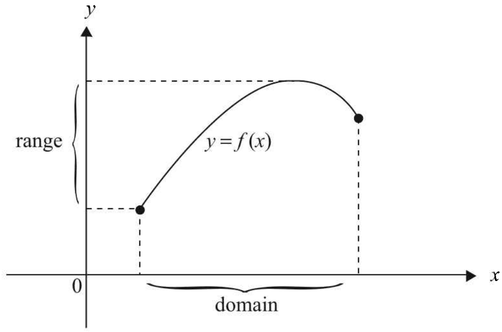
In many cases, the most useful way for visualizing a function is its graph. If f is a function with domain A, then its *graph* is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

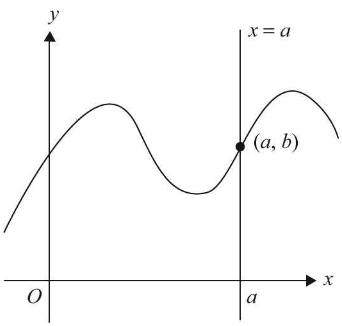
Notice that these are input-output pairs. In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behaviour or 'life history' of a function. Since the y-coordinate of any point (x,y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the *graph above* the point x (see figure on the left below). The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in the figure on the right below.

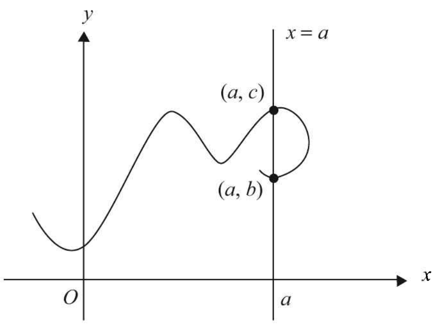




We note that the functions we are studying *must* be single-valued (multi-valued functions do exist, but strictly speaking, they are not exactly functions). This means that for a given value of input, the function *must* produce only one value of output. It is perfectly possible for two or more different inputs to produce the same output, but one input producing multiple outputs is not acceptable if the function is to be single-valued. A graphical techique to check whether a function is single-valued or not is the *Vertical Line Test*: a graph represents a single-valued function if and only if no vertical line intersects the curve more than once:



This is single-valued function.

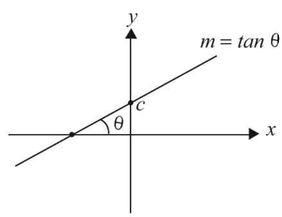


This is not a single valued function; we will consider such functions as invalid functions.

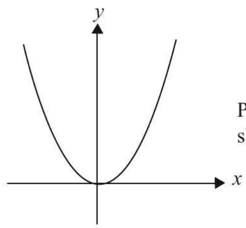
2. Some Commonly Encountered Functions

1. Linear Function $f(x) = mx + c$

Domain = $\mathbb{R}$
Range = $\mathbb{R}$

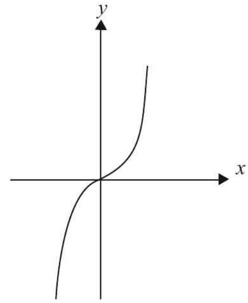


2. Square Function $f(x) = x^2$



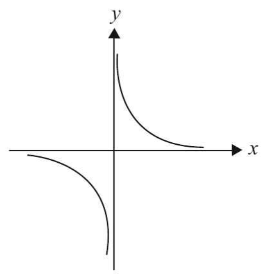
Domain = $\mathbb{R}$
Range = $[0, \infty)$

3. Cube Function $f(x) = x^3$



Domain = $\mathbb{R}$
Range = $\mathbb{R}$

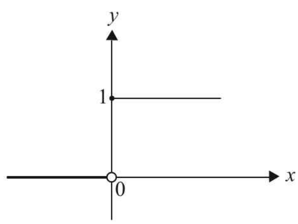
4. Reciprocal Function $f(x) = \dfrac{1}{x}$



Domain = $\mathbb{R} - \{0\}$
Range = $\mathbb{R} - \{0\}$
The range is $\mathbb{R}s - \{0\}$ because for no value of x is $f(x) = 0$.

5. Step Function $f(x) = 0$ if $x < 0$

1 if $x \geq 0$

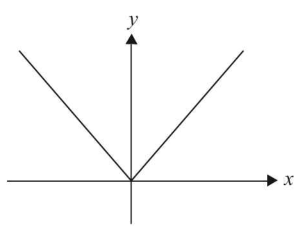


Domain = $\mathbb{R}$
Range = $\{0,1\}$

6. Modulus Function $f(x) = |x| = x$ if $x > 0$

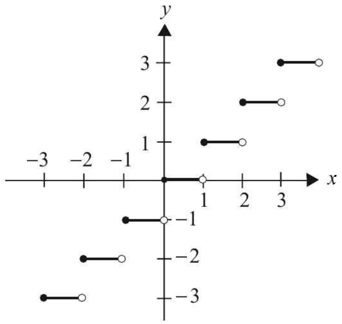
$-x$ if $x < 0$

Basically, this function gives the magnitude of a number and strips it off its negative sign, if it is negative.



Domain = $\mathbb{R}$
Range = $[0, \infty)$

7. Greatest integer function $f(x) = [x]$



Domain = $\mathbb{R}$
Range = $\mathbb{Z}$ (set of all integers)

Properties

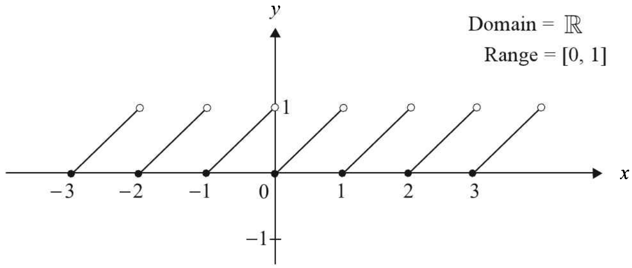
(i) $[x+n]=[x]+n, \quad n\in\mathbb{Z}$

(ii) $[-x]=-[x]-1, \quad x\notin\mathbb{Z}$

$[-x]=-[x], \quad x\in\mathbb{Z}$

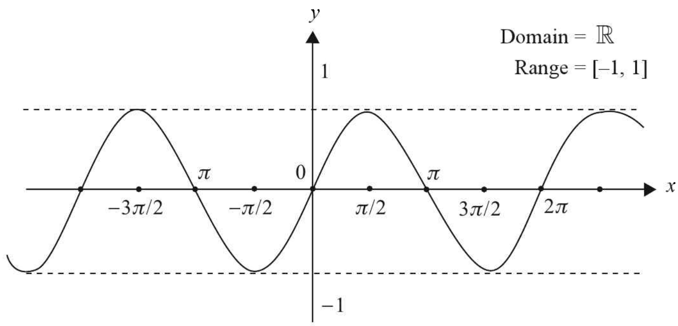
Verify these properties carefully because they are very important.

8. Fractional part $f(x) = \{x\}$



Domain = $\mathbb{R}$
Range = [0, 1]

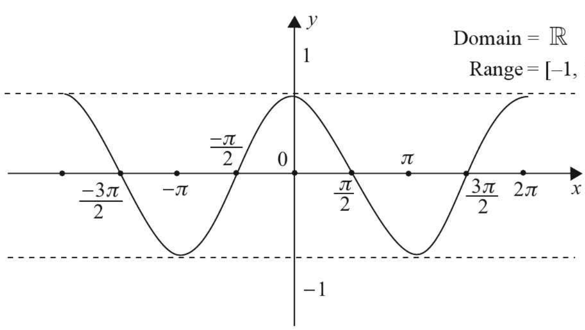
9. $f(x) = \sin x$



Domain = $\mathbb{R}$
Range = [–1, 1]

The graph repeats after every increment (or decrement) of 2π in the angle 'x'. This phenomenon is known as periodicity of the graph. We say that the graph is periodic with period '2π'.

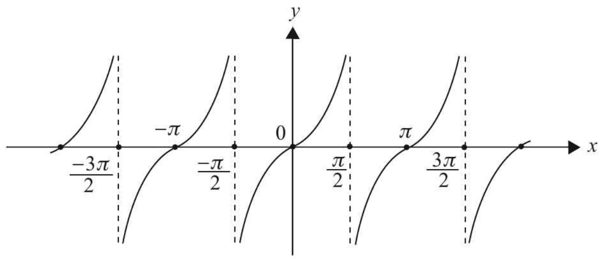
10. $f(x) = \cos x$



Domain = $\mathbb{R}$
Range = [–1, 1]

This function is also periodic with period 2π.

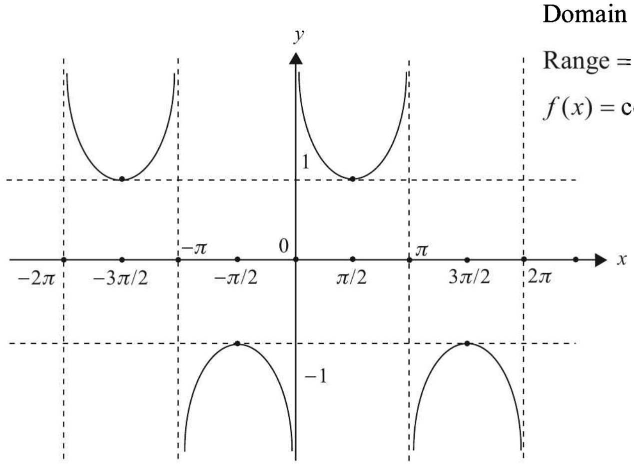
11. $f(x) = \tan x$



Domain = $\mathbb{R} - \{n\pi + \frac{\pi}{2}\}$ (Exclude the set of all points where $\cos x$ becomes 0,
Range = $\mathbb{R}$ *i.e.*, where the angle $x = n\pi + \pi/2$)

This function is periodic with period π.

12. $f(x) = \operatorname{cosec} x$

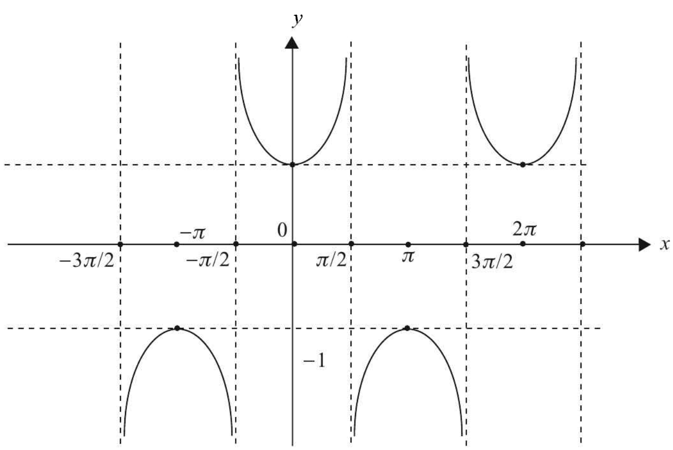


Domain = $\mathbb{R} - \{n\pi\}, n \in \mathbb{Z}$

Range = $(-\infty, -1] \cup [1, \infty)$

$f(x) = \operatorname{cosec} x$ is periodic with period 2π.

13. $f(x) = \sec x$

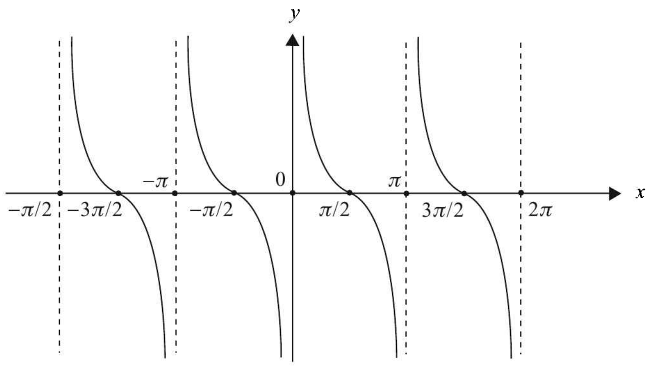


Domain $= \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in \mathbb{Z}$ (We exclude all points where $\cos x = 0$, because $\sec x = \frac{1}{\cos x}$)

Range $= (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$ is again periodic with period 2π.

14. $f(x) = \cot x$



Domain $= \mathbb{R} - \{n\pi\}, n \in \mathbb{Z}$

Range $= \mathbb{R}$

$f(x) = \cot x$ is periodic with period π.

3. Exponential and Logarithmic Functions

In our experience, many students are generally not very clear about the exponential and logarithmic functions. These are therefore being discussed in a dedicated section, and it is imperative that you take time in understanding these functions completely. First of all, at the outset itself, it is helpful to think of the exponential and logarithmic functions as inverses of each other. Examples:

$2^3 = 8$ is the same as saying that, $\log_2 8 = 3$.

$3^{-4} = \frac{1}{81}$ is the same as saying that $\log_3\left(\frac{1}{81}\right) = -4$.

In general,

$a^b = c$ is equivalent to saying that $\log_a c = b$.

Now, we need to understand the exponentiation operation more carefully. We clearly understand the meaning of these terms:

$$2^3 = 2 \times 2 \times 2$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

Going further, terms like $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ also make sense. $2^{\frac{1}{2}}$ is real number which when squared will give $2 \cdot 3^{\frac{1}{3}}$ is a real number which when cubed will give 3. In general, to any term of the form $a^{p/q}$, we can attach the following interpretation:

$$a^{p/q} = (a^{1/q})^p = (q^{th} \text{ root of } a)^p$$

So, we see that we can attach sense to any term of the form a^x where x is rational, of the form $\frac{p}{q}$. Additionally, we generally have $a > 0$ (why should that be?). Now, the following question arises: what if x is irrational? What sense do we attach to a^x then? For example , how will we evaluate $2^{\sqrt{2}}$? Is this quantity defined? The answer is yes, but you have to be clear about how to interpret this quantity:

$$\begin{aligned}\sqrt{2} &= 1.41421...\\ &= 1 + 0.4 + 0.01 + 0.004 + 0.0002 + 0.00001 + \cdots\\ &= 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} + \frac{2}{10000} + \frac{1}{100000} + \cdots\end{aligned}$$

This is how $\sqrt{2}$ can be expressed as a sum of an infinite series of rationals. Now,

$$\begin{aligned}2^{\sqrt{2}} &= 2^{\left(1+\frac{4}{10}+\frac{1}{100}+\frac{4}{1000}+\frac{2}{10000}+\frac{1}{100000}+...\right)}\\ &= 2^1 \cdot 2^{\frac{4}{10}} \cdot 2^{\frac{1}{10}} \cdot 2^{\frac{4}{1000}} \cdot 2^{\frac{2}{10000}} \cdot 2^{\frac{1}{100000}}.\end{aligned}$$

Each of the terms in the product, as we've seen, has a well defined meaning, and therefore so does $2^{\sqrt{2}}$. We can calculate $2^{\sqrt{2}}$ as accurately as we wish by taking as many terms after the decimal point into account.

This discussion should make it clear that for $a > 0$, a^x is defined, whether x is positive or negative or zero, rational or irrational. If a is a fixed constant and we let x be a variable real number, we can define the general exponential function as

$$f(x) = a^x, \quad x \in \mathbb{R}$$

a is called the *base*, and x is the *exponent*. In terms of logarithms, the same relation becomes

$$\log_a f(x) = x$$

We will now understand the behaviour of the exponential and logarithmic functions in more detail by plotting their graphs. For that, it is necessary to get a numerical appreciation of their variation. We do this by calculating exponents and logarithms for a fixed base, say $a = 2$, and particular values of the variable x:

Exp

x	2^x
–4	$\frac{1}{16} = 0.0625$
–3	$\frac{1}{8} = 0.125$
–2	$\frac{1}{4} = 0.25$
–1	$\frac{1}{2} = 0.5$
0	1
1	2
2	4
3	8
4	16

General Behaviour

$x \to -\infty, \quad 2^x \to 0$

$x \to +\infty, \quad 2^x \to \infty$

The output is always positive

Therefore, range is $(0, \infty)$ The function is defined for all inputs, so the domain is $\mathbb{R}$.

log

x	$\log_2 x$
0.0625	– 4
0.125	– 3
0.25	– 2
0.5	– 1
1	0
2	1
4	2
8	3
16	4

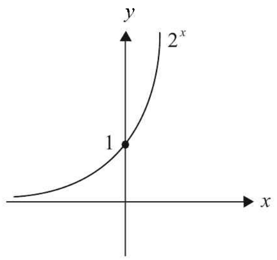
General Behaviour

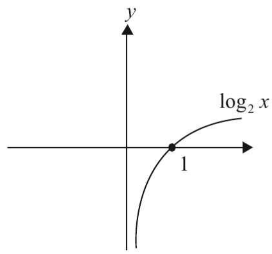
$x \to 0, \quad \log_2 x \to -\infty$

$x \to \infty, \quad \log x \to \infty$

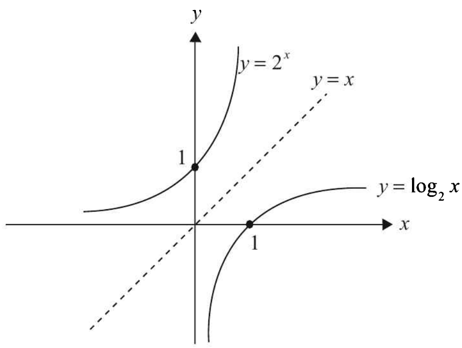
The function is defined for positive inputs only, since the log of a negative number makes no sense. Therefore, the domain is $(0, \infty)$. The output varies from $-\infty$ to $+\infty$, so the range is $\mathbb{R}$.

Finally, let us draw the graphs of the two function:





Notice what happens if we draw $f(x) = 2^x$ and $f(x) = \log_2 x$ on the same axis. The graphs of the two functions seem to be mirror reflections of each other in the line $y = x$ (infact, they are!):



Also note what happens if we operate the $f(x) = \log_2 x$ function and then the $g(x) = 2^x$ function on a particular x, in any order. For the second sequence, $x > 0$ so that $\log_2 x$ is defined:

$$x \longrightarrow \overset{g}{\boxed{2^x}} \xrightarrow{2^x} \overset{f}{\boxed{\log_2 x}} \longrightarrow x$$

$$x \longrightarrow \overset{f}{\boxed{\log x}} \xrightarrow{\log_2 x} \overset{g}{\boxed{2^x}} \longrightarrow x$$

That is, $f(g(x)) = x$ and $g(f(x)) = x$. Such functions are called *inverse functions*:

$$f = g^{-1} \quad \text{or} \quad g = f^{-1}.$$

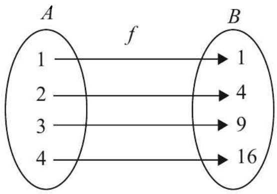
The graphs of f and g are mirror reflections of each other about the line $y = x$. Note that because of this, the domain and range get interchanged. The base in exponential/logarithmic functions is generally taken to be e. The graphs of e^x and $\log_e x$ (denoted generally by $\ln x$) will resemble the graphs of 2^x and $\log_2 x$ respectively.

4. Functions as Maps

Maps are a convenient way to visualise functions, or more generally, the association between two sets. A map relates one set to another using some rule. For example, suppose the rule is:

$$y = f(x) = x^2, \qquad \text{Set } A = \{1, 2, 3, 4\}, \text{ Set } B = \{1,4,9,16\}$$

We can show this as a map.



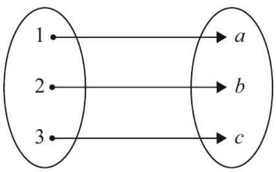
The elements in set B are the images of the elements of set A under the action of f.

Set A would be called the domain of f {the set of all input values}, and set B the co-domain of f {the set in which the output values lie}. Here, the co-domain is the same as the range {precisely the set of all outputs}, but in general, range would be a subset of the co-domain:

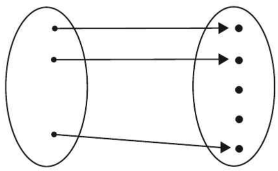
$$\text{Range} \subseteq \text{co-domain}$$

(A) One-one/Many-one

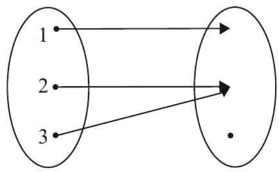
A one-one function (or mapping) is such that each element in the domain set is mapped to only one element in the co-domain set, that is, no two inputs map to the same output.



one-one



one-one



not one-one
(many-one)

The one-one condition can be written mathematically as

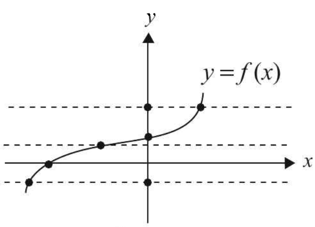
$$f(x_1) = f(x_2) \text{ if and only if } x_1 = x_2$$

This, stated in words, says that two outputs from f can be equal if and only if the corresponding inputs to f are equal. If $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$, then the function is many-one. Therefore, given f, we can find whether its one-one or not by solving the equation $f(x_1) = f(x_2)$. If this equation yields $x_1 = x_2$, then f is one-one; else it is many-one. As an example, consider $f(x) = x^2$. Is this one-one, or many one? We have

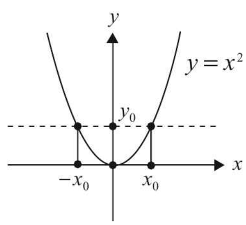
$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad (x_1 + x_2)(x_1 - x_2) = 0 \quad \Rightarrow \quad x_1 = x_2 \text{ or } x_1 = -x_2$$

Hence $f(x_1) = f(x_2)$ does not uniquely imply $x_1 = x_2$ (because another case exists, $x_1 = -x_2$). For example, $f(3) = f(-3) = 9$ (one output for two different inputs). We see that $f(x) = x^2$ is many-one.

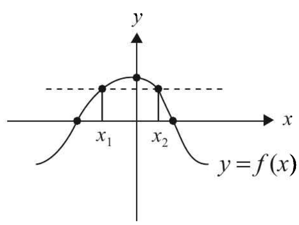
On a graph, this implies that if we draw any horizontal line, and it intersects the graph at the most once, then it is one-one, else it is many-one.



One-one



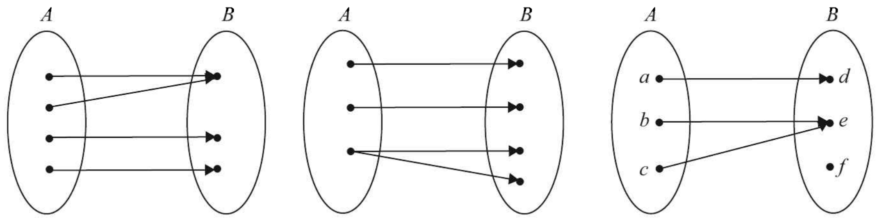
Many-one



Many-one

(B) Into/Onto

Look carefully at the three maps below. The first and the third are many-one, while the second is one-many.



Any other differences? For the first two maps, each element in the co-domain (set B) is *covered*, that is, associated with some element in the domain (set A). For the third map, one element ($\{f\}$) in the co-domain is *left out* (is not associated with any element in the domain, or in other words, does not have a *pre-image*). The first two maps are onto while the third is into.

Stating formally, an onto map is a map in which the range (the set of all images of the elements of the domain) 'covers' the entire co-domain, that is, Range = Co-domain. If Range $\subset$ Co-domain (like the third map above), then it is an into map. An example will make this more clear :

$$f:\underset{\text{Domain}}{\mathbb{R}} \to \underset{\text{Co-domain}}{\mathbb{R}} \qquad f(x) = x^2$$

The range is the set of all non-negative numbers, as $x^2 \geq 0$. Hence, Range $\subset$ Co-domain and the function is into. On the other hand, for

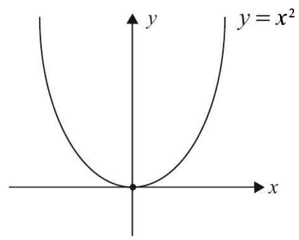
$$f:\underset{\text{Domain}}{\mathbb{R}} \to \underset{\text{Co-domain}}{[0,\infty)} \qquad f(x) = x^2$$

we see that Range = Co-domain and the map is onto.

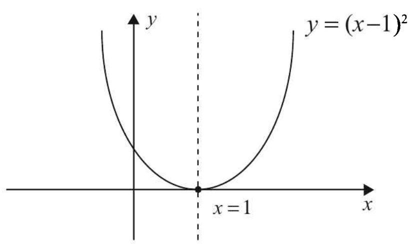
5. Properties of Functions

5.1 Even/Odd Functions

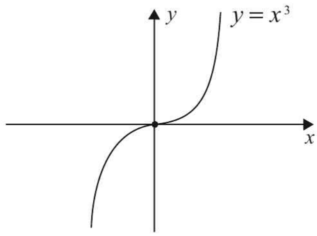
A function is even if $f(x) = f(-x)$. This means that the function is the same for the positive x-axis and the negative x-axis, or graphically, symmetric about the y-axis. For example, $f(x) = x^2$ is even:



A function is even about 'a' if it is symmetric about the line $x = a$. For example, $f(x) = (x-1)^2$ is even about $x = 1$.



For such a function, $f(a-x)=f(a+x)$. A function is odd if $f(x)=-f(-x)$, that is, the function on one side of the x-axis is sign-inverted with respect to the other side, or graphically, it is symmetric about the origin. For example, $f(x)=x^3$ is odd.



As in the even case, $f(x)$ can be odd about 'a'. For example, $f(x)=(x+2)^3$ is odd about $x=-2$.

A function can also be neither even nor odd. For example, $f(x)=x^2+x^3$ is neither even nor odd, as $f(x)\neq f(-x)$ and $f(x)\neq -f(-x)$.

5.2 Increasing/Decreasing Functions

A function $f(x)$ is increasing if f increases as x increases.

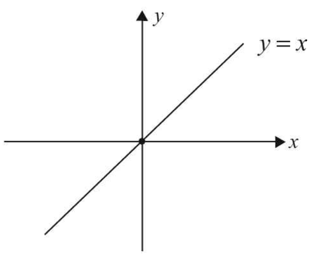
$$\text{If } x_1 < x_2 \Rightarrow f(x_1)\le f(x_2) \Rightarrow \text{function is increasing}$$

$$f(x_1) < f(x_2) \Rightarrow \text{function is strictly increasing}$$

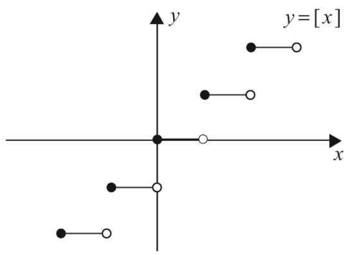
A function $f(x)$ is decreasing if f decreases as x increases.

$$\text{If } x_1 < x_2 \Rightarrow f(x_1)\ge f(x_2) \Rightarrow \text{function is decreasing}$$

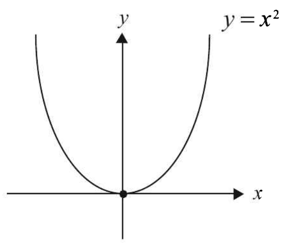
$$f(x_1) > f(x_2) \Rightarrow \text{function is strictly decreasing}$$



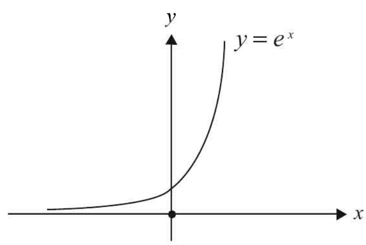
Strictly increasing



Increasing



strictly decreasing on $(-\infty,0]$
strictly increasing on $[0,\infty)$



strictly increasing

5.3 Periodic Functions

We know that some functions, like the trigonometric functions, are repetitive in nature, that is, they repeat after every fixed amount of change in their argument. Such functions are called periodic functions. A periodic function should satisfy

$$f(x+T) = f(x) \text{ for every } x$$

T is a constant called the *period* of f. The smallest such positive value of T is the *fundamental period*. For example, $\sin x$ has periods $2\pi, 4\pi, 6\pi, -4\pi, 2n\pi$ etc, but the fundamental period is 2π. The function $f(x) = \{x\}$ is also periodic with a fundamental period of 1. $f(x) = \tan x$ is periodic with fundamental period π, and so on. Note the following facts about periodic functions:

(a) If $f(x)$ has period T, then $f(kx)$ has a period $T/|k|$. This is because

$$f\left(k\left(x+\frac{T}{|k|}\right)\right) = f(kx \pm T) = f(kx)$$

For example,

$$\sin 2x \text{ has period } 2\pi/2 = \pi,\ \{\pi x\} \text{ has a period } \tfrac{1}{\pi}, \text{ etc.}$$

(b) Consider a function $z(x)$ which is composed of periodic functions $f(x), g(x), h(x), \ldots$, each having a defined period. Then, the period $z(x)$ will (in most cases) be the LCM of the individual periods, because the LCM of the periods of two (or more) functions is the first 'time' when the two (or more) functions start to repeat *simultaneously*, and hence the combination of these two (or more) functions start to repeat. Here are some examples:

(i) $f(x) = \sin x + \cos x$ has a period 2π {LCM (2π, 2π)}.
(ii) $g(x) = \sin 4x + \cos 6x$ has a period $\pi\{LCM(\frac{2\pi}{4}, \frac{2\pi}{6}) \equiv LCM(\frac{\pi}{2}, \frac{\pi}{3})\}$.
(iii) $h(x) = |\sin x|$ has a period π. Its easy to see why; visualise the graph.
(iv) $z(x) = |\sin x| + |\cos x|$ is periodic with period π {LCM (π, π)}, but the fundamental period is $\frac{\pi}{2}$ (verify). Hence, the LCM rule is not applicable everywhere.

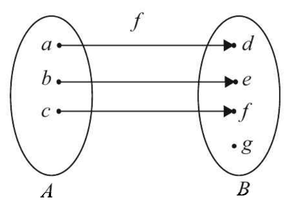
6. Inverse of a Function

The inverse of a function $f(x)$ is a function $g(x)$ such that if f maps an element 'a' to an element 'b', g maps 'b' to 'a'. In mathematical notation, $g(f(x)) = x$. That is, g reverses the action of f on x.

$$x \longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow x$$
$$g = f^{-1}$$

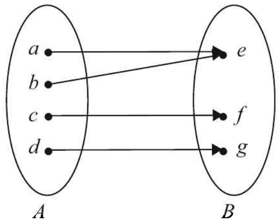
An inverse cannot exist for every function. To see why, consider f: $A \to B$ where A is the domain and B the co-domain. Consider the following cases:

(a) The map below is an into map (into function). If we take the 'inverse map' (from B to A, using the same 'links' as in f), we see that the element 'g' cannot be assigned to any element in A. In other words, for an into function, some values in the co-domain are 'left out', and their 'inverses' do not exist.



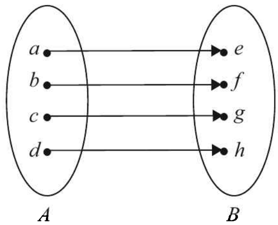
$\Rightarrow$ We cannot defined an inverse function for an into function.

(b) The map below is a many-one map (a many-one function). If we take the inverse map, the element 'e' will be assigned to two elements in A, that is, 'a' and 'b'. Hence, the inverse map cannot be a function.



$\Rightarrow$ We cannot defined an inverse function for a many-one function.

From this discussion, we conclude that for a function to be invertible, it should be one-one and onto. Such functions are known as *bijective* functions.



As is intuitively clear, we can easily define an inverse for the bijective map above:

$$f^{-1}(e) = a, f^{-1}(f) = b, f^{-1}(g) = c, f^{-1}(h) = d$$

Facts about some specific inverse functions are summarized below:

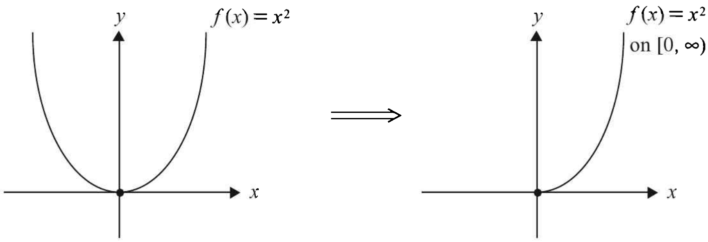
(a) $f(x) = x \quad \mathbb{R} \to \mathbb{R}$

We see that this function is one-one and onto. The inverse exists.

$$f^{-1}(x) = x \quad \mathbb{R} \to \mathbb{R}.$$

(b) $f(x) = x^2 \quad \mathbb{R} \to \mathbb{R}$

This function is many-one and into. To define the inverse, we first need to make f one-one and onto.



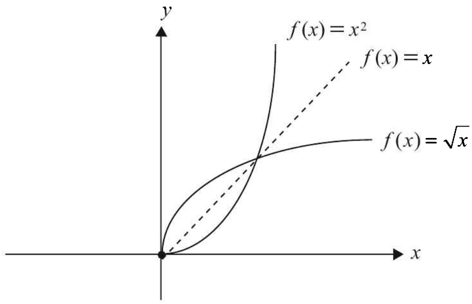
We see that to make the function one-one, we can select the domain as only $[0, \infty)$ (instead of $\mathbb{R}$). In this domain, the function is one-one, as is clear from the second diagram above. Also, the range is $[0, \infty)$. Therefore, we redefine this function:

$$f(x) = x^2 \qquad \underset{A}{[0,\infty)} \to \underset{B}{[0,\infty)}$$

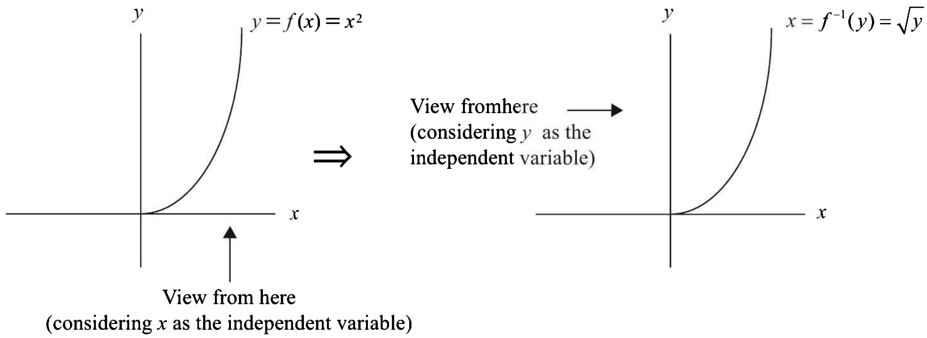
This function is now one-one and onto, and hence invertible:

$$f(x) = \sqrt{x} \qquad \underset{B}{[0,\infty)} \to \underset{A}{[0,\infty)}$$

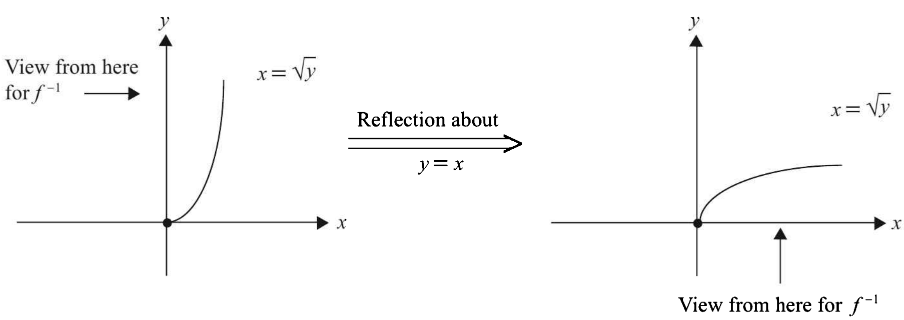
Additionally, note that if we draw the graphs of $\sqrt{x}$ and x^2 on the same axis, they are mirror images of each other in the line $x = y$:



A little thought will show why f and f^{-1} should be mirror images in the mirror $y = x$. In the equation $y = f(x)$, x can be treated as the independent and y the dependent variable. In the equation $x = f^{-1}(y)$, we can reverse the roles. We can treat y-axis as the independent variable axis:

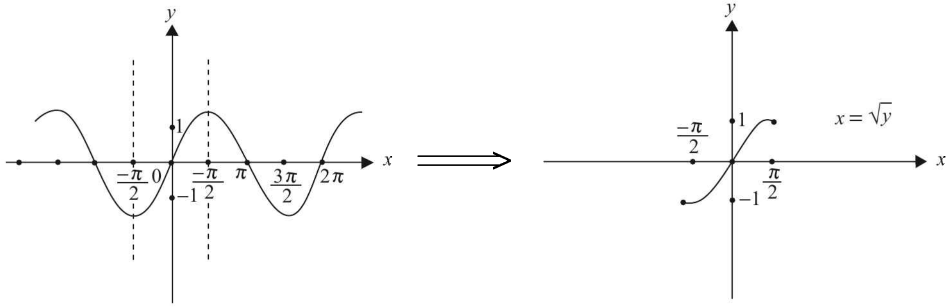


By convention, the independent variable is taken on the horizontal axis. Therefore we convert this horizontal view (the second diagram above) into a conventional vertical view. How? By taking the reflection in $y = x$.



(c) $f(x) = \sin x \quad \mathbb{R} \to [-1, 1]$

This function is onto but not one-one. Lets make it one-one first. Select that interval as the domain, in which $\sin x$ remains one-one.

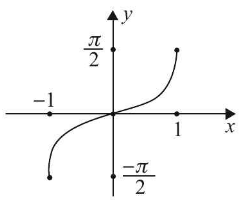


In other words, we redefine f as

$$f(x) = \sin x \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

We could have taken any other interval in which $\sin x$ remains one-one, to define the inverse. But conventionally, we take the interval closest to the origin, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in this case, whose mid-point is the origin. The inverse can now be defined as:

$$f^{-1}(x) = \sin^{-1} x \quad [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Note that these graphs are only representative and not to exact scale.

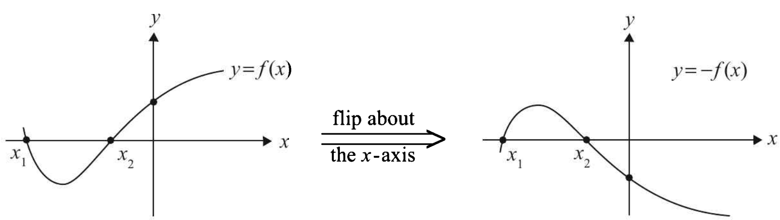
7. Manipulating Graphs

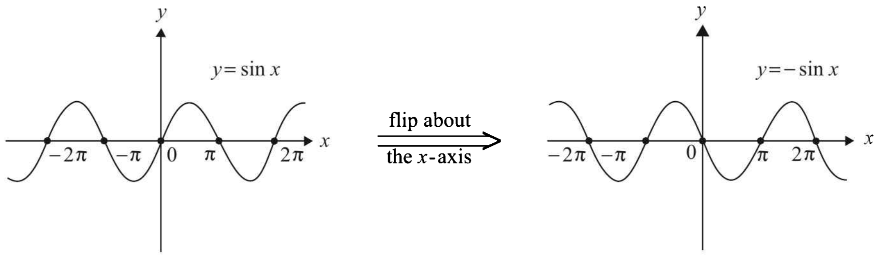
In this section, we will briefly discuss how you can plot the graphs of functions like

$$y = -f(x),\ y = |f(x)|,\ y = f(|x|) \quad \text{etc.}$$

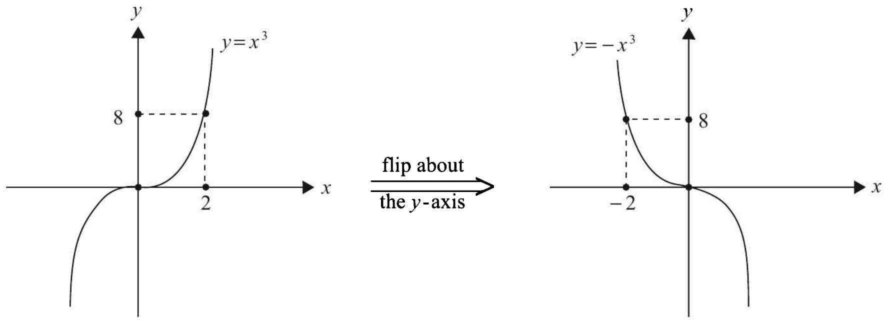
given the graph of $y = f(x)$.

(a) $y = -f(x)$: Flip about the x – axis

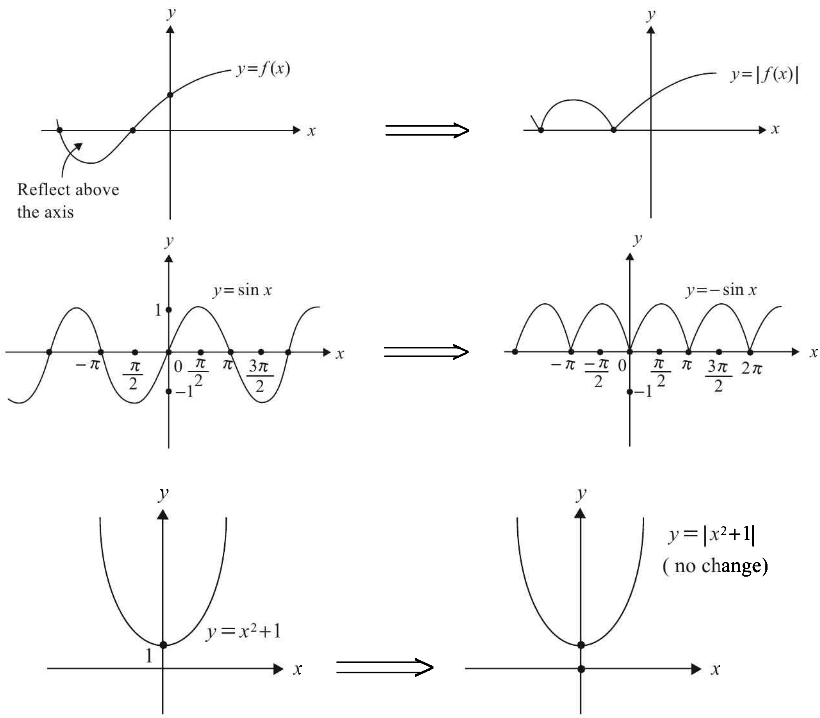


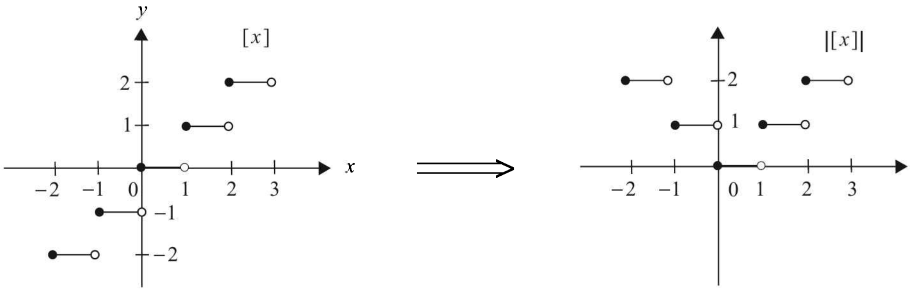


(b) $y = f(-x)$: Flip about the y – axis



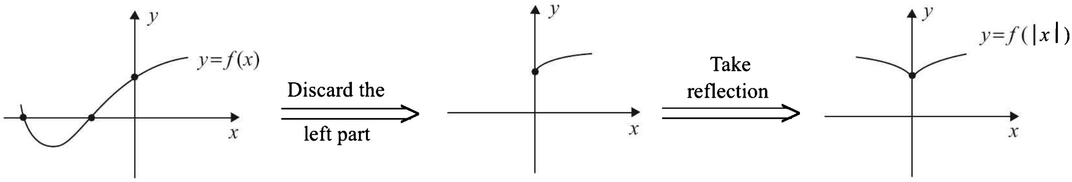
(c) $y = |f(x)|$: Reflect the parts of the graph that lie in the lower half (negative parts) into the upper half of the axes.



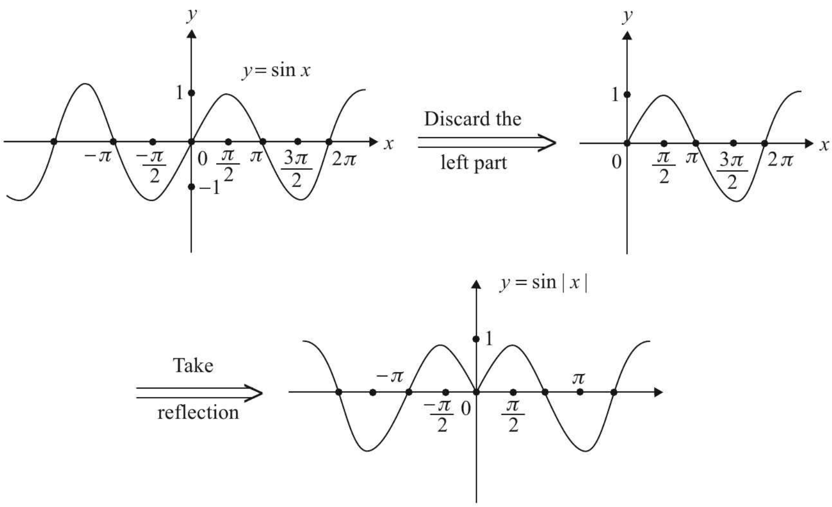


(d) $y = f(|x|)$: Discard the left part of the graph (for $x < 0$) and take a reflection of the right part of the graph into the left half of the axes.

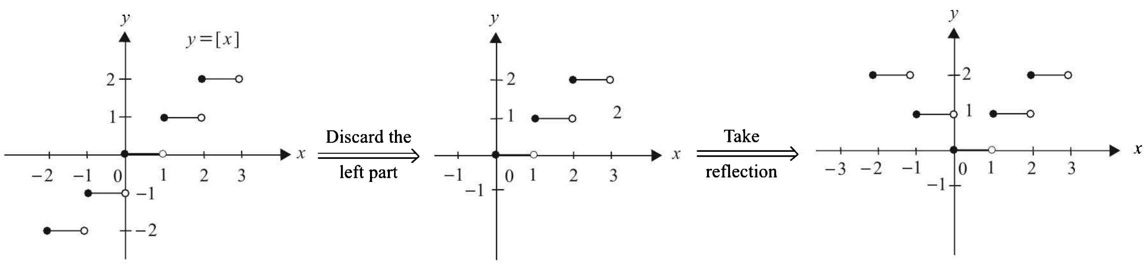
(i)



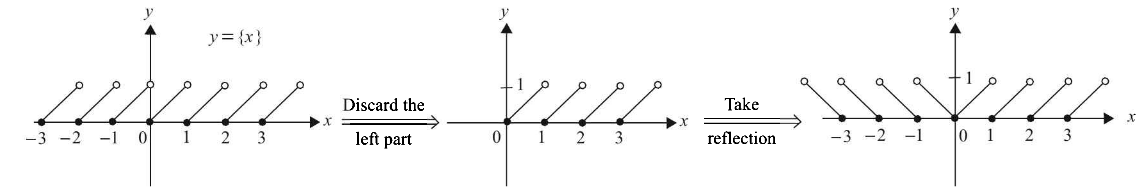
(ii)



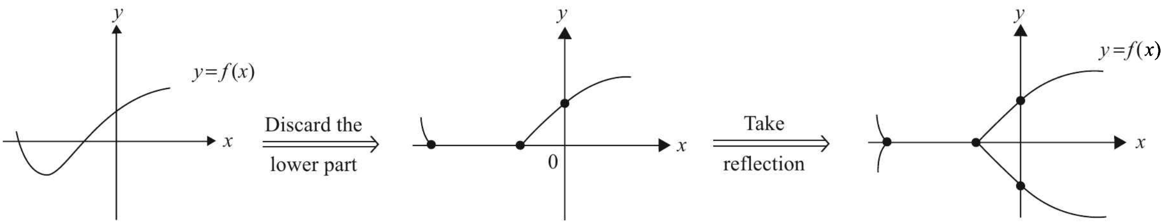
(iii)

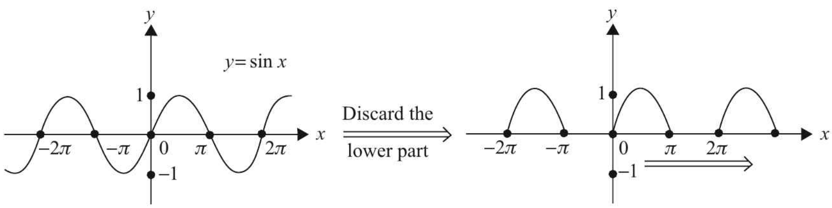


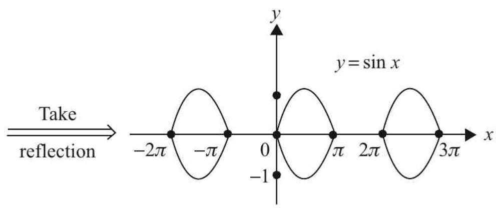
(iv)

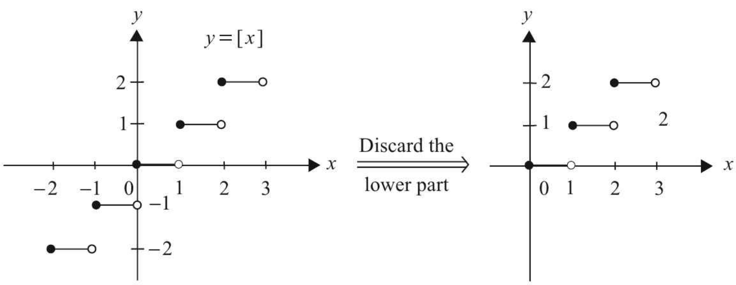


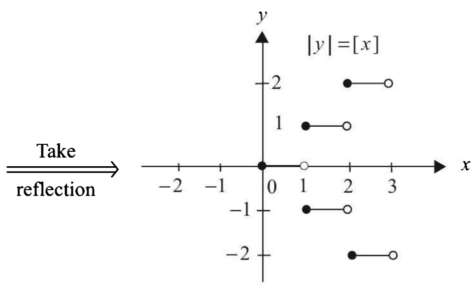
(e) $|y| = f(x)$: Discard the lower part of the graph ($f(x) < 0$) and take a reflection of the upper part of the graph into the lower half of the axes.



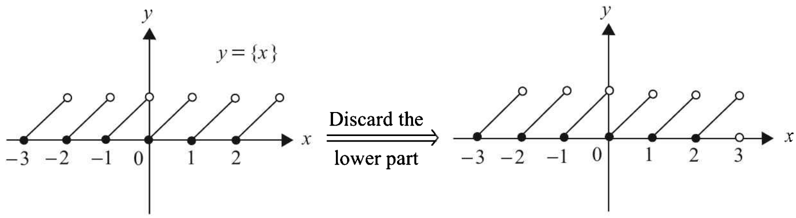


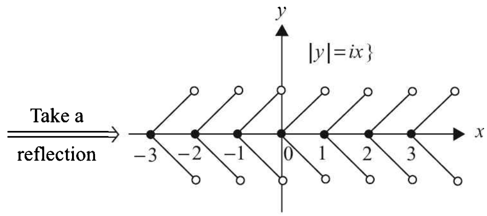




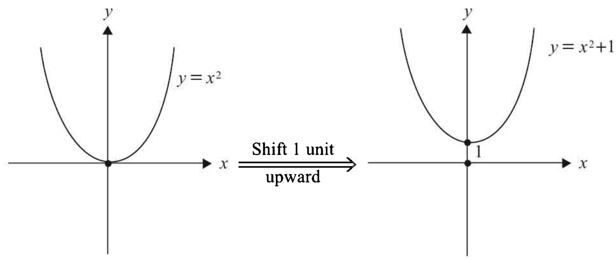


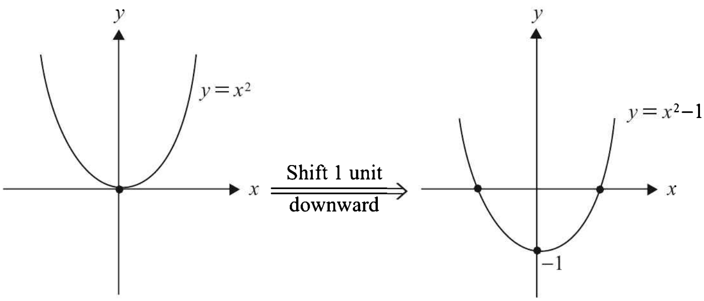
(Same)
No lower part



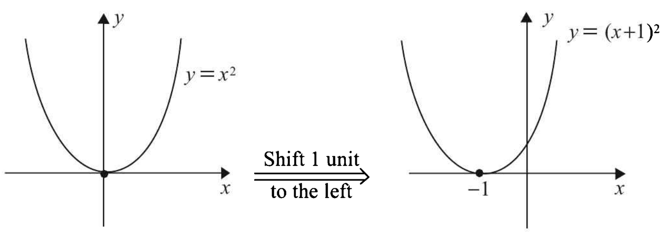


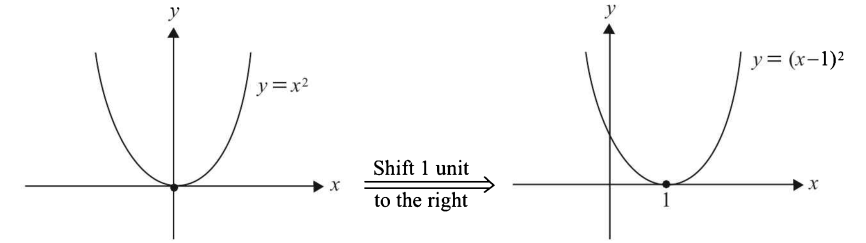
(f) $y = f(x) \pm k$: Shift the graph $|k|$ units upwards or downwards depending on whether k is positive or negative respectively.

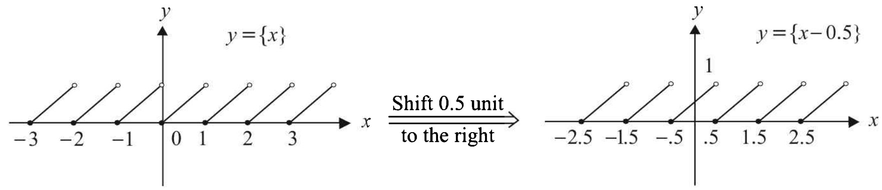


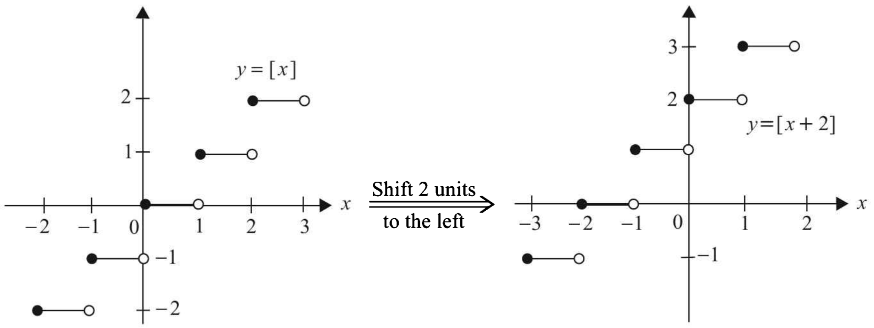


(g) $y = f(x \pm k)$: Advance (shift left) or delay (shift right) the graph by $|k|$ units depending on whether k is positive or negative respectively.

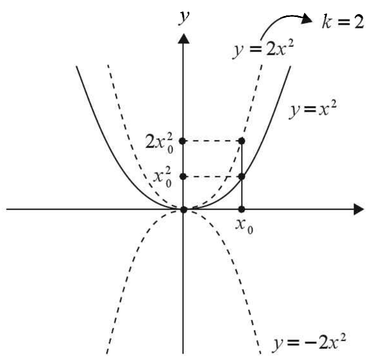




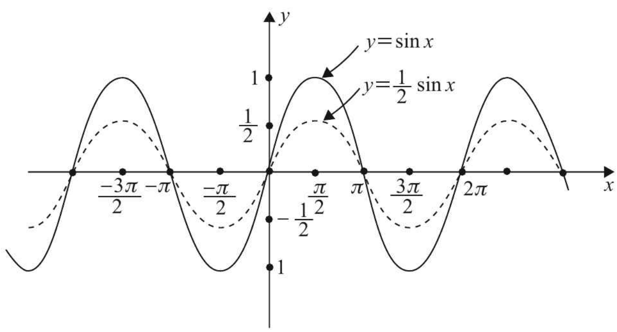




(h) $y = kf(x)$: Stretch or compress the graph along the y-axis depending on whether $|k| > 1$ or $|k| < 1$ respectively. Also, flip it about the x-axis if k is negative.



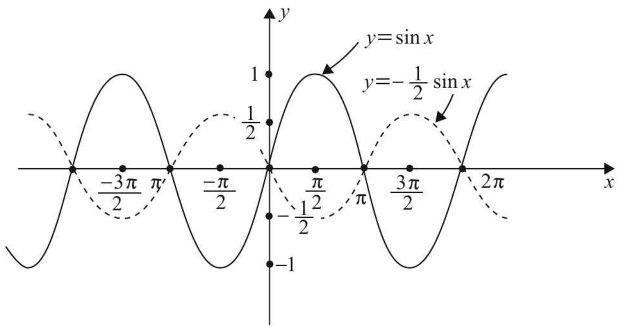
The zero of $y=f(x)$ remains the same ($x=0$)



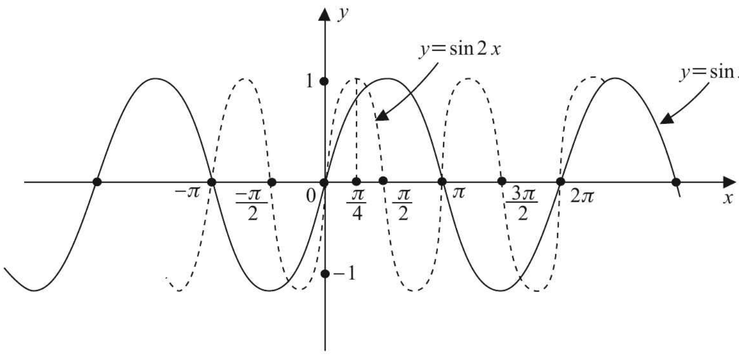
The zeroes for both $y=\pm\frac{1}{2}\sin x$ are the same as the zeroes for $y=\sin x$.

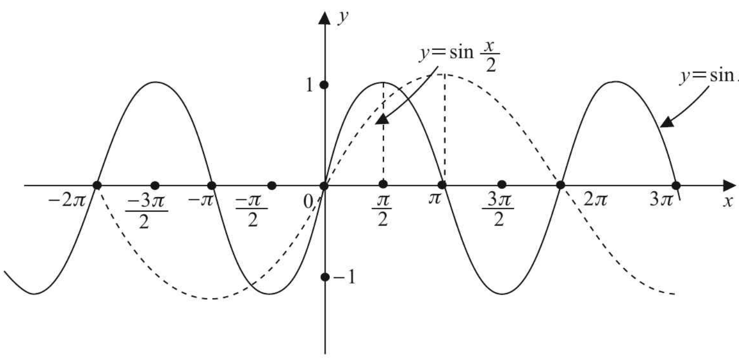
($\{n\pi\}$, $n\in\mathbb{Z}$)

The graphs get 'compressed' though, since $|\frac{1}{2}|<1$.
For $y=-\frac{1}{2}\sin x$, the graph also gets flipped.



(i) $y = f(kx)$: Stretch or compress the graph along the x-axis depending on whether $|k| < 1$ or $|k| > 1$ respectively. Also, flip it about the y-axis if k is negative.





IMPORTANT IDEAS AND TIPS

1. *Real Number System.* One of the most critical concepts to understand in functions, and indeed in Calculus as a whole, is the concept of the real number system, and how the real line is continuous (has no gaps). If you were to plot integers on a straight line, you would get one number every unit distance. If you were to plot rationals, you would get numbers spaced extremely close to each other (extremely densely packed - between any two rationals, no matter how close, you would still be able to find another rational), but there would still be gaps in the line, corresponding to the irrationals. However, if you were to plot the entire real set on the line, there would be absolutely no gap at all, the points would form an absolutely continuous

straight line. Realizing this continuity of the real set (or the real line) is extremely important to be able to obtain a deep understanding of limits and infinitesmal quantities, concepts on which the entire foundation of Calculus rests.

2. *A Matter of Representation*. A function should ideally be denoted as *f* instead $f(x)$, since $f(x)$ represents the specific value of the function at the point *x*. However, following convention, we always use $f(x)$ so that we can name the variable on which *f* depends, although it should be clear in your mind that the function is *f* and $f(x)$ is a specific value of *f*.

3. *Domain Calculation*. The following particular issue related to calculation of domains needs to be carefully understood. Suppose you are asked to calculate the domain of $f(x)=\sqrt{x}$. You would most likely say : *x* should not be negative; the domain is $[0,\infty)$. This is correct, but only if we are restricted to considering real functions, *i.e.*, functions whose output should be real. If we were told that $f(x)$ is a complex function, then there is no problem in *x* taking on negative values as well (or even complex values), because the square root of any complex number is defined (and is another complex number). On the other hand, if we were told that $f(x)$ is an integer function, it should produce integer output, then our domain becomes even more restricted to numbers of the form $0^2, 1^2, 2^2, \ldots$, that is, all perfect squares. Thus, the domain depends on the nature of the function. The same function considered from different perspectives will have different domains.

4. *Specific Cases of Range Calculation*. We have encountered many issues students face while calculating ranges of specific types of functions. For example, suppose you were asked to calculate the range of the function $y=f(x)=\frac{x^2-x+1}{x^2+x+1}$. You would proceed by cross-multiplying and forming a quadratic in *x*, and then make use of the fact that the discriminant of this quadratic is non-negative. Well and good, except, do you know why you would carry out this particular sequence of steps? What is the rationale behind it? This may seem like a minor issue, but it is important to understand the logic behind it. When you are solving the quadratic in *x*, you are essentially expressing *x* in terms of $y: x=g(y)$. When you impose the constraint that the discriminant should be non-negative, you are in fact determining the domain of *g*, the set of allowed values of *y* for which $g(y)$ is defined. This restricts the possible values of *y*, and thus you obtain the range of the original function *f*(*x*). In this and other types of range questions, you must always understand the rationale behind your procedure. Additionally, you must try to think from a graphical point of view: 'If I were to plot the graph of $y=f(x)=\frac{x^2-x+1}{x^2+x+1}$, what curve would I obtain? What would be its variation along the vertical direction?'

5. *Roots and Zeroes*. The difference between writing *f*(*x*) and writing *f*(*x*) = 0 must be kept in mind. The former is an expression, the latter an equation. The former means a function $f(x)$, with a specific domain and range. The latter implies that $f(x)$ is equal to 0, which leads us to *those* particular values of *x* for which $f(x)$ vanishes. The zeroes of $f(x)$ are the same as the roots of $f(x)=0$. But the expression 'the roots of $f(x)$' does not make sense since $f(x)$ in itself is not an equation. On the other hand, expressions like 'the roots of $f(x)=0$' or even 'the roots of $f(x)=\pi$' or 'the roots of $f(x)=1000000000$' make perfect sense.

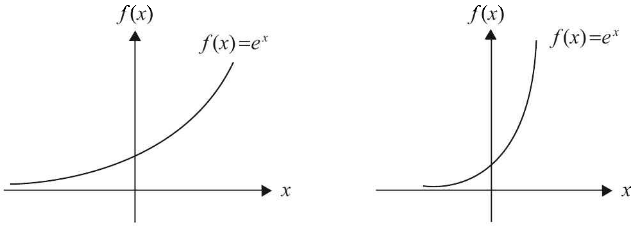
6. *Concept of Graphs*. One of the most widely prevalent pitfalls in functions is to not understand the concept of graphs properly. Here we present two slightly different ways to think about graphs (both are essentially equivalent):

(a) Think of the expression of the function $y=f(x)$ as an equation. If the coordinates of a point (x, y) satisfy this equation, the point will lie on the graph of the function. All other points whose coordinates do not satisfy this equation will not lie on the function's graph. This way of thinking about a function's graph helps in drawing graphs of curves like $|y|=f(x)$ and $y=f(|x|)$ from the graph of $y=f(x)$. For example, to draw the graph of $|y|=f(x)$, you have to consider all points in the plane which satisfy the equation $|y|=f(x)$, and you already know which points satisfy the equation $y=f(x)$.

(b) Think of *f* as an operator, which acts on input *x* and produces output *y*. As *x* is varied in the domain, different *y*-values are obtained. For each *x*, the point (*x*, *y*) is plotted, and the set of all such points constitutes the graph of the function. This way of thinking helps us in correlating the range of a function with its graph. The range is nothing but the variation of the curve of $f(x)$ along the *y*-direction, the set of values that the *y*-coordinate of points on the graph can take.

7. *Graph-Shifting*. Shifting of graphs is one of the most important concepts in functions. In particular, expressions like $y = f(x) \pm k$ and $y = f(x \pm k)$ should be carefully distinguished. The former implies a vertical shift, whereas the latter implies a horizontal shift.

8. *Graph Plotting*. By far the most frequent 'mistake' (bad habit would be a better descriptor) in functions is to plot graphs which are not to scale, and do not show the variations of the curve in sufficient detail. This has a lot of disadvantages, you may not be able to infer properties about the function which you could easily have, had you plotted the graph to a better scale. You will never be able to appreciate the variation in the graph properly with an incorrect scale. One of the best examples is the graph of the function $f(x) = e^x$. Most students would plot a curve resembling the one on the left side below, while the actual variation in e^x is better described by the curve on the right.



With the curve on the left, you can never appreciate how tremendously fast e^x grows; that is something which only starts becoming evident from the curve on the right. If you were to plot the graph of $y = e^x$ such that every unit on the x- and y-axis was equal to 1 cm, then with less than 35 cm of change in x, the height of the graph would exceed the distance of the earth from the sun! Similarly, to appreciate how slowly ln x grows, its graph must be plotted to as good a scale as possible. And this holds true for every function. Graphing a function accurately is one of the most important skills required to master Calculus.

9. *Scalar Multiplication*. Scalar multiplication in graphs is another potential pitfall. In particular, a lot of care should be taken in distinguishing $y = kf(x)$ from $y = f(kx)$. In the former, the original graph gets stretched or compressed along the y direction, while in the latter, it gets stretched or compressed in the x-direction. If k is negative, the graph flips along the vertical or horizontal direction respectively. Also, we note the following:

What happens to the graph of $y = f(x)$?

	$\|k\| > 1$	$\|k\| < 1$
$y = kf(x)$	Stretched vertically	Compressed vertically
$y = f(kx)$	Compressed horizontally	Stretched horizontally

The difference in the two must always be kept in mind.

10. *Even and Odd Functions*. A function can be even or odd about any point $x = a$, and not just $x = 0$.

11. *Inverse Functions*. For the inverse of a function $f(x)$ to be defined on some domain D, $f(x)$ must be one-one and onto for D. A lot of times, this important fact is forgotten, and it leads to incorrect values for the domain and range of the inverse function.

12. *Composition of Functions*. Let f and g be two functions. The composite function $f \circ g$ has to be interpreted as follows: for any input x, first the function g is to be applied. On the resulting quantity $g(x)$, the function f as to be applied, to obtain the final output $f(g(x))$. This also means that $f \circ g$ will be different from $g \circ f$, since the order in which you apply the functions is different in the two cases. A frequent mistake that students make pertains to the calculation of domains of composite functions. Consider $f(x) = \sqrt{x-1}$ and consider $g(x) = x^2$. The composite function $g \circ f(x)$ is $x - 1$. If you consider the function $x - 1$ in itself, its domain is $\mathbb{R}$. But the domain of $g \circ f(x)$ will not be $\mathbb{R}$, since $f(x)$ itself is not defined for all values in $\mathbb{R}$. The actual domain of $g \circ f(x)$ in this case will be the domain of $f(x)$, which is $[1, \infty)$. Therefore, whenever calculating the domain of a composite function, you must check whether your answer is compatible with the domains of all the functions making up the composite.

Functions

PART-B: Illustrative Examples

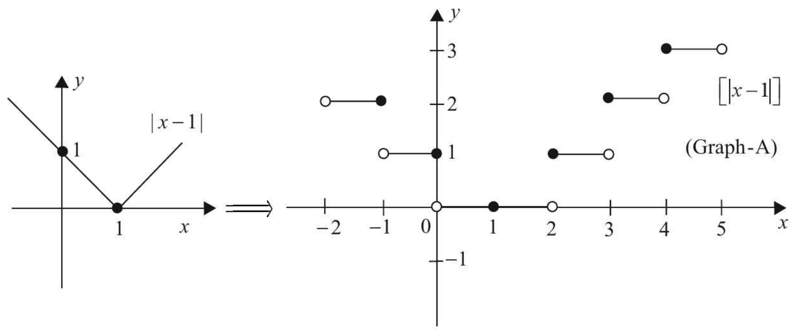
OBJECTIVE TYPE EXAMPLES

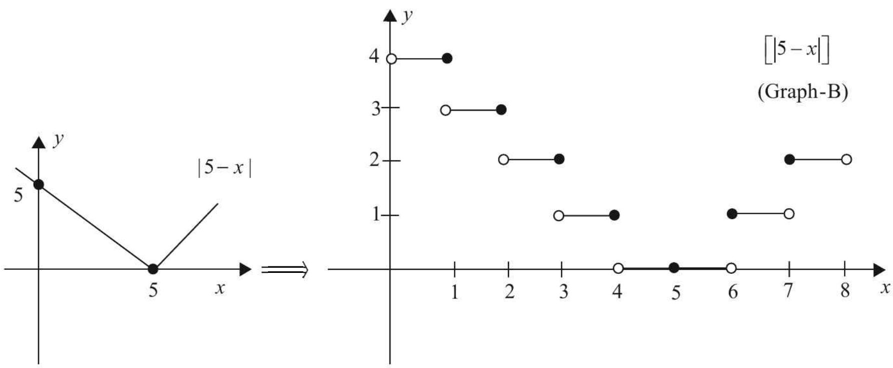
Example 1

How many integers in the interval $[-10,10]$ will not lie in the domain of $f(x)=\frac{1}{[|x-1|]+[|5-x|]-4}$?

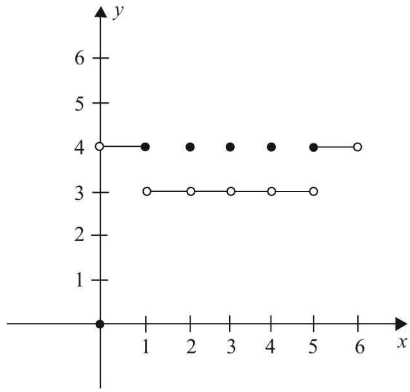
(A) 2 (B) 3 (C) 4 (D) 5 (E) None of these

Solution: The denominator is a bit complicated and we need to analyse it in detail to determine where it can become zero. The fastest and easiest way would be to visualise the graph. Draw the graphs for $|x-1|$ and $|5-x|$, apply the greatest integer function on these graphs separately, then add them and find the values of x for which this sum becomes 4. For these values of x, the denominator of $f(x)$ becomes 0.





Now add the graphs of A and B point by point:



We see that the value of (graph A + graph B) is 4 for the following values of x:

$$x:(0,1],\{2,3,4\},[5,6).$$

Hence, $D=\mathbb{R}\setminus\{(0,1],2,3,4,[5,6)\}$. We see that there are 5 integers which do not lie in the domain of the given function, namely $1,2,3,4,5$. Therefore, the correct option is (D). ■

Example 2

The domain of $f(x)=\sqrt{1-|x^2+3x+2|}$ is

(A) $\left[\dfrac{-3-\sqrt{5}}{2},\dfrac{-3+\sqrt{5}}{2}\right]$ (B) $\left[\dfrac{-1-\sqrt{5}}{2},\dfrac{-1+\sqrt{5}}{2}\right]$ (C) $\left[\dfrac{-1-2\sqrt{5}}{2},\dfrac{-1+2\sqrt{5}}{2}\right]$ (D) None of these

Solution: $f(x)=\sqrt{1-|x^2+3x+2|} \Rightarrow 1-|x^2+3x+2|\geq 0$

$$\Rightarrow |x^2+3x+2|\leq 1$$

$$\Rightarrow \underbrace{-1\leq x^2}_{A}+3x+\overbrace{2\leq 1}^{B}$$

Inequality A: $x^2+3x+2\geq -1 \Rightarrow x^2+3x+3\geq 0$

This is satisfied for all real values of x.

Inequality B: $x^2+3x+2\leq 1 \Rightarrow x^2+3x+1\leq 0$

The roots of LHS are $\frac{-3\pm\sqrt{5}}{2} \Rightarrow \frac{-3-\sqrt{5}}{2}\leq x\leq \frac{-3+\sqrt{5}}{2}$

$$\Rightarrow D=\left[\frac{-3-\sqrt{5}}{2},\frac{-3+\sqrt{5}}{2}\right]$$

Therefore, the correct option is (A). ■

Example 3

How many integers are not included in the domain of $f(x)=\frac{1}{\sqrt{x^2+3x+3}-1}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) Greater than 4

Solution:
$$f(x) = \frac{1}{\sqrt{x^2+3x+3}-1} \Rightarrow \sqrt{x^2+3x+3}-1 \neq 0$$
$$\Rightarrow x^2+3x+3 \neq 1 \Rightarrow x^2+3x+2 \neq 0$$
$$\Rightarrow x \neq -1,-2 \Rightarrow D = \mathbb{R} \setminus \{-1,-2\}$$

We see that two integers are not included in the domain, and so the correct option is (B). ■

Example 4

What is the range of $y = \sin^{-1}(\frac{x^2+1}{x^2+2})$?

(A) $\left[\frac{\pi}{6},\frac{\pi}{2}\right)$ (B) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$ (C) $\left[\frac{\pi}{4},\frac{\pi}{2}\right)$ (D) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ (E) None of these

Solution: Let us first evaluate the range of $\frac{x^2+1}{x^2+2} = z$ (say):

$$z = \frac{x^2+1}{x^2+2} = \frac{x^2+2-1}{x^2+2} = 1-\frac{1}{x^2+2} \geq 1-\frac{1}{2} = \frac{1}{2}$$

Notice the last steps carefully. We have used the following:

$$x^2 \geq 0 \Rightarrow x^2+2 \geq 2 \Rightarrow \frac{1}{x^2+2} \leq \frac{1}{2} \Rightarrow -\frac{1}{x^2+2} \geq -\frac{1}{2}$$

Now, $z \geq \frac{1}{2}$ implies $\sin^{-1} z \geq \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. $\sin^{-1} z$ can take a maximum value of $\frac{\pi}{2}$ for $z = 1$, but we see that for no value of x does z become 1 (although z 'approaches' or 'almost becomes' 1 as x becomes larger and larger, *i.e.*, as $x \to \infty$, $z \to 1$, but $z \neq 1$). Hence $\sin^{-1} 1 = \frac{\pi}{2}$ is not included in the range.

$$R = \left[\frac{\pi}{6},\frac{\pi}{2}\right)$$

The correct option is (A). ■

Example 5

What is the difference between the minimum and maximum elements in the range of $y = \frac{1}{\sin^4 x+\cos^4 x}$?

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 3 (E) None of these

Solution:
$$y = \frac{1}{\sin^4 x+\cos^4 x} = \frac{1}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}$$
$$= \frac{1}{1-2\sin^2\cos^2 x} = \frac{1}{1-\frac{1}{2}(\sin 2x)^2}$$

Now $\quad -1 \leq \sin 2x \leq 1 \Rightarrow 0 \leq (\sin 2x)^2 \leq 1$

$$\Rightarrow 0 \leq \frac{1}{2}(\sin 2x)^2 \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq 1-\frac{1}{2}(\sin 2x)^2 \leq 1$$

$$\Rightarrow \quad 1 \le \frac{1}{1-\frac{1}{2}(\sin 2x)^2} \le 2$$

Therefore, the range is [1, 2]. The difference between the maximum and minimum elements in this set is 1, so the correct option is (B). ■

Example 6

The range of $y = \sqrt{x-1} + 2\sqrt{3-x}$ is

(A) $[\sqrt{2}, 2\sqrt{2}]$ (B) $[\sqrt{2}, 2\sqrt{5}]$ (C) $[\sqrt{2}, \sqrt{10}]$ (D) None of these

Solution: The domain of this function is [1, 3]. Also, $y > 0$. Now,

$$y^2 = (x-1) + 4(3-x) + 4\sqrt{(x-1)(3-x)}$$
$$= 11 - 3x + 4\sqrt{(x-1)(3-x)}$$

Rearranging and squaring gives

$$(y^2 + 3x - 11)^2 + 16(x^2 - 4x + 3) = 0$$
$$\Rightarrow \quad y^4 + (3x-11)^2 + 2y^2(3x-11) + 16(x^2 - 4x + 3) = 0$$
$$\Rightarrow \quad 25x^2 + (6y^2 - 130)x + (y^4 - 22y^2 + 169) = 0$$

For real x, we require $D \ge 0$:

$$\Rightarrow \quad (3y^2 - 65)^2 - 25(y^4 - 22y^2 + 169) \ge 0$$
$$\Rightarrow \quad -16y^4 + 160y^2 \ge 0 \quad \Rightarrow \quad y^4 - 10y^2 \le 0 \quad \Rightarrow \quad y \le \sqrt{10}$$

Also, at $x = 1, y = 2\sqrt{2}$ and at $x = 3, y = \sqrt{2}$, and therefore, the minimum value of y is $\sqrt{2}$. The range is $[\sqrt{2}, \sqrt{10}]$. Thus, the correct option is (C). ■

Example 7

What is the range of $f(x) = \frac{x^2+x-1}{x^2-x+2}$?

(A) $\left[\frac{3-2\sqrt{11}}{7}, \frac{3+2\sqrt{11}}{7}\right]$ (C) $\left[\frac{4-3\sqrt{11}}{7}, \frac{4+3\sqrt{11}}{7}\right]$ (E) None of these

(B) $\left[\frac{3-2\sqrt{11}}{7}, \frac{3+2\sqrt{11}}{7}\right] - \{1\}$ (D) $\left[\frac{4-3\sqrt{11}}{7}, \frac{4+3\sqrt{11}}{7}\right] - \{1\}$

Solution: For the domain, $x^2 - x + 2 \neq 0$. Now, $x^2 - x + 2$ can be rearranged as $(x - \frac{1}{2})^2 + \frac{7}{4}$, which is always positive. Hence, for no value of x is the denominator equal to 0.

$$\Rightarrow \quad D = \mathbb{R}$$

To find the range, we find x in terms of $f(x)$, and see what values of $f(x)$ will make x real. Those values will form our range. We use y instead of $f(x)$ for convenience, *i.e.*, $y = f(x)$:

$$x^2 y - xy + 2y = x^2 + x - 1$$
$$\Rightarrow \quad (1-y)x^2 + (1+y)x - (1+2y) = 0 \qquad (1)$$

By the quadratic formula, we have

$$x=\frac{-(1+y)\pm\sqrt{(1+y)^2+4(1-y)(1+2y)}}{2(1-y)}$$

$$x=\frac{-(1+y)\pm\sqrt{-7y^2+6y+5}}{2(1-y)}$$

The right side is defined when $y \neq 1$ and $-7y^2+6y+5\geq 0 \;\Rightarrow\; 7y^2-6y-5\leq 0$

$$\Rightarrow\quad \frac{3-2\sqrt{11}}{7}\leq y\leq\frac{3+2\sqrt{11}}{7}$$

Therefore, the values that y can take are

$$\frac{3-2\sqrt{11}}{7}\leq y\leq\frac{3+2\sqrt{11}}{7}\quad\text{and}\quad y\neq 1$$

These values should form our range. But we have overlooked something. $y \neq 1$ arises because the quadratic formula gives $(1-y)$ in the denominator. Suppose that in the equation (1) itself, we put $y = 1$, reducing the equation to a linear one:

$$2x-3=0\quad\Rightarrow\quad x=\frac{3}{2}$$

This means that $y = 1$ gives a defined value of x, or in other words, $y=1$ has a pre-image, $x = \frac{3}{2}$ (check this: $f(\frac{3}{2})=1$). Hence $y=1$ should also be in the range.

$$R=\left[\frac{3-2\sqrt{11}}{7},\frac{3+2\sqrt{11}}{7}\right]$$

The correct option is therefore (A). ■

Example 8

What is the range of $f(x)=x^3+3x^2+4x+5$?

(A) $[0,\infty)$ (B) $\mathbb{R}$ (C) $[5,\infty)$ (D) $[-\frac{7}{2},\infty)$ (E) None of these

Solution: We can determine the range without any actual calculation, as follows. As x increases (or as $x\to\infty$), $f(x)$ will keep on increasing in an unbounded fashion $(f(x)\to\infty)$. Similarly, as x decreases (or as $x\to-\infty$), $f(x)$ will keep on decreasing in an unbounded fashion (or $f(x)\to-\infty$). Also, since $f(x)$ is a polynomial function, it is continuous (and hence will vary continuously). Hence, $f(x)$ will assume all values between $-\infty$ and $+\infty$.

$$R=\mathbb{R}$$

■

Example 9

Consider $f(x)=\left\{\begin{smallmatrix} x & \text{if} & x \text{ is rational} \\ 0 & \text{if} & x \text{ is irrational}\end{smallmatrix}\right\}$ and $g(x)=\left\{\begin{smallmatrix} 0 & \text{if} & x \text{ is rational} \\ x & \text{if} & x \text{ is irrational}\end{smallmatrix}\right\}$. What is the nature of the function $f-g$?

(A) One-one and Into (B) One-one and Onto (C) Many-one and Into (D) Many-one and Onto

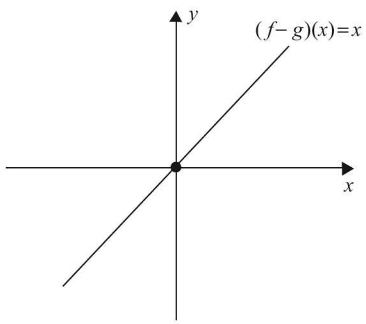
Solution: The co-domain has not been specified explicitly, so we assume it to be $\mathbb{R}$. We have

$$(f-g)(x)=\begin{Bmatrix} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{Bmatrix}$$

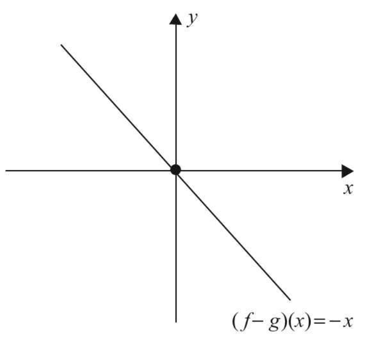
A moment of consideration shows that each input is assigned to only one output. Hence, $f-g$ is one-one. Also, every real number has a corresponding pre-image. For example, the pre-image of $\sqrt{2}$ is $-\sqrt{2}$, of $-\sqrt{2}$ is $\sqrt{2}$, of $\frac{3}{5}$ is $\frac{3}{5}$, and so on. Hence, $f-g$ is onto. The correct option is (B).

If you are not convinced, think of this in the following way. Suppose you construct an axes with two 'rational lines' instead of two real lines. These two lines contain all (and only) the rational points. What will $(f-g)(x)$ look like for this axes?

$$(f-g)(x) = x \text{ if } x \text{ is rational}$$



For the rational axes, $(f-g)$ (x) is one-one and onto. Doing the same for an irrational axes gives us the following graph:



For the irrational axes, $(f-g)(x)$ is one-one and onto as well. Now superimpose the two axes, and you get the normal real axes. The rational and irrational lines are exactly complementary. They do not overlap and together they form the real line. The two parts of $(f-g)(x)$ are separately one-one and onto. When you combine them, the resultant is obviously also one-one and onto. Hence, $f-g$ is one-one and onto. ■

Example 10

The number of solutions to the equation $\frac{1}{[x]}+\frac{1}{[2x]}=\{x\}+\frac{1}{3}$ is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: A solution to this equation is not evident by mere observation. But it can be noted that the RHS is always positive: therefore, the LHS must be positive and hence $x > 0$. Also, for the LHS to be defined, $x \geq 1$ and $2x \geq 1$. Thus, $x \geq 1$ and hence $[x] \geq 1$. Let I be the integral and f the fractional part of x.

$$\Rightarrow \quad \frac{1}{I}+\frac{1}{2I+[2f]}=f+\frac{1}{3}$$

We have retained $[2f]$ since $2f$ could be greater than 1 and hence $[2f]$ is not necessarily 0. We will have to consider two different cases separately:

(i) $\boxed{f<\frac{1}{2}}$ $\quad\Rightarrow\quad \frac{1}{I}+\frac{1}{2I}=f+\frac{1}{3}$

$$\Rightarrow \frac{3}{2I}=f+\frac{1}{3}$$

Now we can substitute different values of I; if these give valid values for f such that $f<\frac{1}{2}$, we accept these solutions.

$I=1 \quad\Rightarrow\quad f=\frac{7}{6}$ [not acceptable]

$I=2 \quad\Rightarrow\quad f=\frac{5}{12}$ [acceptable]

$I=3 \quad\Rightarrow\quad f=\frac{1}{6}$ [acceptable]

$I=4 \quad\Rightarrow\quad f=\frac{1}{24}$ [acceptable]

$I=5 \quad\Rightarrow\quad f<0$ [not acceptable]

No more solutions will exist since f becomes less than 0 for $I \geq 5$.

(ii) $\boxed{f \geq \frac{1}{2}}$ $\quad\Rightarrow\quad \frac{1}{I}+\frac{1}{2I+1}=f+\frac{1}{3}$

$I=1 \quad\Rightarrow\quad f=1$ [not acceptable]

$I=2 \quad\Rightarrow\quad f=\frac{11}{30}$ [not acceptable; we need $f \geq \frac{1}{2}$]

$I=3 \quad\Rightarrow\quad f=\frac{1}{7}$ [not acceptable]

$I=4 \quad\Rightarrow\quad f=\frac{1}{36}$ [not acceptable]

$I=5 \quad\Rightarrow\quad f<0$ [not acceptable]

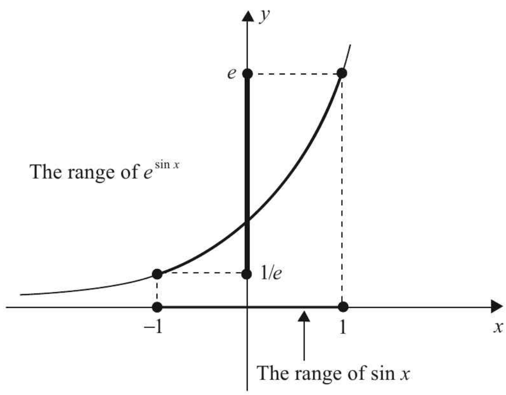
Hence the valid solutions are $x=I+f=\frac{29}{12},\frac{19}{6},\frac{97}{24}$. There are a total of 3 solutions. The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

Example 11

Find the domain and range of $f(x) = e^{\sin x}$.

Solution: As x varies over $\mathbb{R}$, $\sin x$ varies in [–1, 1], so $e^{\sin x}$ will vary from a minimum value of $e^{-1} = \frac{1}{e}$ to a maximum value of $e^1 = e$. Let us depict this graphically.



Therefore,

$$D = \mathbb{R}$$

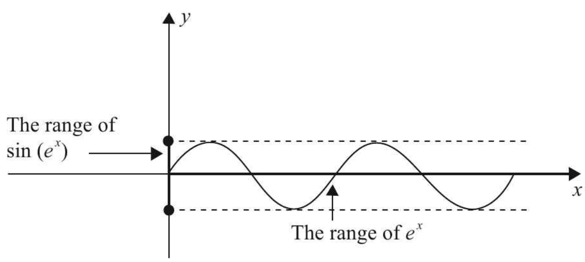
$$R = \left[\frac{1}{e}, e\right]$$

■

Example 12

Find the domain and range of $f(x) = \sin(e^x)$.

Solution: As x varies over $\mathbb{R}$, e^x varies in $(0, \infty)$. The range of $\sin(e^x)$ will be the range of the sin function: [–1, 1]. Convince yourself about this:



Therefore,

$$D = \mathbb{R}$$

$$R = [-1, 1]$$

■

Example 13

Find the domain and range of $f(x) = \ln(\sin(e^x))$.

Solution: In this case, we'll have a constraint on the domain:

$$f(x) = \ln(\sin(e^x))$$

$\Rightarrow$ $\sin(e^x)$ must be positive for the log function to be defined.

$$\Rightarrow \quad \sin(e^x) > 0$$

What values can e^x take then for $\sin(e^x)$ to be positive? Well, e^x can lie in the set of intervals $(0, \pi), (2\pi, 3\pi), (4\pi, 5\pi), \ldots$, and generally in $(2n\pi, (2n+1)\pi)$, $n \ge 0$. Look at the graph of sin x and convince yourself that sin x is positive in these intervals. Intervals on the negative side have not been taken since e^x cannot be negative. So,

If $e^x \in (0, \pi) \quad \Rightarrow \quad x \in (-\infty, \ln \pi)$

If $e^x \in (2\pi, 3\pi) \quad \Rightarrow \quad x \in (\ln 2\pi, \ln 3\pi)$

... and so on

Generalizing, the set of allowed values for x, *i.e.*, the domain, is all intervals of the form

$$(\ln 2n\pi, \ln(2n+1)\pi), \quad n \ge 0, \quad n \in \mathbb{Z}$$

What is the range? For the given domain, $\sin(e^x)$ will take positive values, but $\sin(e^x)$ cannot rise above 1. Thus,

$$\sin(e^x) \in (0, 1]$$

$$\Rightarrow \quad \ln(\sin(e^x)) \in (-\infty, 0]$$

The range is $(-\infty, 0]$. ■

Example 14

Find the domain and range of $f(x) = \ln(\sin(\ln x))$.

■

Solution: We are only providing the answers. The justification is left to the reader as an exercise:

$D =$ All intervals of the form $(e^{2n\pi}, e^{(2n+1)\pi})$, $n \in \mathbb{Z}$

$R = (-\infty, 0]$

Example 15

Find the range of $y = \sqrt{2-x} + \sqrt{1+x}$

Solution: We have $y = \sqrt{2-x} + \sqrt{1+x} = \sqrt{2-x} + \sqrt{x-(-1)}$.

(i) The minimum for y comes at $x = -1, 2$ (why?): $y = \sqrt{3}$
(ii) The maximum comes at $x = \frac{-1+2}{2}$ (why?): $y = \sqrt{6}$

The range is $R = [\sqrt{3}, \sqrt{6}]$. ■

Example 16

Find the range of $f(x) = \frac{x^2+x+1}{x^2+x+2}$.

Solution: As you may already know, to evaluate the range of such (rational) expressions, we can put the expression (in x) equal to some variable y, write x as a function of y ($x = g(y)$) and find the domain of g; this domain consists of values that y can take, or our required range:

$$\Rightarrow \quad y = \frac{x^2+x+1}{x^2+x+2} \quad \Rightarrow \quad (1-y)x^2 + (1-y)x + (1-2y) = 0$$

Note that y can never equal 1 (why?). Solving for x in terms of y, we get

$$x = \frac{-(1-y) \pm \sqrt{(1-y)^2 - 4(1-y)(1-2y)}}{2(1-y)}$$

For x to be real,

$$(1-y)^2 - 4(1-y)(1-2y) \geq 0$$

$$\Rightarrow \quad y^2 - 2y + 1 - 4(2y^2 - 3y + 1) \geq 0 \quad \Rightarrow \quad 7y^2 - 10y + 3 \leq 0$$

$$\Rightarrow \quad (7y-3)(y-1) \leq 0 \quad \Rightarrow \quad \frac{3}{7} \leq y \leq 1$$

Because $y \neq 1$, we have $R = [\frac{3}{7}, 1)$. Observe that the range could, in this particular example, have been evaluated in a simpler manner as follows (by rearrangement of $f(x)$):

$$f(x) = 1 - \frac{1}{x^2+x+2} = 1 - \frac{1}{(x+\frac{1}{2})^2 + \frac{7}{4}}$$

Now $$0 < \frac{1}{(x+\frac{1}{2})^2 + 7/4} \leq \frac{4}{7} \quad \Rightarrow \frac{3}{7} \leq f(x) < 1 \quad \Rightarrow R = \left[\frac{3}{7}, 1\right)$$ ■

Example 17

Find the range for $f(x) = \dfrac{1}{2 + \sin 3x + \cos 3x}$.

Solution: We rearrange the denominator as follows:

$$f(x) = \frac{1}{2 + \sin 3x + \cos 3x} = \frac{1}{2 + \sqrt{2}\sin(3x + \pi/4)}$$

The denominator can vary from $2 - \sqrt{2}$ to $2 + \sqrt{2}$ because $-1 \leq \sin\theta \leq 1$. The denominator is never 0, and hence $D = \mathbb{R}$. Therefore,

$$\frac{1}{2+\sqrt{2}} \leq f(x) \leq \frac{1}{2-\sqrt{2}}.$$ ■

Example 18

Find the range of $f(x) = [x^2] - [x]^2$.

Solution: Let x be expressed as $I + f$, where I is the integral part and f the fractional part of x.

$$f(x) = [(I+f)^2] - I^2 = [I^2 + f^2 + 2If] - I^2 = [f^2 + 2If]$$

A little thinking will show that the right side can take on any integer value, whether positive, zero or negative. Try out some examples:

For $x = 0.5, f(x) = 0$; For $x = -0.5, f(x) = -1$
For $x = 100.9, f(x) = 180$; For $x = -100.9, f(x) = -21$

Also assume any value for $f(x)$ and see whether you can find a corresponding value for x. In addition to that, note that the difference between x^2 and $[x]^2$ can be increased arbitrarily in magnitude due to the term $2If$ (which contains I). Try visualising this in the form of a graph. Hence, $f(x)$ can take on any integer value. That is,

$$R = \mathbb{Z}$$ ■

Example 19

Find the range for $f(x) = \sqrt{a-x} + \sqrt{x-b}, \quad a > b > 0.$

Solution: For the domain, we require $a - x \geq 0$ and $x - b \geq 0$:

$$\Rightarrow \quad D = [b, a]$$

For the range, let $y = f(x) = \sqrt{a-x} + \sqrt{x-b}$. We see that $y > 0$. Observe the expression for $f(x)$ carefully. The expression is somewhat symmetric with respect to a and b. This indicates that $f(x)$ should attain an extremum at $x = \frac{a+b}{2}$ (we can of course, prove this).

At $x = a, f(x) = \sqrt{a-b}$

At $x = b, f(x) = \sqrt{a-b}$

At $x = \frac{a+b}{2}, f(x) = \sqrt{2(a-b)}$

Hence, the observation of some symmetry in the expression directly allows us to write the range as

$$R = [\sqrt{a-b}, \sqrt{2(a-b)}]$$

Let us recalculate the answer analytically:

$$y = \sqrt{a-x} + \sqrt{x-b}$$

Squaring and rearranging gives

$$y^2 = (a-b) + \underbrace{2\sqrt{(a-x)(x-b)}}_{\text{always non-negative}} \geq a-b$$

$$\Rightarrow y \geq \sqrt{a-b} \tag{1}$$

Now,

$$y^2 - (a-b) = 2\sqrt{(a-x)(x-b)}$$

Squaring and arranging in the form of a quadratic in x gives

$$4x^2 - 4(a+b)x + y^4 - 2(a-b)y^2 + (a+b)^2 = 0$$

Since x is real, the discriminant for this equation should satisfy $D \geq 0$ (this gives a constraint on y, or the range):

$$\Rightarrow \quad (a+b)^2 \geq y^4 - 2(a-b)y^2 + (a+b)^2$$

$$\Rightarrow \quad y^2 \leq 2(a-b)$$

$$\Rightarrow \quad y \leq \sqrt{2(a-b)} \tag{2}$$

Combining (1) and (2) gives $R = [\sqrt{a-b}, \sqrt{2(a-b)}]$. ■

Example 20

Find the range for $f(x) = \dfrac{1}{\sin x + 2\cos x + 3}$

Solution: To evaluate the range, our approach should be to somehow determine the range of the variable term {sin x + 2 cos x} in the denominator; this can be determined by reducing this term to a simpler form. The general approach to reduce A sin x + B cos x (to a single variable term) is as follows:

$$A\sin x + B\cos x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin x + \frac{B}{\sqrt{A^2+B^2}}\cos x\right)$$

Now, do the following substitutions:

$$\frac{A}{\sqrt{A^2+B^2}} = \cos\phi \quad \text{and} \quad \frac{B}{\sqrt{A^2+B^2}} = \sin\phi \quad \text{where} \quad \tan\phi = \frac{B}{A}$$

Therefore,

$$A\sin x + B\cos x = \sqrt{A^2+B^2}\,\sin(x+\phi)$$

For this particular question, we have

$$\sin x + 2\cos x = \sqrt{5}\sin(x+\phi) \quad \text{where} \quad \tan\phi = 2$$

$$\Rightarrow \quad -\sqrt{5} \leq \sqrt{5}\sin(x+\phi) \leq \sqrt{5}$$

$$\Rightarrow \quad -\sqrt{5} \leq \sin x + 2\cos x \leq \sqrt{5}$$

$$\Rightarrow \quad 3-\sqrt{5} \leq \sin x + 2\cos x + 3 \leq 3+\sqrt{5}$$

$$\Rightarrow \quad \frac{1}{3+\sqrt{5}} \leq \frac{1}{\sin x + 2\cos x + 3} \leq \frac{1}{3-\sqrt{5}} \quad \Rightarrow \quad R = \left[\frac{1}{3+\sqrt{5}}, \frac{1}{3-\sqrt{5}}\right]$$ ■

Example 21

Find the range of $f(x) = \sqrt{3x^2 - 4x + 5}$.

Solution: The expression inside the square root function is

$$3x^2 - 4x + 5 = 3\left(x^2 - \frac{4}{3}x + \frac{5}{3}\right) = 3\left(x - \frac{2}{3}\right)^2 + \frac{11}{3} \geq \frac{11}{3}$$

Therefore,

$$\sqrt{3x^2-4x+5} \geq \sqrt{\frac{11}{3}} \quad \Rightarrow \quad R = \left[\sqrt{\frac{11}{3}}, \infty\right)$$ ■

Example 22

Find the range of $f(x) = \dfrac{1}{x^4 + 2x^2 + 2}$.

Solution: We have

$$f(x) = \frac{1}{x^4 + 2x^2 + 2} = \frac{1}{(x^2+1)^2 + 1}$$

Now, $x^2 + 1 \geq 1$:

$$\Rightarrow \quad (x^2+1)^2 + 1 \geq 2 \quad \Rightarrow \quad 0 < \frac{1}{(x^2+1)^2+1} \leq \frac{1}{2} \quad \Rightarrow \quad R = \left(0, \frac{1}{2}\right] \qquad \blacksquare$$

Example 23

Find the range of $f(x) = \frac{1}{1+3\{x\}^2}$.

Solution: We have

$$0 \leq \{x\} < 1 \quad \Rightarrow \quad 0 \leq 3\{x\}^2 < 3 \quad \Rightarrow \quad 1 \leq 1 + 3\{x\}^2 < 4$$

$$\Rightarrow \quad \frac{1}{4} < \frac{1}{1+3\{x\}^2} \leq 1 \quad \Rightarrow \quad R = \left(\frac{1}{4}, 1\right] \qquad \blacksquare$$

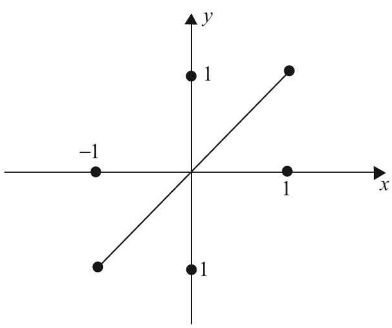
Example 24

Plot the graphs for the following:

(a) $y = \sin(\sin^{-1} x)$ (b) $y = \sin^{-1}(\sin x)$

Solution: By observation, you might be tempted to say that since y is a composition of two inverse functions, which should cancel out, the output should be $y = x$, which is a straight line. But we have to be more cautious:

(a) The inner function, $\sin^1 x$, is defined only for $x \in [-1, 1]$. For these values of x, $\sin(\sin^{-1} x)$ will give back x again. Hence, the graph is the identity function but only in the interval $[-1, 1]$.



(b) The inner function, $\sin x$, is defined for all $x \in \mathbb{R}$ and gives an output in $[-1, 1]$. The outer function, $\sin^{-1}(\ )$, will now give a value in $[\frac{-\pi}{2}, \frac{\pi}{2}]$. For example, if $x = \pi$, $\sin x = 0$, and $\sin^{-1}(\sin x) = 0$, and not π. Similarly, if $x = \frac{3\pi}{2}$, $\sin x = -1$ and $\sin^{-1}(\sin x) = \frac{-\pi}{2}$ and not $\frac{3\pi}{2}$, and so on. Summarizing these facts, we can give a piecewise definition of this function as follows:

$$f(x) = \sin^{-1}(\sin x) = x \quad \text{if } \frac{-\pi}{2} \le x < \frac{\pi}{2}$$

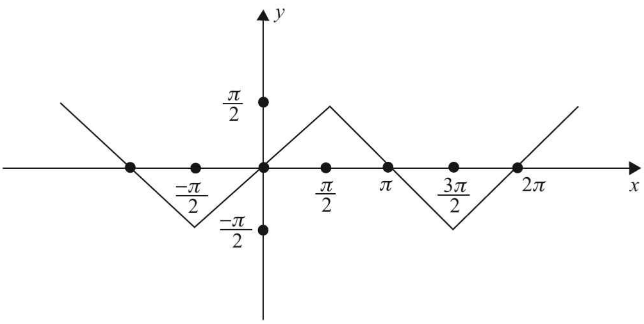
$$\pi - x \quad \text{if } \tfrac{-\pi}{2} + \pi \le x < \tfrac{\pi}{2} + \pi$$

$$+ (x - 2\pi) \quad \text{if } \frac{-\pi}{2} + 2\pi \le x < \frac{\pi}{2} + 2\pi$$

$$3\pi - x \quad \text{if } \frac{-\pi}{2} + 3\pi \le x \le \frac{\pi}{2} + 3\pi$$

and so on

The graph of this function will be as follows:



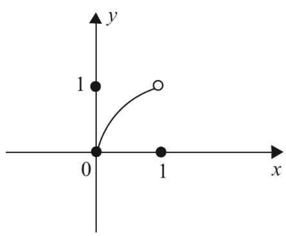
■

Example 25

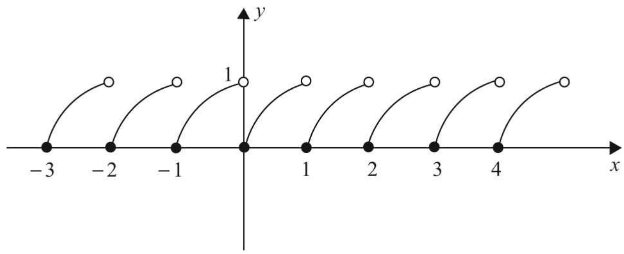
Draw the graphs for:

(a) $y = \sqrt{\{x\}}, |y| = \sqrt{\{x\}}$ **(b)** $y = [x] + \sqrt{\{x\}}$ **(c)** $y = -[x] + \sqrt{\{x\}}$

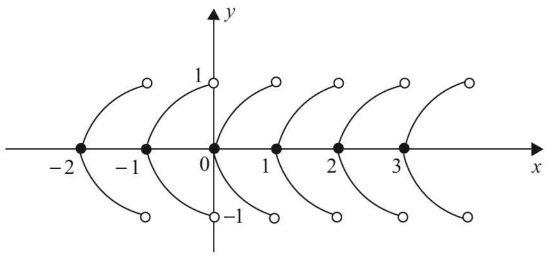
Solution: **(a)** Let us first draw the graph for $y = \sqrt{\{x\}}$. Now, $\{x\}$ is the same as x for $0 \le x < 1$. In this interval, $\sqrt{\{x\}}$ will be the same as $\sqrt{x}$:



We can easily see that this same curve will be repeated in every previous and subsequent unit interval, since $\{x\}$ is the same in all such intervals. Hence, we obtain the graph of $y=\sqrt{\{x\}}$:



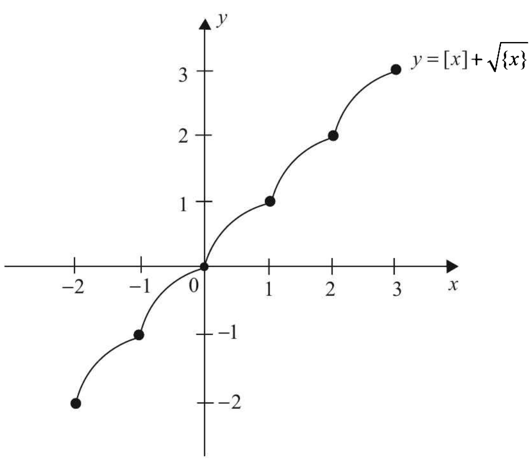
Now we can easily draw $|y|=\sqrt{\{x\}}$ by taking a reflection in the x-axis:



(b) In any interval $n \le x < n+1$ (where n is an integer), $[x]$ has the value n. Therefore, in any such interval, the graph of $y=[x]+\sqrt{\{x\}}$ will be the graph of $\sqrt{\{x\}}$ + (integer) n. This will be the graph segment of $\sqrt{\{x\}}$ shifted vertical upwards by n units. For example,

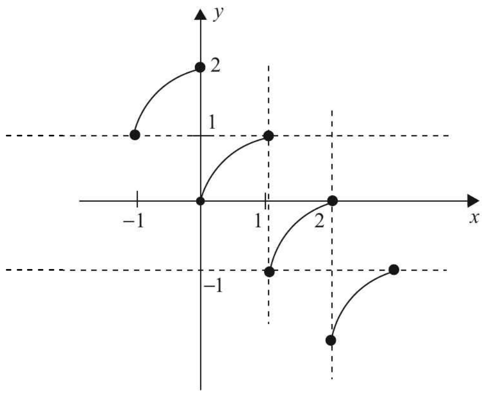
$$0 \le x < 1 \quad \Rightarrow \quad y=\sqrt{\{x\}}$$
$$1 \le x < 2 \quad \Rightarrow \quad y=1+\sqrt{\{x\}}$$
$$-1 \le x < 0 \quad \Rightarrow \quad y=-1+\sqrt{\{x\}}$$

The graph is drawn below:



Note that this graph has no holes (or breaks). We express this fact by saying that this function is continuous. The function $y=\sqrt{\{x\}}$, is, on the other hand, discontinuous.

(c) Using a similar approach as in part (b), we obtain the graph of $y=-[x]+\sqrt{\{x\}}$.



This function, as we can see, is discontinuous.

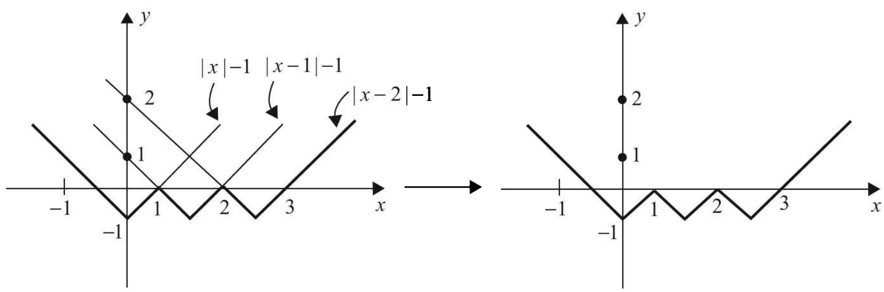
For further practice, draw the graphs of the following:

$$y=\sqrt{\{|x|\}},\ y=|[x]+\sqrt{\{x\}}|,\ |y|=|[x]+\sqrt{\{x\}}|,\ |y|=[|x|]+\sqrt{|\{x\}|},\ y=[x]-\{x\},\ y=x+\{x\}$$ ■

Example 26

Plot the curve $y=\min\{|x|-1,\ |x-1|-1,\ |x-2|-1\}$.

Solution: What does such a statement mean? It means that for each value of x, we evaluate all the three quantities on the RHS, select the minimum of the three, and plot that value of y. What we can do is plot the separate graphs for these three quantities on the same axes and select those portions that lie *lowermost*, out of all the three.



The darker line segments on the left hand side diagram show the minimum value out of all the three, considered at each point x. These therefore give us the required graph, as shown in the second figure. ■

Functions

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. What is the domain of $f(x)=\sqrt{\log_{1/2}(\log_2[x^2+4x+5])}$ ([] denotes the GIF)?

(A) $(-2,0)$ (B) $(-3,-1)$ (C) $(-3,0)$ (D) $(-4,1)$ (E) None of these

P2. What is the range of $f(x)=\log_2\left(\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)$?

(A) $(0,1]$ (B) $[1,2]$ (C) $[1,3]$ (D) $[2,3]$ (E) None of these

P3. What is the minimum value of $y=|x-1|+|x-2|+|x-3|+|x-4|$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) The minimum value is not an integer

P4. Let $f(x)=\frac{1}{1-x}$. How many discontinuities exist in the function $\underbrace{f\circ f\circ f\circ\ldots\circ f(x)}_{3n\text{ times}}$?

(A) 2 (B) 3 (C) 4 (D) More than 4 (E) None of these

P5. Which of the following functions are even?

(A) $2\tan^{-1}(x+\sqrt{1+x^2})-\dfrac{\pi}{2}$ (C) $x\log|\sec x+\tan x|+\sin x\log\left|\dfrac{1-x}{1+x}\right|$

(B) $\sin^{-1}\left(\dfrac{\sqrt{1-x}}{2}\right)+\cos^{-1}\left(\dfrac{\sqrt{1+x}}{2}\right)-\dfrac{\pi}{2}$ (D) $x\cos^{-1}\left(\dfrac{1-x}{4}\right)-x\sin^{-1}\left(\dfrac{1+x}{4}\right)$

(E) $\cos^2(e^x)-\sin^2(e^{-x})$

P6. Suppose that $a\in(0,1)$ and $b>1$. The function $f(x)=a^x-b^x$ is

(A) increasing on $\mathbb{R}$ (C) increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$

(B) decreasing on $\mathbb{R}$ (D) decreasing on $(-0,\infty)$ and increasing on $(0,\infty)$

P7. Consider a function f which satisfies

$$f(x,y)=f(2x+2y,2y-2x)$$

Now, consider the function

$$g(x)=f(2^x,0)$$

(a) Show that $g(x)$ is periodic.
(b) The period of $g(x)$ is

(A) 4 (B) 8 (C) 12 (D) 16

SUBJECTIVE TYPE EXAMPLES

P8. Suppose that the domain of $f(x)$ is (0, 1). What is the domain of the following?

(a) $f(\tan x)$ (b) $f(2\sin x), x \in [0, \pi]$

P9. What is the domain of the function $f(x) = \sqrt{\sin(\sin(\sin x))}$?

P10. Let $f(x) = \ln(\frac{x^2+e}{x^2+1})$. Find the range of $g(x) = \sqrt{\sin(f(x))} + \sqrt{\cos(f(x))}$.

P11. Suppose that $f(x)$ satisfies the following:

$$10^x + 10^{f(x)} = 10$$

What is the range of $f(x)$?

P12. Let n be an odd positive integer and let $x_1, x_2, \ldots, x_n$ be distinct real numbers. Find all one-to-one functions:

$$f : \{x_1, x_2, \ldots, x_n\} \to \{x_1, x_2, \ldots, x_n\}$$

such that

$$|f(x_1) - x_1| = |f(x_2) - x_2| = \ldots = |f(x_n) - x_n|$$

P13. For real x, find the condition on a, b, c such that the function

$$f(x) = \frac{(x-a)(x-b)}{(x-c)} \qquad (\mathbb{R} - \{c\} \to \mathbb{R})$$

is onto.

P14. Let f be a real-valued function defined for all real numbers x such that $f(x) \in [0, \frac{1}{2}]$ and for some fixed $a > 0$,

$$f(x+a) = \frac{1}{2} - \sqrt{f(x) - (f(x))^2}$$

Show that $f(x)$ is periodic and hence find its period.

P15. Let f be a real-valued function on $\mathbb{R}$ such that

$$f(x+p) = 1 + (2 - 3f(x) + 3f^2() - f^3(x))^{1/3}$$

holds true $\forall$ $x \in \mathbb{R}$ and some positive real constant p. Show that $f(x)$ is periodic and hence find its period.

P16. Let us define $f(x)$ and $g(x)$ as follows:

$$f(x) = \begin{Bmatrix} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{Bmatrix}, \; g(x) = (x^2 + 2) f(x)$$

Define the inverse of $g(x)$.

P17. How many roots will the following equation have?

$$2^{\sin x} = 3 - 2x + x^2$$

Functions

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. We must have $\log_{\frac{1}{2}}(\log_2[x^2+4x+5]) \geq 0$, which implies that

$$0 < \log_2[x^2+4x+5] \leq 1$$
$$\Rightarrow \quad 1 < [x^2+4x+5] \leq 2$$
$$\Rightarrow \quad 2 \leq x^2+4x+5 < 3$$
$$\Rightarrow \quad 2 \leq (x+2)^2 < 3$$
$$\Rightarrow \quad x \in (-2-\sqrt{3}, -2-\sqrt{2}] \cup [-2+\sqrt{2}, -2+\sqrt{3})$$

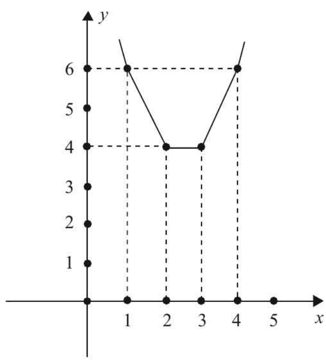
None of the first four options is correct, and so we choose (E).

S2. Since $\sin x - \cos x$ can be written as $\sqrt{2}\,\sin(x-\frac{\pi}{4})$, its range is $[-\sqrt{2}, +\sqrt{2}]$. Therefore, the range of the entire expression is [1, 2]. The correct option is (B).

S3. We define the function piecewise as follows:

$$\begin{array}{ll} x < 1 & y = 1-x+2-x+3-x+4-x = 10-4x \\ 1 < x < 2 & y = x-1+2-x+3-x+4-x = 8-2x \\ 2 < x < 3 & y = x-1+x-2+3-x+4-x = 4 \\ 3 < x < 4 & y = x-1+x-2+x-3+4-x = 2x-2 \\ x > 4 & y = x-1+x-2+x-3+x-4 = 4x-10 \end{array}$$

Now, we draw the graph:



The minimum is $y = 4$ for $2 \leq x \leq 3$. The range is $[4, \infty)$. The correct option is (C).

S4. We note that $f(x)=\frac{1}{1-x}$ has a discontinuity at $x=1$. Now, for $x\neq 1$, we have

$$f(f(x))=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x}\ S$$

Thus, $f(f(x))$ has a discontinuity at $x = 0$, while it is not defined for $x = 1$, *i.e.*, it has two discontinuities. Further, for $x\neq 0,1$, we have

$$f(f(f(x)))=\frac{1}{1-\frac{x-1}{x}}=x$$

Thus,

$$\underbrace{f\circ f\circ f\circ\cdots\circ f(x)}_{3n\text{ times}}=x,\quad x\neq 0,1$$

We see that the function has two discontinuities. The correct option is (A).

S5. (A) $f(-x)=2\tan^{-1}(-x+\sqrt{1+x^2})-\dfrac{\pi}{2}=2\left(\dfrac{\pi}{2}-\tan^{-1}(x+\sqrt{1+x^2}\right)-\dfrac{\pi}{2}$

$$=\frac{\pi}{2}-2\tan^{-1}\left(x+\sqrt{1+x^2}\right)=-f(x)$$

$f(x)$ is therefore odd.

(B) $f(-x)=\sin^{-1}\left(\dfrac{\sqrt{1+x}}{2}\right)+\cos^{-1}\left(\dfrac{\sqrt{1-x}}{2}\right)-\dfrac{\pi}{2}=\dfrac{\pi}{2}-\cos^{-1}\left(\dfrac{\sqrt{1+x}}{2}\right)+\dfrac{\pi}{2}-\sin^{-1}\left(\dfrac{\sqrt{1-x}}{2}\right)-\dfrac{\pi}{2}$

$$=\frac{\pi}{2}-\sin^{-1}\left(\frac{\sqrt{1-x}}{2}\right)-\cos^{-1}\left(\frac{\sqrt{1+x}}{2}\right)=-f(x)$$

$f(x)$ is odd.

(C) $f(-x)=-x\log|\sec x-\tan x|-\sin x\log\left|\dfrac{1+x}{1-x}\right|=x\log|\sec x+\tan x|+\sin x\log\left|\dfrac{1-x}{1+x}\right|=f(x)$

$f(x)$ is even in this case.

(D) $f(-x)=-x\cos^{-1}\left(\dfrac{1+x}{4}\right)+x\sin^{-1}\left(\dfrac{1-x}{4}\right)=-x\left(\dfrac{\pi}{2}-\sin^{-1}\left(\dfrac{1+x}{4}\right)\right)+x\left(\dfrac{\pi}{2}-\cos^{-1}\left(\dfrac{1-x}{4}\right)\right)$

$$=-x\cos^{-1}\left(\frac{1-x}{4}\right)+x\sin^{-1}\left(\frac{1+x}{4}\right)=-f(x)$$

$f(x)$ is odd.

(E) $f(x)=\cos^2(e^{-x})-\sin^2(e^x)=(1-\sin^2(e^{-x}))-(1-\cos^2(e^x))=\cos^2(e^x)-\sin^2(e^{-x})=f(x)$

$f(x)$ is even.

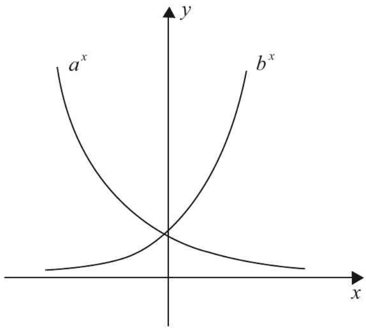
The correct options are (C) and (E).

S6. We note that

$$x \to -\infty \quad \Rightarrow \quad \begin{cases} a^x \to \infty \\ b^x \to 0 \end{cases} \Rightarrow \quad a^x - b^x \to \infty$$

$$x \to +\infty \quad \Rightarrow \quad \begin{cases} a^x \to 0 \\ b^x \to \infty \end{cases} \Rightarrow \quad a^x - b^x \to -\infty$$

Also, $f(0) = 0$. Thus, the function is decreasing on $\mathbb{R}$. This can be verified from the graphs of the two functions a^x and b^x.



S7. Consider the first relation given in the problem carefully. We will apply this relation to *itself*, so that

$$\begin{aligned} f(x, y) &= f(2x+2y, 2y-2x) \\ &= f(a, b) \quad (\text{where } a = 2x+2y, b = 2y-2x) \\ &= f(2a+2b, 2b-2a) \\ &= f(8y, -8x) \end{aligned}$$

Again,

$$\begin{aligned} f(x, y) &= f(8y, -8x) \\ &= f(c, d) \quad (\text{where } c = 8y, d = -8x) \\ &= f(8d, -8c) \\ &= f(-64x, -64y) \\ &= f(4096x, 4096y) \\ &= f(2^{12}x, 2^{12}y) \end{aligned}$$

Now, if we put $y = 0$ and substitute 2^x in place of x, we have

$$\begin{aligned} & f(2^x, 0) = f(2^{12+x}, 0) \\ \Rightarrow \quad & g(x) = g(x+12) \end{aligned}$$

Thus, $g(x)$ is periodic with period 12. The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

S8. (a) We have

$$\tan x \in (0, 1)$$
$$\Rightarrow \quad x \in \left(n\pi, n\pi + \frac{\pi}{4}\right), n \in \mathbb{Z}$$

(b) We have

$$2\sin x \in (0, 1)$$
$$\Rightarrow \quad \sin x \in \left(0, \frac{1}{2}\right)$$

Since x can only take values in $[0, \pi]$, we have

$$x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$$

S9. We require $\sin(\sin(\sin x)) \geq 0$, which means that

$$\sin(\sin x) \in [2n\pi, (2n+1)\pi], n \in \mathbb{Z} \qquad (1)$$

However, the range of the LHS in (1) is [–1, 1], so the only set of values it can take so that it satisfies (1) is [0, 1], which as you may observe, is a subset of the RHS in (1). Thus, we have

$$\sin(\sin x) \in [0, 1]$$
$$\Rightarrow \quad \sin(\sin x) \geq 0$$

This will happen if

$$\sin x \in [2n\pi, (2n+1)\pi], n \in \mathbb{Z} \qquad (2)$$

(2) is similar to (1), and we conclude that

$$\sin x \in [0, 1]$$

Finally, this implies that

$$\sin x \geq 0$$
$$\Rightarrow \quad x \in [2n\pi, (2n+1)\pi], n \in \mathbb{Z}$$

This is the required domain.

S10. Since $\frac{x^2+e}{x^2+1} \in (1, e]$, $f(x) = \ln\left(\frac{x^2+e}{x^2+1}\right) \in (0, 1]$. Thus, we have to find the range of the function

$$g(\theta) = \sqrt{\sin\theta} + \sqrt{\cos\theta} \text{ where } \theta \in (0, 1].$$

Now,

$$g'(\theta) = \frac{\cos\theta}{2\sqrt{\sin\theta}} - \frac{\sin\theta}{2\sqrt{\cos\theta}} = 0 \text{ when } \theta = \frac{\pi}{4},$$

which means that $g(\theta)$ has an extremum point at $\theta = \frac{\pi}{4}$. We observe that $g(0) = 1$, $g(1) = \sqrt{\sin 1} + \sqrt{\cos 1}$, while $g(\frac{\pi}{4}) = 2^{\frac{3}{4}}$. Also, we note that $g(1) > g(0) = 1$ (since $g(\frac{\pi}{2}) = g(0) = 1$). Thus, $g(x)$ has a range of $(1, 2^{\frac{3}{4}}]$.

S11. We have

$$f(x) = \log_{10}(10 - 10^x)$$

To calculate the domain, we have

$$10 - 10^x > 0$$

$$\Rightarrow \quad x < 1$$

For this domain, we note that the range of the expression $10 - 10^x$ is (0, 10) (why?), and so the range of $f(x)$ is $(-\infty, 1)$.

S12. Let $|f(x_i) - x_i| = \lambda$ for every i:

$$\Rightarrow \quad f(x_i) - x_i = \lambda \in_i \quad \text{where} \quad \in_i = +1 \text{ or } -1$$

$$\Rightarrow \quad f(x_i) = x_i + \lambda \in_i$$

Now, if we sum over all i, we have

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} x_i + \lambda_i \sum_{i=1}^{n} \in_i$$

But the sum of $f(x_i)$ and the sum of x_i over all i *must* be the same, since f maps the set $\{x_i\}$ to itself. Thus,

$$\lambda \sum_{i=1}^{n} \in_i = 0$$

Since n is odd and $\in_i = +1$ or -1, the only way this is possible is that $\lambda = 0$:

$$\Rightarrow \quad |f(x_i) - x_i| = 0 \quad \forall\, i$$

$$\Rightarrow \quad f(x_i) = x_i \quad \forall\, i$$

This is the required one-to-one function.

S 13. Let $y = f(x) = \frac{(x-a)(x-b)}{(x-c)}$. Since the co-domain for this function has been specified has $\mathbb{R}$, we require the range also to be $\mathbb{R}$ if $f(x)$ is to be onto. Hence we require y to take on all real values. Now,

$$y(x - c) = (x - a)(x - b)$$

$$\Rightarrow \quad x^2 - (a + b + y)x + (ab + cy) = 0$$

For x to be real, the discriminant should be non-negative.

$$\Rightarrow \quad (a + b + y)^2 - 4(ab + cy) \geq 0 \tag{1}$$

For the function f to be onto, we require that each y have a real pre-image x. This is only possible if that y satisfies the constraint (1). Hence, this constraint, or this inequality, should be true for all real y. Rearranging as a quadratic in y, we have

$$y^2 + 2(a + b - 2c)y + (a - b)^2 \geq 0 \tag{2}$$

For the LHS of (2) to be always non-negative, we require its graph to lie above the x-axis (or touching it, at the most). If it goes below the axis, the LHS will become negative. Hence we require the discriminant for (2) to be non-positive, *i.e.*, $D \leq 0$:

$$\Rightarrow \quad (2(a + b - 2c))^2 - 4(a - b)^2 \leq 0$$

$$\Rightarrow \quad (a+b)^2 + 4c^2 - 4(a+b)c - (a-b)^2 \le 0$$

$$\Rightarrow \quad ab + c^2 - (a+b)c \le 0$$

We can treat the LHS above as a quadratic in c. LHS ≤ 0 implies that c must lie within the roots of this quadratic expression, which are $\frac{(a+b)\pm\sqrt{(a+b)^2-4ab}}{2} = \frac{(a+b)\pm(a-b)}{2} = a, b$. There is no loss of generality in assuming that $a > b$ since the expression for $f(x)$ is symmetric about a and b. Hence, we get the constraint on a, b, c as

$$a < c < b$$

A question for the reader: Why cannot c be equal to a or b?

S 14. We have

$$\left(f(x+a) - \frac{1}{2}\right)^2 = \left(f(x+a)\right)^2 - f(x+a) + \frac{1}{4} = f(x) - \left(f(x)\right)^2$$

Using $x \to x + a$ in the given relation, we have

$$f(x+2a) = \frac{1}{2} - \sqrt{f(x+a) - f((x+a))^2} = \frac{1}{2} - \sqrt{\frac{1}{4} - f(x) + \left(f(x)\right)^2} = \frac{1}{2} - |\frac{1}{2} - f(x)|$$

Since $f(x) \in [0, \frac{1}{2}]$, this implies that $f(x+2a) = f(x)$. Thus, $f(x)$ is periodic with period $2a$.

S15. The trick is to express the bracket inside the cube root in terms of $(1 - f(x))^3$.

$$f(x+p) = 1 + (1 + (1 - f(x))^3)^{\frac{1}{3}} \quad \Rightarrow \quad (f(x+p) - 1)^3 = 1 + (1 - f(x))^3$$

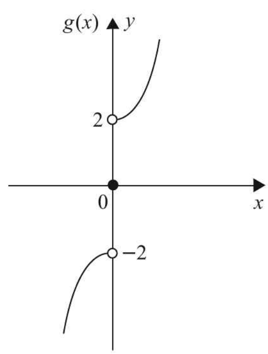
$$\Rightarrow \quad (f(x+p) - 1)^3 + (f(x) - 1)^3 = 1 \quad \Rightarrow \quad (f(x+2p) - 1)^3 + (f(x+p) - 1)^3 = 1$$

This implies that $f(x+2p) = f(x)$, *i.e.*, $f(x)$ is periodic with period $2p$.

S16. We have

$$g(x) = \begin{Bmatrix} x^2 + 2, & x > 0 \\ 0, & x = 0 \\ -(x^2 + 2) & x < 0 \end{Bmatrix}$$

The approximate plot of $g(x)$ is as follows:



We observe that $g(x)$ is onto over the interval $(-\infty, -2) \cup \{0\} \cup (2, \infty)$. The inverse function will also be defined piecewise, as follows:

$$g^{-1}(x) = \begin{Bmatrix} \sqrt{x-2}, & x > 2 \\ 0, & x = 0 \\ \sqrt{-x-2}, & x < -2 \end{Bmatrix}$$

S17. We note that $\text{LHS}_{\text{max}} = 2$, when $x = 2n\pi + \frac{\pi}{2}$, while

$$\text{RHS} = (x-1)^2 + 2$$

$$\Rightarrow \quad \text{RHS}_{\text{min}} = 2, \text{ when } x = 1$$

The only way the two sides can be equal is if the LHS attains its maximum value and the RHS attains its minimum value for the same value of x, but as is evident, this is not possible. Thus, no roots exist for this equation.

Limits, Continuity, Differentiability and Differentiation

PART-A: Summary of Important Concepts

1. Fundamentals of Limits

1.1 The concept of Limits

A limit describes the behaviour of some quantity that depends on an independent variable, as that independent variable *approaches* or *comes close to* a particular value. For example, how does $\frac{1}{x}$ behave when x becomes larger and larger? $\frac{1}{x}$ becomes smaller and smaller and *tends to* 0. We write this as

$$\lim_{x\to\infty}\frac{1}{x}=0$$

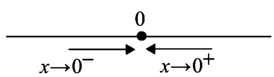
How does $\frac{1}{x}$ behave when x becomes smaller and smaller and approaches 0? $\frac{1}{x}$ obviously becomes larger and larger and *tends to* infinity. We write this as:

$$\lim_{x\to 0}\frac{1}{x}=\infty$$

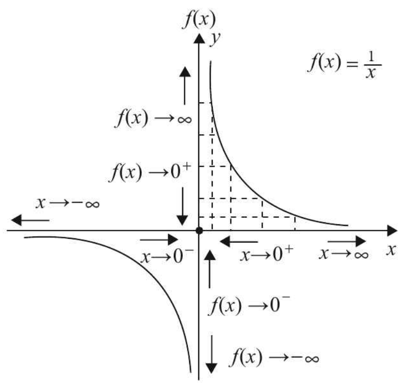
The picture is not yet complete. In the example above, x can *approach* 0 in two ways, either from the left hand side or from the right hand side:

$x \to 0^-$: approach is from left side of 0

$x \to 0^+$: approach is from right side of 0



How do we differentiate between the two possible approaches? Consider the graph of $f(x)=\frac{1}{x}$ carefully:



As we can see in the graph above, as x increase in value or as $x \to \infty$, $f(x)$ decreases in value and approaches 0 (but it remains positive, or in other words, it approaches 0 from the positive side). This can be written as

$$\lim_{x\to\infty} f(x) = 0^+$$

Similarly,

$$\lim_{x\to 0^+} f(x) = +\infty$$

What if x approaches 0, but from the left hand side $(x \to 0^-)$? From the graph, we see that as $x \to 0^-$, $\frac{1}{x}$ increases in magnitude but it also has a negative sign, that is $\frac{1}{x} \to -\infty$. What if $x \to -\infty$? $\frac{1}{x}$ decreases in magnitude (approaches 0) but it still remains negative, that is, $\frac{1}{x}$ approaches 0 from the negative side or $\frac{1}{x} \to 0^-$. These concepts and results are summarized below:

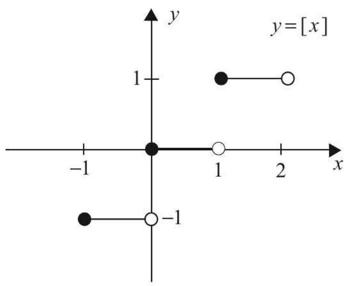
(i) $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0^+$ $\quad \dfrac{1}{x}$ approaches 0 from the positive side

(ii) $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0^-$ $\quad \dfrac{1}{x}$ approaches 0 from the negative side

(iii) $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$ $\quad \dfrac{1}{x}$ remains negative and increases in magnitude

(iv) $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$ $\quad \dfrac{1}{x}$ remains positive and increases in magnitude

Lets consider another example now. We analyse the behaviour of $f(x) = [x]$, as x approaches 0.



What happens when x approaches 0 from the right hand side? We see that $[x]$ remains 0. What happens when x approaches 0 from the left hand side? $[x]$ has a value –1. Note that we are not talking about what value $[x]$ takes *at* $x = 0$. We are concerned with the behaviour of $[x]$ in the *neighbourhood of* $x = 0$, that is, to the immediate left and right of $x = 0$. Hence, we have:

$$\lim_{x\to 0^+}[x] = 0 \qquad \lim_{x\to 0^-}[x] = -1 \qquad \left.\begin{aligned} &\lim_{x\to I^+}[x] = I \\ &\lim_{x\to I^-}[x] = I-1 \end{aligned}\right\} I \text{ is any integer}$$

What about $f(x) = \{x\}$?

$$\lim_{x\to 0^+}\{x\} = 0 \qquad \lim_{x\to 0^-}\{x\} = 1 \qquad \left.\begin{array}{l}\lim\limits_{x\to I^+}\{x\} = 0\\ \lim\limits_{x\to I^-}\{x\} = 1\end{array}\right\} I \text{ is any integer}$$

We have now seen three different quantities regarding $f(x)$:

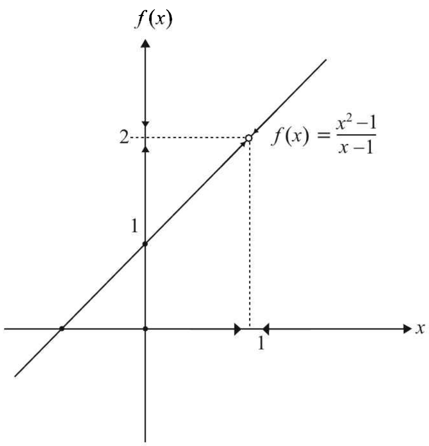
(i)	The left hand limit (LHL) at $x = a$: $\lim\limits_{x\to a^-} f(x)$	Describes the behaviour of $f(x)$ to the immediate left of $x = a$
(ii)	The right hand limit (RHL) at $x = a$: $\lim\limits_{x\to a^+} f(x)$	Describes the behaviour of $f(x)$ to the immediate right of $x = a$
(iii)	The value of $f(x)$ at $x = a$: $f(a)$	Gives the precise value that $f(x)$ takes $x = a$

Its possible that the value of the function at $x = a$ is undefined, and yet the LHL or RHL (or both) exist.

For example, consider $f(x) = \frac{x^2-1}{x-1}$. We observe that $f(x)$ is not defined at $x = 1$. Everywhere else, $f(x)$ can be written in a simple form as

$$f(x) = \frac{(x-1)(x+1)}{(x-1)} = x+1$$

which is a straight line:



This line has a hole at $x = 1$ because $f(x)$ is undefined there. In the neighbourhood of

$$x = 1 (\text{as } x \to 1^+ \text{ or } x \to 1^-),$$

$f(x)$ approaches the value 2 (though it never becomes 2, because to become 2, x has to have the value 1, which is not possible). We see that

$$\text{LHL} = \lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x) = \text{RHL}$$

To emphasize once again, in evaluating a limit at $x = a$, we are not concerned with what value $f(x)$ assumes *precisely at* $x = a$; we are concerned with only how $f(x)$ behaves as x approaches or nearly becomes a, whether from the left hand or right hand side, giving rise to the LHL and the RHL respectively.

And finally, the limit of $f(x)$ at $x = a$ is *said to exist* if the function approaches the same value from both sides:

$$\text{LHL} = \text{RHL} \quad \text{at } x = a$$

$$\text{implies} \quad \lim_{x\to a} f(x) \text{exists} \quad \text{and} \quad \lim_{x\to a} f(x) = \text{LHL} = \text{RHL}$$

1.2 The concept of infinity in limits

We have used the concept of ∞ (infinity) in the discussions above. Lets discuss this concept in somewhat more detail:

- Infinity does not stand for any particular real number. In fact, it cannot be defined precisely. This is a deep concept. For any number you can think of, no matter how large, infinity is still larger. When we say that $x \to \infty$, we mean that x increase in an *unbounded* fashion, that is, becomes indefinitely large.
- We cannot apply the normal rules of arithmetic to infinity. For example, saying that

$$\infty - \infty = 0 \quad \text{or} \quad \frac{\infty}{\infty} = 1$$

  is absurd because such quantities are not defined.
- It should be clear to you that an expression like $x \to \infty$ defines a tendency (of unbounded increase). Consider a fraction $f = \frac{\text{Num}}{\text{Den}}$. As $x \to a$, if (Num) $\to$ (a finite number) and (Den) $\to \infty$, then f tends to become infinitesmally small or $f \to 0$. Observe other such similar cases for $f = \frac{\text{Num}}{\text{Den}}$:

(i) $$\left.\begin{array}{l} \lim\limits_{x\to a}(\text{Num}) = \text{finite} \\ \lim\limits_{x\to a}(\text{Den}) = \text{infinity} \end{array}\right\} \Rightarrow \lim_{x\to a} f = 0$$

(ii) $$\left.\begin{array}{l} \lim\limits_{x\to a}(\text{Num}) = \text{infinity} \\ \lim\limits_{x\to a}(\text{Den}) = \text{finite} \end{array}\right\} \Rightarrow \lim_{x\to a} f = \infty$$

(iii) $$\left.\begin{array}{l} \lim\limits_{x\to a}(\text{Num}) = 0 \\ \lim\limits_{x\to a}(\text{Den}) = \text{non-zero} \end{array}\right\} \Rightarrow \lim_{x\to a} f = 0$$

(iv) $$\left.\begin{array}{l} \lim\limits_{x\to a}(\text{Num}) = p \\ \lim\limits_{x\to a}(\text{Den}) = q \neq 0 \end{array}\right\} \Rightarrow \lim_{x\to a} f = \frac{p}{q}$$

$$\text{Suppose that } \lim_{x\to a}(\text{Num}) = 0 \text{ and } \lim_{x\to a}(\text{Den}) = 0$$

$$\text{or} \quad \lim_{x\to a}(\text{Num}) = \infty \text{ and } \lim_{x\to a}(\text{Den}) = \infty,$$

Then f is of the form:

$$f = \frac{\to 0}{\to 0} \text{ or } \frac{\to \infty}{\to \infty}$$

The limits of such forms may or may not exist. For example, if (Num) = $x^2 - 1$ and (Den) = $x - 1$, then $\lim\limits_{x\to 1}(\text{Num}) = 0$ and $\lim\limits_{x\to 1}(\text{Den}) = 0$, so that f is of the form $\frac{0}{0}$. But as we have seen earlier, the limit of f exists, as $x \to 1$:

$$\lim_{x\to 1} f = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Limits of the form $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \infty \times 0, 1^{\infty}, 0^{\circ}$ are called *indeterminate* forms. The limits of such forms may exist but they cannot be determined by simple observation (hence the name indeterminate). Such forms need to be reduced into *determinate* forms for which the limit can be determined. In the above case, for example, an indeterminate form $(\lim\limits_{x\to 1} \frac{x^2-1}{x-1})$ was reduced into a determinate form $(\lim\limits_{x\to 1}(x+1))$, whose limit can be determined by the substitution $x = 1$.

2. Frequently Used Results in Limits

2.1 Some standard limits

We are now going to summarize some widely used indeterminate limits:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1;\quad \lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

(b) $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^x = e;\quad \lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1;\quad \lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \dfrac{1}{\log a}$

(c) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln a;\quad \lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

(d) $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x - 1} = m;\quad \lim\limits_{x\to a} \dfrac{x^m - a^m}{x - a} = ma^{m-1}$

(e) $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = \lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1$

2.2 Important series expansions

The following series expansions are frequently used in the evaluation of limits:

(a) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots\infty,\quad |x| < 1$

(b) $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots\infty,\quad |x| < 1$

(c) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\infty,\quad x \in \mathbb{R}$

(d) $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots\infty,\quad x \in \mathbb{R}$

(e) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\infty$

(f) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots\infty$

(g) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{11x^7}{315} + \cdots\infty$

(h) $(1+x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \cdots\infty \quad \{|x| < 1 \text{ and } n \in Q\}$

2.3 Limits of the form $\lim\limits_{x\to a}(f(x))^{g(x)}$

(i) If $\lim\limits_{x\to a} f(x) = l > 0$, then we may write

$$\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim\limits_{x\to a} g(x)\ln f(x)}$$

(ii) If $\lim\limits_{x\to a} f(x) = 1$, then let $f(x) = 1+h(x)$, where $\lim\limits_{x\to a} h(x) = 0$

Now, $\lim\limits_{x\to a}(f(x))^{g(x)} = \lim\limits_{x\to a}(1+h(x))^{g(x)} = \lim\limits_{x\to a}(1+h(x))^{\frac{1}{h(x)}\cdot g(x)\cdot h(x)} = e^{\lim\limits_{x\to a} g(x)\cdot h(x)}$

3. Techniques for evaluation of limits

3.1 Direct substitution

For a continuous function with a determinate expression, the limit can be obtained by direct substitution, as in the following examples:

(i) $\lim\limits_{x\to 1}(x^3+1) = 2$ (ii) $\lim\limits_{x\to 2}(5x^2+3x+1) = 27$

(iii) $\lim\limits_{x\to -1}(4x^3+4) = 0$ (iv) $\lim\limits_{y\to 1}|y|+1 = 2$

(v) $\lim\limits_{x\to 5}\dfrac{5x^2+4}{2x+7} = \dfrac{129}{17}$ (vi) $\lim\limits_{x\to 8}\dfrac{x^3+1}{x+1} = \dfrac{513}{9} = 57$

3.2 Factorization

In many cases, factorization leads to changing of the limit from an indeterminate to a determinate form, as in the following examples:

(i) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{x-1} = \lim\limits_{x\to 1}(x^2+x+1) = 3$

Note that this limit is also of the form $\lim\limits_{x\to 1}\frac{x^m-1}{x-1}$ (whose limit is m).

(ii) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2-4} = \lim\limits_{x\to 2}\dfrac{(x-1)(x-2)}{(x-1)(x+2)} = \lim\limits_{x\to 2}\dfrac{x-1}{x+2} = \dfrac{1}{4}$

(iii) $\lim\limits_{x\to 0}\dfrac{(1+x)(1+2x)(1+3x)-1}{x} = \lim\limits_{x\to 0}\dfrac{1+6x+11x^2+6x^3-1}{x}$

$$= \lim_{x\to 0}\frac{6x+11x^2+6x^3}{x} = \lim_{x\to 0}(6+11x+6x^2) = 6$$

3.3 Rationalization

In this method, the rationalization of an indeterminate expression leads to a determinate one. The following example elaborates this method:

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}\quad\left(\text{of the indeterminate form } \frac{0}{0}\right)$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}\times\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\times\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}\quad\left(\begin{array}{l}\text{Rationalizing both the numerator}\\ \text{and the denominator}\end{array}\right)$$

$$=\lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}}\times\frac{\sqrt{x^2+16}+4}{1}\quad(\text{A determinate form now!})$$

$$=\frac{8}{2}=4$$

3.4 Reduction to standard forms

In this method, we try to reduce the given limit to one of the standard forms we have encountered. Consider the following example:

$$\lim_{x\to 0}(1+\sin x)^{2\cot x}$$

This limit is of the indeterminate form 1^∞ (as $x \to 0$, $\sin x \to 0$ and $\cot x \to \infty$). We proceed as follows:

$$\lim_{x\to 0}(1+\sin x)^{2\cot x}=\lim_{x\to 0}\boxed{(1+\sin x)^{\frac{1}{\sin x}}}^{2\cos x}$$

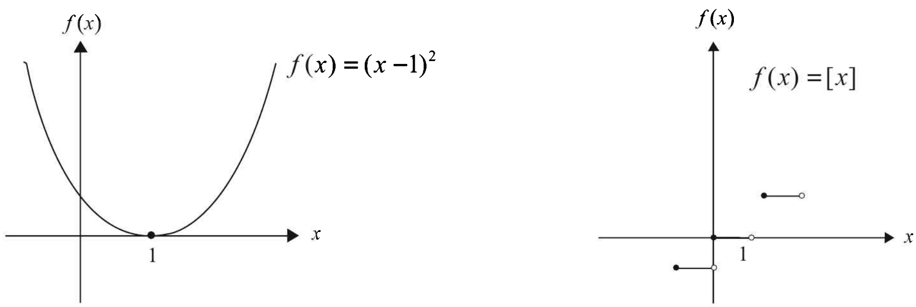
$$=e^{\lim\limits_{x\to 0}2\cos x}=e^2(\cos x\to 1 \text{ as } x\to 0)$$

We have used the fact that the boxed limit is equal to e.

4. Continuity

4.1 The concept of continuity

Consider the two graphs given in the figure below:



Our purpose is to analyse the behaviour of these functions around the region $x = 1$. The obvious visual difference between the two graphs around $x = 1$ is that whereas the first graph passes uninterrupted (without a break) through $x = 1$, the second function suffers a break at $x = 1$ (there is a jump). This visual difference, put into mathematical language, gives us the concept and definition of continuity. Mathematically, we say that the function $f(x)=(x-1)^2$ is continuous at $x = 1$ while $f(x)=[x]$ is discontinuous at $x = 1$

For $\quad f(x)=(x-1)^2$,

$$\text{LHL (at } x=1)=\lim_{x\to 1^-}(x-1)^2=0$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+}(x-1)^2 = 0$$

$$f(1)=0$$

$$\Rightarrow \quad \text{LHL} = \text{RHL} = f(1)$$

For $f(x)=[x]$,

$$\text{LHL (at } x=1) = \lim_{x\to 1^-}[x]=0$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+}[x]=1$$

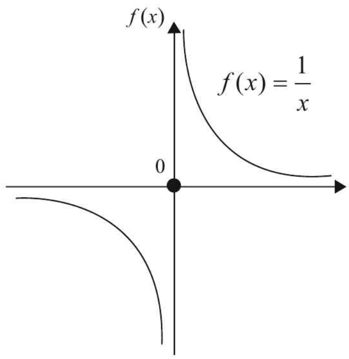
$$f(1)=1$$

$$\Rightarrow \quad \text{LHL} \neq \text{RHL} = f(1)$$

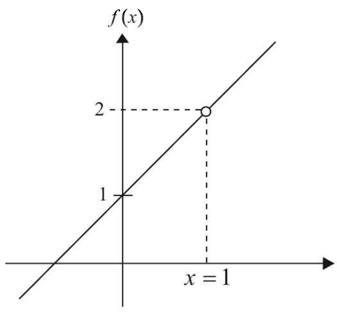
We note that for a function to be continuous at $x = a$, all the three quantities, namely, LHL, RHL and $f(a)$ should be equal. In any other scenario, the function becomes discontinuous. Discontinuities therefore arise in the following cases:

(a) **One or more than one of the three quantities, LHL, RHL and *f*(*a*) is not defined**. Lets consider some examples:

(i) $f(x)=\frac{1}{x}$ around $x = 0$. We see that LHL $=-\infty$, RHL $=+\infty$, $f(0)$ is not defined. Therefore, $f(x)=\frac{1}{x}$ is discontinuous at $x = 0$, which is obvious from the graph:



(ii) $f(x)=\{\frac{x^2-1}{x-1}$ for $x\neq 1\}$ around $x = 1$. We see that LHL = RHL = 2 but $f(1)$ is not defined. Therefore, this function's graph has a hole at $x = 1$; it is discontinuous at $x = 1$:



(b) **All the three quantities are defined, but any pair of them is unequal** (or all three are unequal). Lets go over some examples again:

(i) $f(x) = [x]$ around any integer I:

$$\text{LHL} = I-1,\ \text{RHL} = I,\ f(I) = I$$

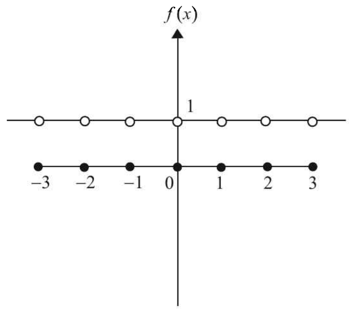
$\Rightarrow$ LHL $\neq$ RHL $= f(I)$ so this function is discontinuous at all integers as we already know.

(ii) $f(x) = \{x\}$ around any integer I:

$$\text{LHL} = 1, \text{RHL} = 0, f(I) = 0$$

$\Rightarrow$ LHL $\neq$ RHL $= f(I)$ so this function is also discontinuous at all integers.

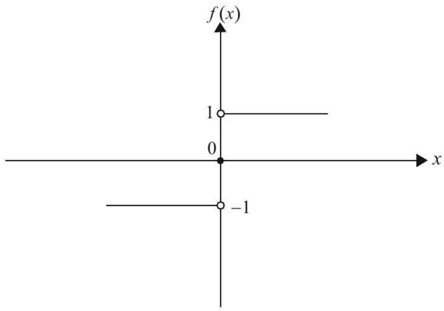
(iii) $f(x) = \left\{\begin{smallmatrix}1, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z}\end{smallmatrix}\right\}$ around any integer I:



From the figure, we notice that at any integer I, LHL = 1, RHL = 1, $f(I) = 0$.

$\Rightarrow$ LHL = RHL $\neq f(I)$ so that this function is again discontinuous.

(iv) $f(x) = \begin{Bmatrix} \frac{|x|}{x}, x \neq 0 \\ 0 \quad x = 0 \end{Bmatrix}$ around $x = 0$



At $x = 0$, we see that

$$\text{LHL} = -1, \text{RHL} = 1, f(0) = 0$$

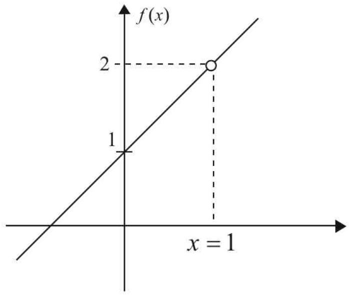
$\Rightarrow$ LHL $\neq$ RHL $\neq f(0)$ and this function is discontinuous.

To summarize, if we intend to evaluate the continuity of a function at $x = a$, which means that we want to determine whether $f(x)$ will be continuous at $x = a$ or not, we have to evaluate all the three quantities, LHL, RHL and $f(a)$. If these three quantities are finite and equal, $f(x)$ is continuous at $x = a$. In all other cases, it is discontinuous at $x = a$. Therefore, for continuity at $x = a$, we have

$$\text{LHL(at } x = a) = \text{RHL(at } x = a) = f(a)$$

4.2 Important notes

(i) Some authors talk about removable and irremovable discontinuities. Let us discuss what this means. Consider $f(x) = \frac{x^2-1}{x-1}, x \neq 1$. The LHL and RHL at $x = 1$ exist and both are equal to 2.



There is a hole in the graph at $x=1$ and therefore the function is discontinuous at $x=1$. We can, if we want to, fill this hole by redefining the function in the following manner:

$$f(x)=\begin{Bmatrix} \dfrac{x^2-1}{x-1} & x\neq 1 \\ 2 & x=1 \end{Bmatrix}$$

The additional definition at $x=1$ fills the hole and removes the discontinuity. Hence, such a discontinuity would be called a *removable discontinuity*. It is not difficult to see that a discontinuity is removable only if the LHL and RHL are equal, since only then we can redefine $f(a)$ to make all the three quantities equal. If the LHL and RHL are themselves non-equal, no redefinition of $f(a)$ could possibly make the function continuous and hence, such a discontinuity would be called *irremovable*. For example, $f(x)=[x]$ and $f(x)=\{x\}$ suffer from irremovable discontinuities at all integers.

(ii) If f and g are two continuous functions at a point $x=a$ (which is common to their domains), then $f\pm g$ and fg will also be continuous at $x=a$. Furthermore, if $g(a)\neq 0$, then $\frac{f}{g}$ will also be continuous at $x=a$.

(iii) If g is continuous at $x=a$ and f is continuous at $x=g(a)$, then $f(g(x))$ will be continuous at $x=a$.

(iv) Any polynomial function is continuous for all values of x.

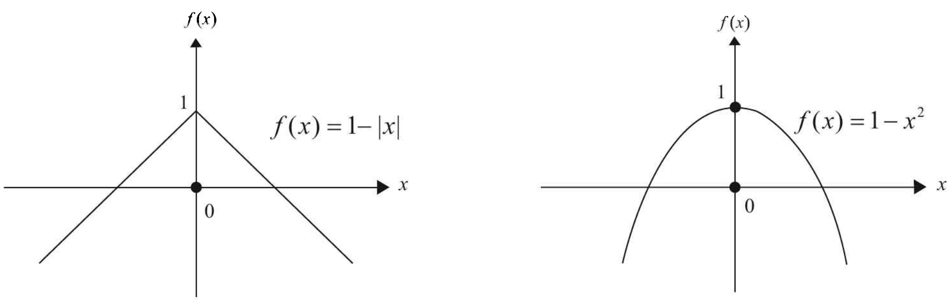
(v) The functions $\sin x$, $\cos x$ and e^x (or a^x) are continuous for all values of x. The function $\ln x$ (or $\log_a x$) is continuous for all $x>0$.

5. Differentiability

It is extremely important to understand differentiability from a geometrical point of view, and therefore this concept will be discussed in some detail.

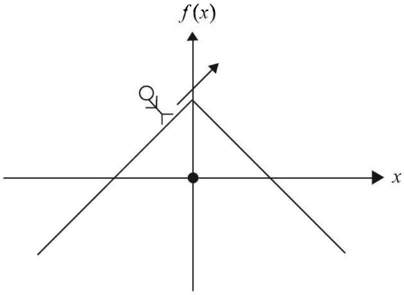
5.1 The Concept of Differentiability

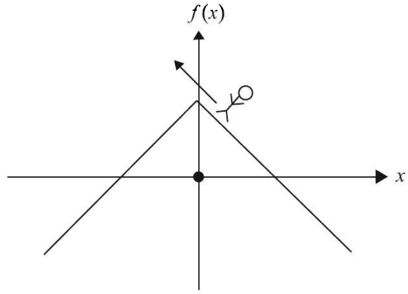
Consider the two graphs given in the figure below:



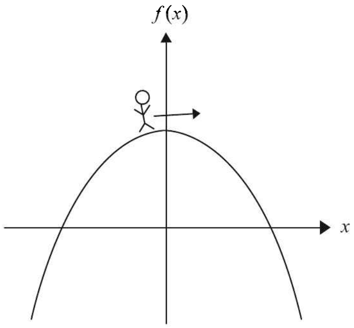
What is the striking difference between the two graphs at the origin (apart from one being linear and the other, non-linear)? The first has a sharp, sudden turn at the y-axis while the second passes

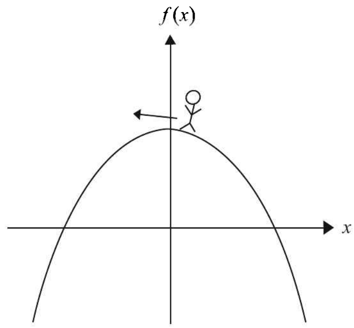
smoothly through the y-axis. Lets make this idea more concrete. Imagine a person called Theta walking on the graph of $f(x) = 1 - |x|$, towards the y-axis, once from the left and once from the right:



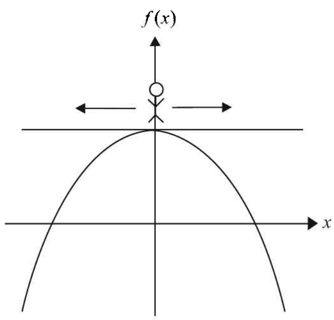


While walking from the left, Theta will be moving in a north-east direction as he approaches the y-axis; while walking from the right, he will be walking in a north-west direction. Now consider Theta walking on $f(x) = 1 - x^2$, once from the left and once from the right, towards $x = 0$.





As Theta approaches the y-axis, we see that he moves *almost* horizontally near the y-axis, in both the cases. The line of travel becomes almost the same from either side near $x = 0$. At $x = 0$, the line of travel becomes precisely horizontal (for an instant), whether Theta is walking from the left or the right. This unique line of travel is obviously the tangent drawn at $x = 0$:



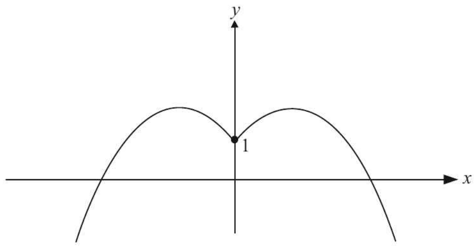
In mathematical language, since the lines of travel from both sides tend to become the same as the y-axis is approached, or more precisely, the tangent drawn to the immediate left of $x = 0$ and the one to the immediate right, become precisely the same at $x = 0$ (a unique tangent), we say that the function $f(x) = 1 - x^2$ is differentiable at $x = 0$. This means that the graph is *smoothly varying* around $x = 0$ or there is no sharp turn at $x = 0$.

In the case of $f(x) = 1 - |x|$, the lines of travel from the left hand and the right hand sides are different. The line of travel (or tangent) to the immediate left of $x = 0$ is inclined at 45° to the x-axis

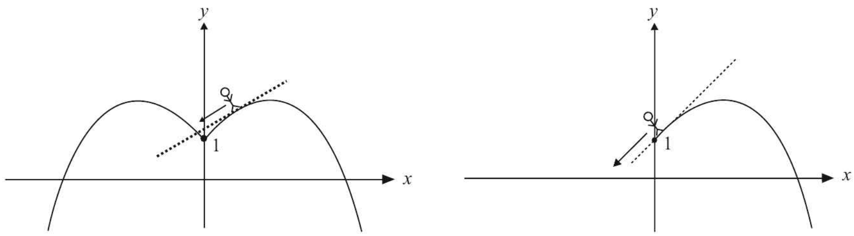
while the one to the immediate right is inclined at 135°. Precisely at $x = 0$, there is no unique tangent that can be drawn to $f(x)$. We therefore say that $f(x) = 1 - |x|$ is non-differentiable at $x = 0$. This means that the graph has a sharp point (or turn) at $x = 0$, as is evident from the previous figure.

5.2 Left and right derivatives

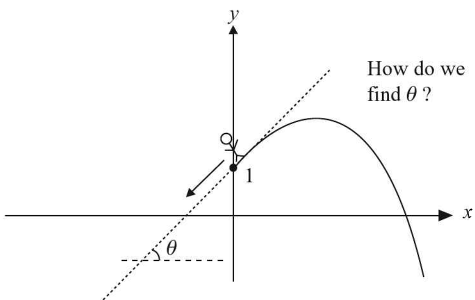
Consider the curve $f(x) = 1 + |x| - x^2$.



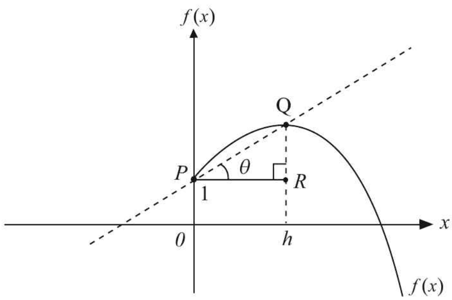
Theta is walking on this curve towards the y-axis from the right. When he is infinitesmally close to the y-axis, his direction of travel will be along the tangent drawn to the right segment of the graph, at an x-coordinate in the immediate right neighbourhood of the origin; or equivalently, at a point on the right segment of the graph which is infinitesmally close to the point (0, 1).



How do we find out this direction of travel near the point (0, 1)? In other words, how do we find out the slope of a tangent drawn to the right part of the graph, at a point extremely (infinitesmally) near to (0, 1)?



To evaluate this slope, we first drawn a secant on this graph, passing through (0, 1), as shown in the figure below:



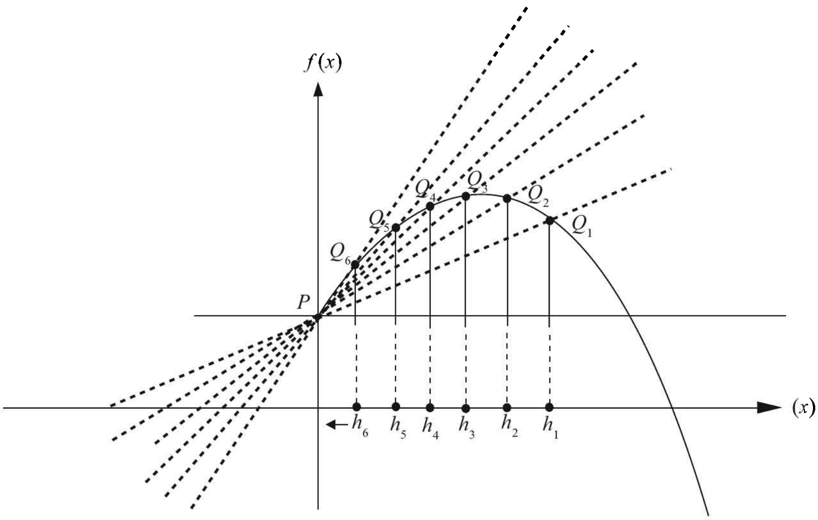
Let the x-coordinate of the point Q be h. The slope of the secant PQ is

$$\tan\theta = \frac{QR}{PR}$$

Notice that QR is $f(h) - f(0)$ and PR is h. Therefore,

$$\tan\theta = \frac{f(h) - f(0)}{h} \tag{1}$$

Now we make this secant closer to a tangent by reducing h: look at the figure below; as h is reduced or as $h \to 0$, the secant PQ *tends* to become a tangent drawn 'at' P (or more accurately, a tangent at a point infinitesmally close to the point P):



We see that as the point $Q \to P$ or as $h \to 0$, the secant PQ tends to become a tangent to the curve; to find the slope of this tangent, we find $\lim\limits_{h\to 0}(\tan\theta)$ where $\tan\theta$ is given by (1):

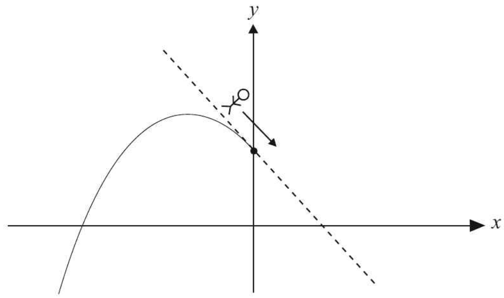
$$\text{Slope of tangent} = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

This limit gives us the slope of the tangent 'at' the point P (by 'at' we mean 'just near'). Lets evaluate this limit for this particular function:

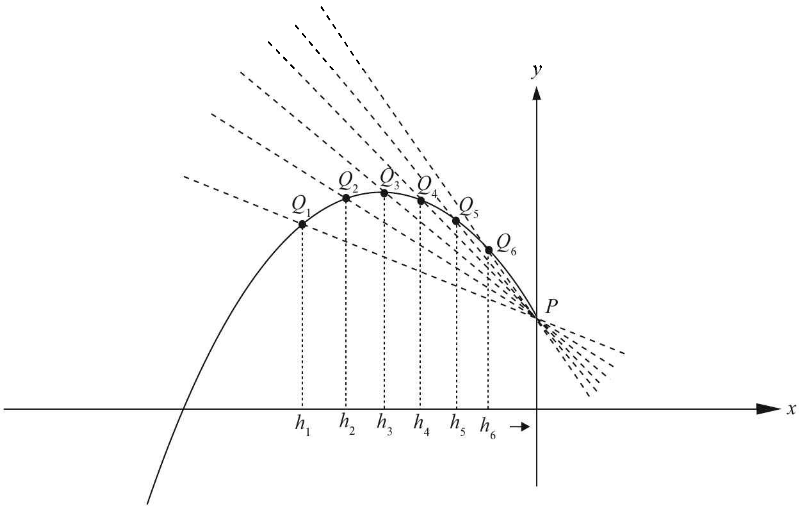
$$f(x) = 1 + |x| - x^2 \quad \text{so } f(h) = 1 + h - h^2 \quad \text{and} \quad f(0) = 1$$

$$\text{Slope of tangent} = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{h - h^2}{h} = 1$$

This means that the tangent drawn at P (on the right part of the graph) is inclined at 45° to the x-axis. In mathematical jargon, the limit we have just evaluated is called the Right Hand Derivative (RHD) of $f(x)$ at $x = 0$. This quantity, as we have seen, gives us the behaviour of the curve (its slope) in the immediate right side vicinity of $x = 0$. Obviously, there will exist a Left Hand Derivative (LHD) also that will give us the behaviour of the curve in the immediate left side vicinity of $x = 0$. In other words, the LHD will give us the direction of travel of Theta as he is 'just about' to reach the point (0, 1) travelling from the left towards the y-axis.



To evaluate the LHD, we follow a procedure similar to the one we used to evaluate the RHD; only this time we will drawn the secant PQ on the left side of the graph.



As the point $Q \to P$ or as $h \to 0$, the secant PQ again tends to become a tangent. As for the previous case, the slope of this tangent will be given by:

$$\underset{\text{(LHD)}}{\text{Slope of tangent}} = -\lim_{h \to 0}\left(\frac{f(h)-f(0)}{h}\right) \text{(Note the negative sign on the RHS)}$$

$$= \lim_{h \to 0}\frac{f(h)-f(0)}{-h}$$

For this particular case:

$$\text{LHD} = \lim_{h \to 0}\frac{(1+h-h^2)-1}{-h} \quad = -1$$

The tangent drawn to the left part of the graph 'just near' P will be inclined at 135° to the x-axis.

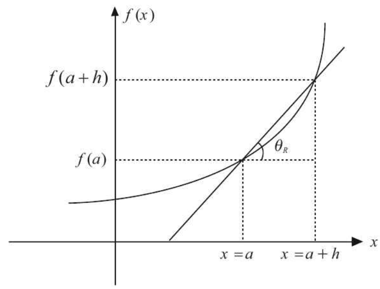
Notice that because there is a sharp point at $x = 0$, or in other words, Theta's direction of travel will change when crossing the y-axis, the LHD and RHD have different values. We would say that this function is *non-differentiable* at $x = 0$. No tangent can be drawn to $f(x)$ precisely at $x = 0$. On the other hand, for a *smooth* function, the LHD and RHD at that point will be equal and such a function would be *differentiable* at that point. This means that a unique tangent can be drawn at that point.

Now, let us write down the general expressions for the LHD and RHD:

(i) LHD at $x = a$ for $y = f(x)$

$$\tan\theta = \frac{f(a)-f(a-h)}{h} = \frac{f(a-h)-f(a)}{-h}$$

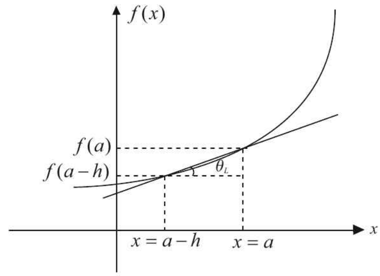
$$\Rightarrow \quad \text{LHD(at } x = a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$



(ii) RHD at $x = a$ for $y = f(x)$

$$\tan\theta = \frac{f(a+h)-f(a)}{h}$$

$$\Rightarrow \quad \text{RHD(at } x = a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



Lets conclude this discussion by summarizing. Consider a continuous function $y = f(x)$:

- If at a particular point $x = a$, the LHD and RHD have equal numerical values, we say that $f(x)$ is differentiable at $x = a$. In graphical terms, this means that the graph crosses $x = a$ smoothly, without any sharp turn. This also means that a unique tangent can be drawn to the curve $y = f(x)$ at $x = a$.

  The *derivative at* $x = a$ implies the slope of the tangent at $x = a$; *i.e.*,

  $$\text{Derivative (at } x = a) = \text{LHD} = \text{RHD}$$

  Obviously, the derivative exists only if $f(x)$ is differentiable at $x = a$. The derivative at $x = a$ is denoted by $f'(a)$.

- If at a particular point $x = a$, the LHD and RHD have non-equal values or one (or both) of them does not exist, we say that $f(x)$ is non differentiable at $x = a$. Graphically, this means that the graph does not pass through $x = a$ smoothly, which could be for a number of reasons, the function might be discontinuous at that point, or there might be a sharp turn. For any discontinuous function at $x = a$, $f(x)$ would always be non differentiable at $x = a$ since no unique tangent could be drawn to $f(x)$ at $x = a$. Therefore, for differentiability at $x = a$ the necessary and sufficient conditions that $f(x)$ has to satisfy are:

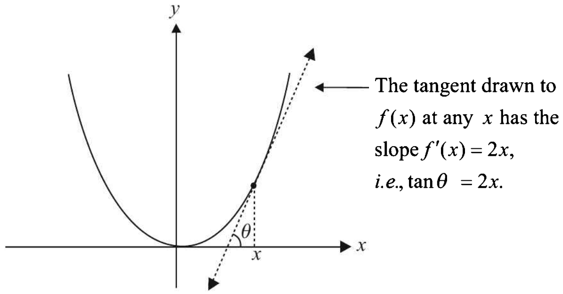
  (i) $f(x)$ must be continuous at $x = a$ (ii) LHD = RHD at $x = a$.

6. Differentiation

6.1 What is differentiation?

We have seen that if $f(x)$ is a differentiable function for a given x, this means that we can draw a unique tangent to $f(x)$ for that given x. The slope of this unique tangent is called the derivative of $f(x)$ for that given x. *The process of finding the derivative is known as differentiation.*

For example, for $f(x) = x^2$, the derivative at any given x has the value $2x$ (you can easily show this by *first principles*). This means that the slope of the tangent drawn to $f(x)$ at any given x has the numerical value $2x$. Equivalently stated, we can *differentiate* $f(x) = x^2$ to get $f'(x) = 2x$. Note that $f'(x)$ represents the derivative of $f(x)$.

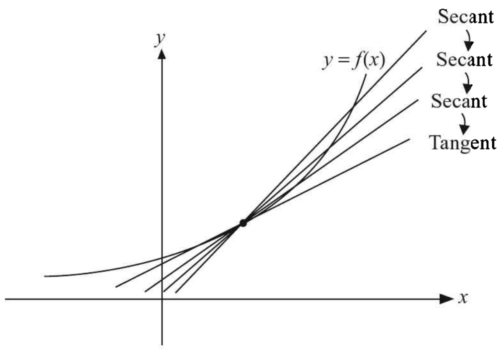


We have seen that we can differentiate a function at a given point only if the LHD and RHD at that point have equal values. If they do, then

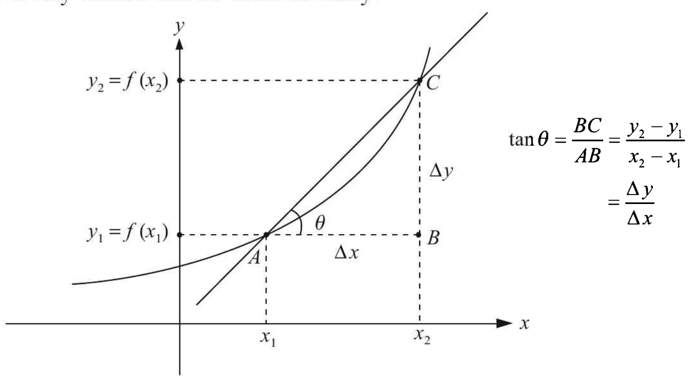
$$f'(x) = \frac{d(f(x))}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

The notation that we use to signify the derivative of $y = f(x)$ is either $f'(x)$(or y') or $\frac{df(x)}{dx}$ (or $\frac{dy}{dx}$).

Here, we need to understand the significance of the notation $\frac{dy}{dx}$ (the derivative of y with respect to the variable x). To evaluate the derivative (slope of the tangent) of $y = f(x)$ at a given point, we first drew a secant passing through that point and then let that secant tend to a tangent as follows:



The slope of any secant can be written easily:



The slope is obvious from the figure:

$$\tan\theta = \frac{BC}{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Now, to evaluate the derivative at x_1, we need to make the secant AC tend to a tangent at A by letting x_2 approach $x_1 (x_2 \to x_1)$ or equivalently, by letting $\Delta x \to 0$. As Δx becomes an infinitesmally small

quantity (approaches 0), the corresponding Δy will also become infinitesmally small (will approach 0), but the ratio $\frac{\Delta y}{\Delta x}$ will become an increasingly accurate representation of the slope of the tangent at A. An infinitesmally small change in the x value is represented by dx instead of Δx. Similarly, an infinitesmally small change in the y value would be represented by dy instead of Δy. Therefore,

$$\lim_{x_2 \to x_1} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y'$$

You should now be clear about the notation $\frac{dy}{dx}$. We will use $\frac{dy}{dx}, \frac{df(x)}{dx}, f'(x)$ or y' interchangeably to represent the derivative of $y = f(x)$ at any given x.

Note: Keep in mind that d(*variable*) represents an infinitesmally small change in the variable value while Δ (*variable*) represents a finite change in the variable value.

6.2 Derivatives of some commonly encountered functions

Below, we have listed some commonly encountered functions and their derivatives, and also shown the evaluation of the derivatives by first principles in some cases:

(1) $f(x) = k \quad \Rightarrow \quad f'(x) = 0$

(2) $f(x) = mx + c \quad \Rightarrow \quad f'(x) = m$

(3) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{x^n (1 + \frac{h}{x})^n - x^n}{h} \right\} = x^n \lim_{h \to 0} \left\{ \frac{(1 + \frac{h}{x})^n - 1}{h} \right\}$$

$$= x^n \lim_{h \to 0} \frac{\left\{ 1 + \frac{nh}{x} + n\frac{(n-1)}{2!}\frac{h^2}{x^2} + \ldots. \right\} - 1}{h} \quad \text{[By the binomial expansion]}$$

$$= x^n \cdot \frac{n}{x} = nx^{n-1}$$

So, for example, $\frac{d(x^2)}{dx} = 2.x^{2-1} = 2x$ and $\frac{d(x^2)}{dx} = 3.x^{3-1} = 3x^2$ and so on.

(4) $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

(5) $f(x) = \cos x \quad \Rightarrow \quad f'(x) = -\sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{-2\sin(x + \frac{h}{2})\sin\frac{h}{2}}{h} \right\} = \lim_{h \to 0} \left\{ -\sin\left(x + \frac{h}{2}\right) \cdot \frac{\sin(\frac{h}{2})}{(\frac{h}{2})} \right\} = -\sin x$$

(6) $f(x) = \tan x \Rightarrow \quad f'(x) = \sec^2 x$

(7) $f(x) = \sec x \Rightarrow \quad f'(x) = \sec x \tan x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x - \cos(x+h)}{h\cos x\cos(x+h)} = \lim_{h\to 0}\left\{\frac{2\sin(x+\frac{h}{2})\sin\dfrac{h}{2}}{h\cos x\cos(x+h)}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sin(x+\frac{h}{2})}{\cos x\cos(x+h)}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\right\} = \sec x \tan x$$

(8) $f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$

(9) $f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$

(10) $f(x) = a^x \Rightarrow f'(x) = a^x \ln a$

(11) $f(x) = \log_a x \Rightarrow f'(x) = \dfrac{1}{x\ln a}$

(12) $f(x) = \sin^{-1} x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

(13) $f(x) = \cos^{-1} x \Rightarrow f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$

(14) $f(x) = \tan^{-1} x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan^{-1}(x+h)-\tan^{-1} x}{h}$$

$$= \lim_{h\to 0}\left\{\frac{\tan^{1}\frac{h}{1+x(x-h)}}{h}\right\} \quad \left(\begin{array}{l}\text{we used } \tan^{-1} A - \tan^{-1} B \\ = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right); \text{verify this}\end{array}\right)$$

$$= \lim_{h\to 0}\left\{\frac{\tan^{-1}\frac{h}{1+x(x+h)}}{\frac{h}{1+x(x+h)}}\right\}\cdot\frac{1}{1+x(x+h)} = \frac{1}{1+x^2}$$

(15) $f(x) = \operatorname{cosec}^{-1} x \Rightarrow f'(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

(16) $f(x) = \sec^{-1} x \Rightarrow f'(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

(17) $f(x) = \cot^{-1} x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$

7. Rules of Differentiation

We will now summarize certain general rules pertaining to differentiation that will help us in calculating the derivative of an arbitrary function without using first principles. In the discussion that follows, we assume that $f(x)$ and $g(x)$ are two differentiable functions.

Rule 1: $\dfrac{d(kf(x))}{dx} = \dfrac{kd(f(x))}{dx}$

This rule says that a constant can be taken out from the argument of the differentiation operator. The proof is very straight forward:

$$\frac{d(kf(x))}{dx} = \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$$

$$= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{kd(f(x))}{dx}$$

Rule 2: $$\frac{d(f(x)\pm g(x))}{dx} = \frac{d(f(x))}{dx} \pm \frac{d(g(x))}{dx}$$

This rule says that the differentiation operator is distributive over addition and subtraction. The proof for this is again quite simple:

$$\frac{d(f(x)\pm g(x))}{dx} = \lim_{h\to 0}\frac{\{f(x+h)\pm g(x+h)\}-\{f(x)\pm g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x+h)-f(x)\}\pm\{g(x+h)-g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \frac{d(f(x))}{dx} \pm \frac{d(g(x))}{dx}$$

Rule 3: Product rule: $$\frac{d\{f(x)g(x)\}}{dx} = f(x)\frac{d(g(x))}{dx} + g(x)\frac{d(f(x))}{dx}$$

This rule, called the Product rule, is of great help in evaluating the derivative of the product of two (or more) functions. The proof is as follows:

$$\frac{d\{f(x)g(x)\}}{dx} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{A\to 0}\frac{f(x+h)g(x+h)\overset{\text{New terms}}{\boxed{-f(x+h)g(x)+f(x+h)g(x)}}-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\{g(x+h)-g(x)\}+g(x)\{f(x+h)-f(x)\}}{h}$$

$$= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\frac{d(g(x))}{dx} + g(x)\frac{d(f(x))}{dx}$$

Rule 4: Quotient Rule: $$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\dfrac{d(f(x))}{dx} - f(x)\dfrac{d(g(x))}{dx}}{(g(x))^2}$$; wherever $g(x) \neq 0$

This rule, called the Quotient rule, helps us evaluate the derivative of the ratio of two functions $f(x)/g(x)$, wherever $g(x) \neq 0$.

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Introduction of an extra term

$$= \lim_{A\to 0}\frac{g(x)f(x+h)\boxed{-g(x)f(x)+g(x)f(x)} - f(x)g(x+h)}{h.g(x).g(x+h)}$$

$$= \lim_{h\to 0}\frac{1}{g(x)\cdot g(x+h)}\lim_{h\to 0}\left[\frac{g(x)\{f(x+h)-f(x)\}}{h} + f(x)\frac{\{g(x)-g(x+h)\}}{h}\right]$$

$$= \frac{1}{(g(x))^2}\cdot\left[g(x)\frac{d(f(x))}{dx} - f(x)\frac{d(g(x))}{dx}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Rule 5: Chain Rule: $\dfrac{d\,f(g(x))}{dx} = \dfrac{d\,f(g(x))}{d(g(x))}\cdot\dfrac{d(g(x))}{dx}$

This can be stated more conveniently as

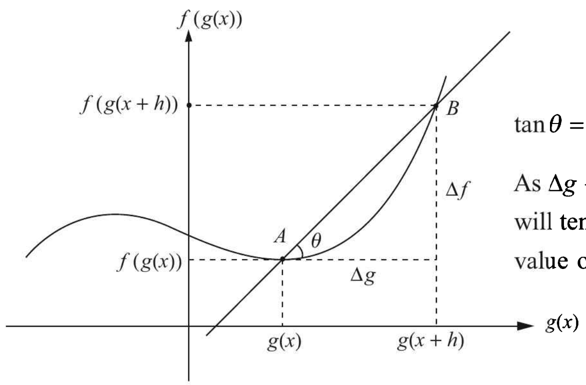
$$(f(g(x)))' = f'(g(x))g'(x)$$

This rule, called the Chain rule, is extremely useful to differentiate composite functions, and will be used extensively. It says that to differentiate $f(g(x))$, we first differentiate f with respect to $g(x)$ and not x (*i.e.*, we treat $g(x)$ as a variable y and differentiate f with respect to y) and then we multiply this by the derivative of $g(x)$(or y) with respect to x.

$$\frac{d[f(g(x))]}{dx} = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\cdot\frac{g(x+h) - g(x)}{h}$$

Notice carefully that the first ratio in the limit above is actually the derivative of f, but with respect to $g(x)$ as the variable $\{\text{as } h\to 0, (g(x+h) - g(x))\to 0\}$. You can view this graphically as follows:



$\tan\theta = \dfrac{\Delta f}{\Delta g}$

As $\Delta g \to 0,\quad \tan\theta$ will tend to the instantaneous value of slope at A.

Therefore,

$$\frac{d(f(g(x)))}{dx} = \frac{d\,f(g(x))}{d(g(x))}\cdot\frac{d(g(x))}{dx} \quad = f'(g(x))\cdot g'(x)$$

For example,

$$\frac{d(\sin(x^2))}{dx} = \underbrace{\cos(x^2)}_{\substack{\uparrow\\ \text{Derivative of}\\ \sin(x^2)\,w.r.t\,(x^2)}} \cdot \underbrace{2x}_{\substack{\uparrow\\ \text{Derivative of}\\ (x^2)\,w.r.t\,x}}$$

$$\frac{d(\log(\sin x))}{dx} = \frac{1}{\underbrace{\sin x}_{\substack{\downarrow\\ \text{Derivative of log}\\ (\sin x)\,w.r.t\,\sin x}}} \cdot \underbrace{\cos x}_{\substack{\downarrow\\ \text{Derivative of}\\ \sin x\,w.r.t\,x}}$$

and so on. You must note here that for this rule to be applicable at any $x = a$, the function $f(x)$ must be differentiable at $x = g(a)$, since the variable, that is, the argument (input) to f is $g(x)$ {and not x}.

Rule 6: Parametric Differentiation

If a function $y = f(x)$ is specified in parametric form:

$$y = g(t),\ x = h(t),$$

then we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{h'(t)}$$

Rule 7: L'Hospital's Rule (LH Rule)

The LH rule says that if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or if } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty,$$

then $\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$ given that the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

IMPORTANT IDEAS AND TIPS

1. $\lim_{x\to a} f(x)$ and $f(a)$. The limit of the function at a point has nothing to do with the value of the function at that point. The function may have any value at that point (or it may not even be defined at that point) - as long as the left hand and right hand limits exist and are equal to each other, we will say that the limit of the function at that point exists. On the other hand, for a function to be continuous at a point, the value of the function at that point is absolutely critical - it should be equal to the value of the limit of the function at that point.

$$\text{Limit of } f(x) \text{ exists at } x = a\text{: } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

$$f(x) \text{ is continuous at } x = a\text{: } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

2. *Indeterminate Limits:* A common mistake made by students pertains to indeterminate limits. Consider a function $f(x) = g(x)h(x)$. We are given that as $x \to 0$, $g(x) \to 0$. What is the value of $\lim_{x\to 0} f(x)$? Many students would say that it is 0. However, the actual answer depends on the limit of $h(x)$ as $x \to 0$. If $\lim_{x\to 0} h(x)$ is not finite, then the limit on $f(x)$ is indeterminate. The point we are trying to make is that in calculating the limit of a function which is the product of two or more functions, if one of the function tends to 0 under the relevant limit, that does not make it necessary for the entire limit to be 0 as well. Similar remarks hold for other indeterminate forms. For example, if $f(x) = g(x)^{h(x)}$, and $g(x) \to 1$ as $x \to a$, it does not necessarily imply that $f(x) \to 1$ as $x \to a$, because if $\lim_{x\to a} h(x)$ is not finite, then the limit on $f(x)$ is indeterminate. Recognizing that a limit is indeterminate is one of the most basic aspects of studying this topic, and Calculus as a whole.

3. *L'Hospital's Rule:* You are familiar with calculating an indeterminate limit using the LH rule. What you may overlook is the fact that the LH rule is applicable only when the modified limit (obtained by differentiating the numerator and denominator) also exists. Let's consider an example to illustrate this point. Consider the limit $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2}$. We see that the limit is of the indeterminate form $\frac{\infty}{\infty}$. Applying the LH rule two times in succession, we obtain the result that the limit does not exist:

$$\lim_{x\to\infty} \frac{x^2+\sin x}{x^2} = \lim_{x\to\infty} \frac{2x+\cos x}{2x} \left(\text{again } \frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x\to\infty} \frac{2-\sin x}{2}$$

$$= 1 - \frac{1}{2} \lim_{x\to\infty} \sin x, \text{ which does not exist}$$

However, only a few moments of consideration are required to conclude that the limit must exist, because the numerator is $x^2+\sin x$, and since x tends to infinity, the term $\sin x$ can be ignored in comparison to x^2, the denominator is x^2, and so the limit must be 1. Why did the LH rule go wrong? The mistake was in the second application of the LH rule, because the resulting limit which we obtain does not exist, so the LH rule cannot be applied the second time:

$$\underbrace{\lim_{x\to\infty} \frac{2x+\cos x}{2x} = \lim_{x\to\infty} \frac{2-\sin x}{2}}_{\substack{\text{This step is wrong, because the limit on} \\ \text{the RHS does not exist}}}$$

4. *Continuity:* Functions constructed using continuous functions will in general be continuous. For example, the sum, difference and product of continuous functions will be continuous. If we have rational functions of the form $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are continuous, then discontinuities will arise at the zeroes of $g(x)$, if any. Functions constructed using discontinuous functions (for example, compositions of piecewise-defined functions which are discontinuous) will in general be discontinuous. Inferring that a given function could have discontinuities and identifying potential points where the function could be discontinuous is an important skill which is tested in a lot of problems.

5. *Composition of Piecewise Functions:* A major difficulty students encounter is to evaluate the composite function of two (or more) piecewise-defined functions, and questions on evaluating the continuity and differentiability of such composite functions are common. Therefore, this is an important concept to be mastered. The only remedy is to practice a lot of relevant problems. In particular, the following exercise should be helpful: construct a lot of piecewise functions on your own. Form random pairs from this set and try determining their composites (do get your answers verified from your teacher!).

6. *Geometrical Interpretations:* It will be extremely helpful to retain the following geometrical associations in your mind:

Fact about $f(x)$	**Interpretation**
The limit of $f(x)$ exists at $x = a$	The graph of $f(x)$ approaches the same value from both sides of $x = a$
$f(x)$ is continuous at $x = a$	There is no break in the graph as x moves from a^- to a to a^+
$f(x)$ is differentiable at $x = a$	The graph is smooth as x moves from a^- to a to a^+

These are not technical definitions but help a lot in intuitive thinking. For example, thinking of a differentiable function's graph as a smooth curve helps us picture easily the fact that tangents can be drawn to the curve (wherever it is differentiable).

7. *A matter of Terminology.*

Derivative: Many students interpret the term derivative incorrectly. What is the derivative of a function $f(x)$? What do we mean when we say things like "the derivative of x^2 is $2x$"? In the simplest terms, the derivative is the slope of the tangent to the function. If $f(x) = x^2$, then the derivative (slope of tangent) at $x = 1$ will have the value 2, the derivative (slope of tangent) at $x = 2$ will have the value 4, and the derivative (slope of tangent) at any point x will have the value $2x$. This is what we mean by saying that the derivative of x^2 is $2x$. Essentially, the derivative of $f(x)$ is a new function $g(x)$, and plugging in an x-value in $g(x)$ gives us the slope of the tangent to the curve of f at that particular x-value.

Differentiation: Once again, the term differentiation is frequently not understood properly. In simple terms, differentiation is the process by which you find the derivative of a function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ or } f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

Thus, for example, when we say something like 'differentiate $f(x) = x^2$ to get $2x$', we are essentially talking about carrying out the process to find the derivative of $f(x)$ at any x-value, which comes out to be $2x$. The various rules we have made for differentiation - from the simpler standard formulae of differentiation to the product, quotient and chain rule - all have the same basis and meaning - finding the slope of the tangent to the curve at any x-value. The derivative of f obtained at any point x is dependent on the value of x, and so the derivative is itself a function of x.

We strongly urge you to remember this significance of the terms derivative and differentiation. It is suggested that you practice proving a few differentiation formulae from first principles, so that the connection between derivatives and slopes of tangents is clear in your mind (that is the reason that the proofs of the product, chain and quotient rule are all given in the theory - they are important!). In addition, whenever you carry out the process of differentiation, don't think of it as a mechanical rule with no basis - think of it as an activity to find the slopes of tangents to the given function at various points.

Limits, Continuity, Differentiability and Differentiation

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

The value of $\lim\limits_{n\to\infty} \frac{[x]+[2^2x]+[3^2x]+\cdots+[n^2x]}{n^3}$ (where $[\cdot]$ represents the greatest integer function) is

(A) x (B) $\dfrac{x}{2}$ (C) $\dfrac{x}{3}$ (D) $\dfrac{2x}{3}$ (E) none of these

Solution: One might say that since the terms in the numerator are all integers, the numerator is not continuous and hence the limit will not exist. However, note first of all that the limit is on n, and that secondly, the addition of a large number of integer terms in the numerator $(n\to\infty)$ would tend to 'overcome' or 'make negligible' the effect of fractional parts that would otherwise have been present had there been no greatest integer functions on any of the terms. This implies that whether we consider $([x]+[2^2x]+\cdots+[n^2x])$, or $(x+2^2x+\cdots+n^2x)$, as n becomes larger and $\to\infty$, the difference between these two terms becomes negligible in comparison to their own magnitudes. Hence, the limit in question will be the same as

$$\lim_{n\to\infty}\frac{x+2^2x+3^2x+\cdots+n^2x}{n^3}=\lim_{n\to\infty}\frac{x(1+2^2+3^2+\cdots+n^2)}{n^3}$$

$$=\frac{x}{6}\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{n^3}=\frac{x}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)=\frac{x}{3}$$

Therefore, the correct option is (C). For students who like more rigor, here is the proof of the above result using Sandwich theorem (in this proof, it will become clear that the effect of the fractional part is negligible as $n\to\infty$):

$$\left.\begin{aligned} x-1&<[x]\le x\\ 2^2x-1&<[2^2x]\le 2^2x\\ 3^2x-1&<[3^2x]\le 3^2x\\ n^2x-1&<[n^2x]\le n^2x \end{aligned}\right\}\text{ By definition of the greatest integer function}$$

Addition of these inequalities yields

$$(x+2^2x+3^2x+\cdots+n^2x-n)<[x]+[2^2x]+\cdots+[n^2x]\le(x+2^2x+\cdots+n^2x)$$

Division by n^3 and application of $\lim\limits_{n\to\infty}$ on all three terms yields:

$$\lim_{n\to\infty}\left\{\frac{\frac{n(n+1)(2n+1)}{6}x-n}{n^3}\right\}<\lim_{n\to\infty}\left\{\frac{[x]+[2^2x]+\cdots+[n^2x]}{n^3}\right\}\le\lim_{n\to\infty}\left\{\frac{\frac{n(n+1)(2n+1)}{6}x}{n^3}\right\}$$

It is easy to that the left and right limits are both $\frac{x}{3}$, and hence the centre limit is also $\frac{x}{3}$. ■

Exmple 2

What are the values of the following limits?

(a) $\lim\limits_{x\to0}\sin\dfrac{1}{x}$

(A) 0 (B) 1 (C) A number between 0 and 1 (D) Limit does not exist (E) None of these

(b) $\lim\limits_{x\to0} x\sin\dfrac{1}{x}$

(A) 0 (B) 1 (C) A number between 0 and 1 (D) Limit does not exist (E) None of these

(c) $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$

(A) 0 (B) 1 (C) e (D) $\dfrac{1}{e}$ (E) None of these

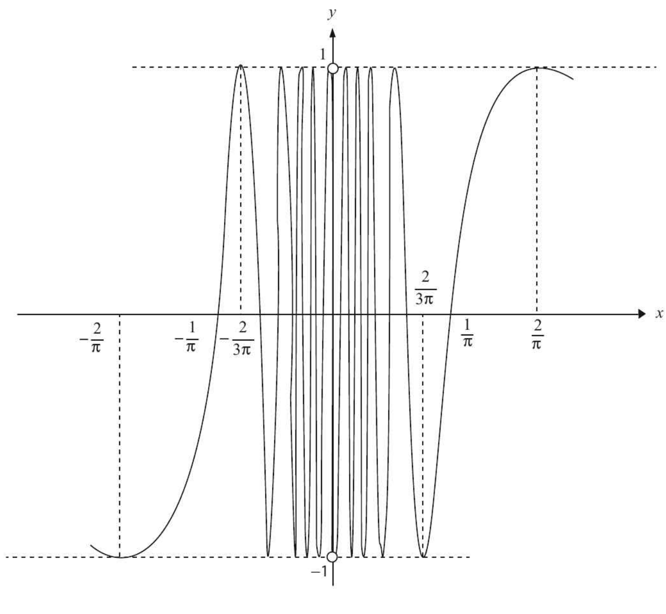
(d) $\lim\limits_{x\to0^+} x\ln x$

(A) 0 (B) 1 (C) e (D) $\dfrac{1}{e}$ (E) None of these

(e) $\lim\limits_{x\to+\infty}\dfrac{x^n}{n!}$

(A) 0 (B) 1 (C) n (D) $\ln n$ (E) None of these

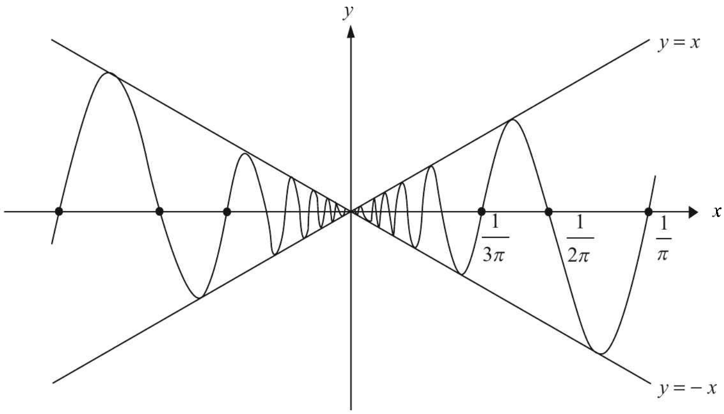
Solution: **(a)** The correct option is (D). Notice that as $x\to0$, $\frac{1}{x}\to\infty$, that is, $\frac{1}{x}$ has no particular limit to which it converges. Hence $\sin\frac{1}{x}$ keeps oscillating between $+1$ and -1 as $x\to0$. Therefore, the limit for this function does not exist. This is also clear from the graph (approximate) of $\sin\frac{1}{x}$ sketched below:



(b) The correct option is (A). In this limit, in addition to $\sin \frac{1}{x}$, 'x' is also present. Thus, although $\sin \frac{1}{x}$ remains oscillating and does not approach any particular limit, it nevertheless remains somewhere between $+1$ and -1, and when it gets multiplied by x (where $x \to 0$), the whole product gets infinitesimally small. That is,

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

Again, this is evident from the graph below (we have shown only the part of the graph near the origin):



(c) The correct option is (A). This limit can be evaluated purely by observation as follows. Although $\ln x$ and x are both tending to infinity, $\ln x$ increases *very slowly* as compared to x. For example, when $x = e^{10}$, $\ln x$ is just 10. When $x = e^{10000}$ (a very large number indeed !), $\ln x$ is just 10000. Therefore, $\frac{\ln x}{x}$ decreases and becomes infinitesimally small as $x \to \infty$, *i.e.*,

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0$$

We could also have used the LH rule to evaluate this limit.

(d) The correct option is (A). Consider $x \ln x$. As $x \to 0^+$, $\ln x \to -\infty$, so that this limit is of the indeterminate form $0 \times \infty$. But as in parts (b) and (c), try to see that the product becomes infinitesimally small as $x \to 0$. For example,

$$\text{At } x = e^{-10}, \quad \ln x = -10 \text{ and } x \ln x = \frac{-10}{e^{10}}.$$

$$\text{At } x = e^{-1000}, x \ln x = \frac{-1000}{e^{1000}} \text{ (which is very, very small).}$$

Hence, here again, $\lim_{x \to 0^+} x \ln x = 0$.

(e) The correction option is (A). If $|x| < 1$, then as $n \to \infty$, $Num \to 0$ and $Den \to \infty$, so that the limit is 0. For $|x| = 1$, also, the limit is obviously 0. For $|x| > 1$ we write $\frac{x^n}{n!}$ as

$$\frac{x^n}{n!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdot \cdots \cdot \frac{x}{n}$$

Now, since x is finite, let N be the integer just less than or equal to x; $N = [x]$. Hence,

$$\frac{x^n}{n!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \cdots \cdot \frac{x}{N} \cdot \frac{x}{N+1} \cdot \frac{x}{N+2} \cdot \cdots \cdot \frac{x}{n}$$

The product of the first N terms is finite; let it be equal to P. Thus,

$$\lim_{n\to\infty}\frac{x^n}{n!} = P\lim_{n\to\infty}\left\{\frac{x}{N+1}\cdot\frac{x}{N+2}\cdot\dots\cdot\frac{x}{n}\right\}$$

The product inside the limit consists of all terms less than 1. Also successive terms become smaller and smaller and tend to 0 as $n\to\infty$. Therefore, this product tends to 0 and hence the value of the overall limit is $P\times 0 = 0$:

$$\lim_{n\to\infty}\frac{x^n}{n!} = 0$$

■

Example 3

The value of $\lim\limits_{n\to\infty} n^{-n^2}\left((n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\dots\left(n+\frac{1}{2^{n-1}}\right)\right)^n =$

(A) 1 (B) e (C) e^2 (D) none of these

Solution: The limit L can be written as

$$L = \lim_{n\to\infty}\left(\frac{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\dots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^n\left(1+\frac{1}{2^2 n}\right)^n\dots\left(1+\frac{1}{2^{n-1}n}\right)^n$$

$$= e\cdot e^{\frac{1}{2}}\cdot e^{\frac{1}{2^2}}\cdots e^{\frac{1}{2^{n-1}}}\cdots\infty = e^2 \text{ (how?)}$$

The correct option is (C). ■

Example 4

The value of $\lim\limits_{n\to\infty}\frac{48}{n^4}\left[1\left(\sum_{k=1}^{n}k\right)+2\left(\sum_{k=1}^{n-1}k\right)+3\left(\sum_{k=1}^{n-2}k\right)+\dots+n\cdot 1\right] =$

(A) 0 (B) 1 (C) 2 (D) none of these

Solution: The general, r^{th} term in the series is $t_r = r(\sum_{k=1}^{n-r+1}k)$ which equals $r\cdot\frac{(n-r+1)(n-r+2)}{2}$. Thus,

$$\sum_{r=1}^{n}t_r = \sum_{r=1}^{n}\frac{r(n-r+1)(n-r+2)}{2} = \sum_{s=0}^{n-1}\frac{(s+1)(n-s)(n-s+1)}{2}$$

$$= \sum_{s=0}^{n-1}\frac{1}{2}\left(s^3-2ns^2+(n^2-n-1)s+n^2+n\right)$$

The highest degree term in this summation is $\frac{n^4}{24}$. Thus, the limit is $\frac{48}{n^4}\times\frac{n^4}{24} = 2$. The correct option is (C). ■

Example 5

The value of $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Using $t = \sin x$, we have

$$L = \lim_{t\to 1}\frac{t-t^t}{1-t+\ln t} = \lim_{h\to 0}\frac{(1+h)-(1+h)^{1+h}}{-h+\ln(1+h)} = \lim_{h\to 0}\frac{(1+h)\{1-(1+h)^h\}}{-h+\{h-\frac{h^2}{2}+\frac{h^3}{3}-\cdots\}}$$

$$= \lim_{h\to 0}\frac{(1+h)\{1-(1+h^2+h\frac{(h-1)h^2}{2!}+\cdots)\}}{-\frac{h^2}{2}+\frac{h^3}{3}h-+-\cdots} = 2$$

Thus, the correct option is (B). ■

Example 6

The value of $\lim_{k\to\infty}\left\{\dfrac{e^{\frac{1}{k}}+2e^{\frac{2}{k}}+3e^{\frac{3}{k}}+\cdots+ke^{\frac{k}{k}}}{k^2}\right\}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: The numerator is an AGP. which sums to (verify this!)

$$s = -\frac{e^{1/k}(e-1)}{\left(e^{1/k}-1\right)^2}+\frac{ke^{1+1/k}}{e^{1/k}-1}$$

Now, $\lim_{k\to\infty}\frac{s}{k^2}$ can be reduced to $-(e-1)+e$, which is equal to 1. This is left to the reader to verify. Therefore, the required limit is 1. ■

Example 7

The value of $\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{\sin x}$

(A) 0 (B) 1 (C) e (D) None of these

Solution: This limit is of the indeterminate form ∞^0. Lets first convert it into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking its logarithm:

$$\ln L = \lim_{x\to 0^+}\ln\left(\frac{1}{x}\right)^{\sin x} = \lim_{x\to 0^+}\sin x\cdot\ln\left(\frac{1}{x}\right) = -\lim_{x\to 0^+}\sin x\cdot\ln x$$

$$= -\lim_{x\to 0^+}\frac{\ln x}{\operatorname{cosec} x} = -\lim_{x\to 0^+}\frac{1/x}{-\operatorname{cosec}\cot x} \text{ (By applying the LH rule)}$$

$$= \lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)\cdot\tan x = 0$$

This implies that $L = e^0 = 1$. The correct option is (B). ■

Example 8

The value of $f(1)$ for which the function $f(x) = \frac{x^{m+1}-(m+1)x+m}{(x-1)^2}$ is continuous at $x=1$ is

(A) $\dfrac{m(m-1)}{4}$ (B) $\dfrac{m(m+1)}{4}$ (C) $\dfrac{m(m-1)}{2}$ (D) $\dfrac{m(m+1)}{2}$

Solution: For continuity at $x=a$, the following must be satisfied:

$$\lim_{x\to a} f(x) = f(a)$$

We have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^{m+1}-(m+1)x+m}{(x-1)^2} = \lim_{x\to 1}\frac{(x^{m+1}-x)-m(x-1)}{(x-1)^2}$$

$$= \lim_{x\to 1}\frac{x(x^m-1)-m(x-1)}{(x-1)^2} = \lim_{x\to 1}\frac{x(x^{m-1}+x^{m-2}+\cdots+1)-m}{(x-1)}$$

$$= \lim_{x\to 1}\frac{(x^m-1)+(x^{m-1}-1)+\cdots+(x-1)}{(x-1)} = m+(m-1)+\cdots+1 = \frac{m(m+1)}{2}$$

Therefore,

$$f(1) = \frac{m(m+1)}{2}$$

The correct option is (D). ■

Example 9

The value of $f\left(\frac{\pi}{4}\right)$ for which the function

$$f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, \qquad x \neq \frac{\pi}{4}$$

is continuous at $x = \frac{\pi}{4}$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

Solution: We have

$$\lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x - 1}{\cot x - 1} = \lim_{x\to\pi/4}\frac{\left(\sqrt{2}\cos x - 1\right)\cdot(\sin x)}{(\cos x - \sin x)}$$

$$= \lim_{x\to\pi/4}\frac{(\sqrt{2}\cos x - 1)}{(\cos x - \sin x)}\cdot\frac{(\sqrt{2}\cos x + 1)}{(\sqrt{2}\cos x + 1)}\cdot\frac{(\cos x + \sin x)}{(\cos x + \sin x)}.\sin x$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x}\cdot\frac{(\cos x + \sin x)}{(\sqrt{2}\cos x + 1)}\cdot\sin x$$

$$= \lim_{x\to\frac{\pi}{4}}\cdot\frac{\cos x + \sin x}{\sqrt{2}\cos x + 1}\cdot\sin x = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{\sqrt{2}.\frac{1}{\sqrt{2}}+1}.\frac{1}{\sqrt{2}} = \frac{1}{2}$$

Therefore, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$. The correct option is (B). ■

Example 10

The value of $a + b$ for which the function

$$f(x) = \begin{cases} \dfrac{x(1+a\cos x) - b\sin x}{x^3}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous at $x = 0$ is

(A) −1 (B) −2 (C) −3 (D) − 4

Solution: We obviously require

$$\lim_{x\to a} f(x) = f(0) = 1$$

We use the expansion series for sin x and cos x to get:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x + ax(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) - b(x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots)}{x^3}$$

$$= \lim_{x\to 0} \frac{\overset{\text{Must be 0}}{\overset{\downarrow}{\boxed{(1+a-b)}}}x - (\frac{a}{2!} - \frac{b}{3!})x^3 + (\frac{a}{4!} - \frac{b}{5!})x^5 \cdots}{x^3}$$

Now notice that (1 + a – b) must be necessarily 0, otherwise $\lim_{x\to 0} \frac{(1+a-b)x}{x^3}$ will become infinite. Hence,

$$1 + a - b = 0 \tag{1}$$

Also,

$$\lim_{x\to 0} f(x) = -\left(\frac{a}{2!} - \frac{b}{3!}\right) = 1$$

$$\Rightarrow \quad 6a - 2b + 12 = 0 \tag{2}$$

Solving (1) and (2), we get

$$a = \frac{-5}{2} \text{ and } b = \frac{-3}{2} \Rightarrow a + b = -4$$

The correct option is (D). ■

Example 11

Let $f(x) = \begin{Bmatrix} \frac{\sin 2x + \sin x}{x} & x<0 \\ a & x=0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{\frac{3}{2}}} & x>0 \end{Bmatrix}$. The value of a and b so that $f(x)$ is continuous at $x = 0$ is

(A) 0 (B) 1 (C) 2 (D) None of these

Solution: We have

$$\text{LHL(at } x = 0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 2x + \sin x}{x} = \lim_{x\to 0} \left\{\frac{2.\sin 2x}{2x} + \frac{\sin x}{x}\right\} = 3$$

$$f(0) = a$$

$$\text{RHL(at } x = 0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} \times \frac{(\sqrt{x+bx^2} + \sqrt{x})}{(\sqrt{x+bx^2} + \sqrt{x})}$$

$$= \lim_{x\to 0} \frac{bx^2}{bx^{3/2}(\sqrt{x+bx^2} + \sqrt{x})}$$

$$= \lim_{x\to 0} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{x+bx^2}} = \lim_{x\to 0} \frac{1}{1+\sqrt{1+bx}} = \frac{1}{2}$$

Since LHL ≠ RHL, no such values of a and b exist that could make the function continuous at $x = 0$. The correct option is (D). ■

Example 12

If $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}}, & \frac{-\pi}{6}<x<0\\ b, & x=0\\ e^{\frac{\tan 2x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$ is continuous at $x=0$, then which of the following is correct?

(A) $a=\frac{1}{3},\ b=e^{\frac{2}{3}}$ (B) $a=\frac{2}{3},\ b=e^{\frac{1}{3}}$ (C) $a=\frac{2}{3},\ b=e^{\frac{2}{3}}$ (D) None of these

Solution: For continuity at $x=0$, LHL (at $x=0$) $=f(0)=$ RHL (at $x=0$). Now, we have

$$\text{LHL(at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(1+|\sin x|)^{\frac{a}{|\sin x}}$$

$$= \lim_{y\to 0}(1+y)^{\frac{1}{y}\cdot a}\left\{\begin{matrix}\text{where } |\sin x|=y;\\ \text{as } x\to 0^-, y\to 0\end{matrix}\right\} = e^a$$

$$\text{RHL(at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\frac{\tan 2x}{\tan 3x}} = \lim_{x\to 0^+} e^{\frac{2}{3}\cdot\frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}} = e^{2/3}$$

$$\Rightarrow\ e^a=b=e^{\frac{2}{3}} \Rightarrow\ a=\frac{2}{3} \text{ and } b=e^{\frac{2}{3}}$$

The correct option is (C). ■

Example 13

Let

$$F(x)=\begin{Bmatrix}\left(1+x+\dfrac{f(x)}{x}\right)^{\frac{1}{x}}, & x\neq 0\\ e^3 & , x=0\end{Bmatrix} \text{ and } G(x)=\begin{Bmatrix}\left(1+\dfrac{f(x)}{x}\right)^{\frac{1}{x}} & , x\neq 0\\ k & , x=0\end{Bmatrix},$$

where $f(x)$ is some function of x. If $F(x)$ is continuous at $x=0$, the value of $\ln k$ so that $G(x)$ is also continuous at $x=0$ is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: Since $F(x)$ is continuous at $x=0$, we have

$$\lim_{x\to 0} F(x) = F(0) = e^3$$

$$\Rightarrow\ \lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}} = e^3 \ \Rightarrow\ \lim_{x\to 0} e^{\frac{1}{x}\cdot\left\{x+\frac{f(x)}{x}\right\}} = e^3$$

$$\Rightarrow\ \lim_{x\to 0}\left(1+\frac{f(x)}{x^2}\right)=3 \Rightarrow\ \lim_{x\to 0}\frac{f(x)}{x^2}=2$$

For $G(x)$ to be continuous at $x=0$, k should be equal to $\lim_{x\to 0} G(x)$:

$$\lim_{x\to 0} G(x) = \lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{f(x)}{x}\right)} = e^{\lim_{x\to 0}\frac{f(x)}{x^2}} = e^2$$

Therefore, $k=e^2 \Rightarrow \ln k = 2$. The correct option is (C). ■

Example 14

If $f(x)=\begin{Bmatrix} \frac{\sin 3x + a\sin 2x + b\sin x}{x^5}, & x\neq 0 \\ c & ,x=0 \end{Bmatrix}$ is continuous at $x = 0$, the value of $a+b+c$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: For continuity, we require $\lim\limits_{x\to 0} f(x) = c$. Now,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left\{\frac{\sin 3x + a\sin 2x + b\sin x}{x^5}\right\}$$

$$= \lim_{x\to 0}\frac{\left\{3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots\right\} + a\left\{2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right\} + b\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right\}}{x^5}$$

$$= \lim_{x\to 0}\frac{(3+2a+b)x - \left(\frac{27}{6} + \frac{8a}{6} + \frac{b}{6}\right)x^3 + \left(\frac{243}{120} + \frac{32a}{120} + \frac{b}{120}\right)x^5 - \ldots}{x^5}$$

In order for this limit to be finite, we must have

$$3 + 2a + b = 0 \qquad (1)$$

$$27 + 8a + b = 0 \qquad (2)$$

If this does not hold, $\lim\limits_{x\to 0}\frac{(3+2a+b)x}{x^5}$ and $\lim\limits_{x\to 0}\frac{(27+8a+b)x^3}{x^5}$ will become infinite. Now, solving (1) and (2), we get

$$a = -4,\ b = 5$$

The limit now reduces to

$$\lim_{x\to 0}\frac{\left(\frac{243}{120} + \frac{32a}{120} + \frac{b}{120}\right)x^5 + \ldots(\text{higher order terms})}{x^5} = \frac{243 + 32a + b}{120} = 1$$

Hence,

$$c = 1$$

Finally, we have $a + b + c = 2$. The correct option is (B). ■

Example 15

A function $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x) - f(y)| \le |x - y|^3$ for all $x, y \in \mathbb{R}$. The function $f(x)$ is

(A) a constant function
(B) a non-constant linear function
(C) a square function
(D) a cubic function
(E) none of these

Solution: Let us evaluate the RHD at any x:

$$\text{RHD at } x = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Therefore,

$$|\text{RHD at } x| = \lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h}\right| = \lim_{h\to 0}\frac{|f(x+h) - f(x)|}{|h|}$$

$$\le \lim_{h\to 0}\frac{|(x+h) - (x)|^3}{|h|} \quad \{\text{Using the given relation}\}$$

$$= \lim_{h \to 0} |h|^2 = 0$$

$$\Rightarrow \quad |\text{RHD}| \le 0$$

Since |RHD| cannot be negative, it has to be 0.

$$\Rightarrow \quad \text{RHD} = 0 \text{ for all } x$$

Similarly, LHD = 0 for all x. This implies that $f(x)$ is a constant function. The correct option is (A). ■

Example 16

Let $f(x) = \left\{ \begin{matrix} |x^2-1|-1 & , & x \le 1 \\ |2x-3|-|x-2| & , & x > 1 \end{matrix} \right\}$.

(a) The number of points at which $f(x)$ is not continuous is

(A) 0 (B) 1 (C) 2 (D) More than 2

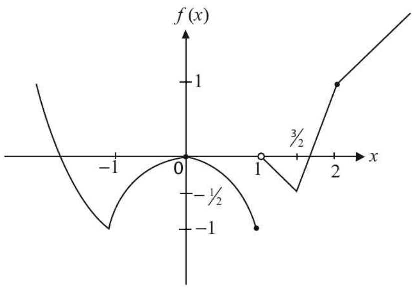
(b) The number of points at which $f(x)$ is non-differentiable is

(A) 1 (B) 2 (C) 3 (D) More than 3

Solution: Note that $f(x)$ can equivalently be rewritten as:

$$f(x) = \left\{ \begin{matrix} x^2 - 2 & x \le -1 \\ -x^2 & -1 < x \le 1 \\ 1 - x & 1 < x \le 3/2 \\ 3x - 5 & 3/2 < x < 2 \\ x - 1 & 2 \le x \end{matrix} \right\}$$

The graph of this function will be as follows:



We see that $f(x)$ is non-continuous at $x = 1$ and non-differentiable at 4 points, $x = -1, 1, \frac{3}{2}, 2$. The correct options for the two parts of the question are (B) and (D) respectively. ■

Example 17

Let $f(x) = x^2 - 2|x|$ and $g(x) = \left\{ \begin{matrix} \text{Min}\{f(t) : -2 \le t \le x, & -2 \le x < 0\} \\ \text{Max}\{f(t) : \ 0 \le t \le x, & 0 \le x \le 3\} \end{matrix} \right\}$.

(a) Draw the graph of $g(x)$.

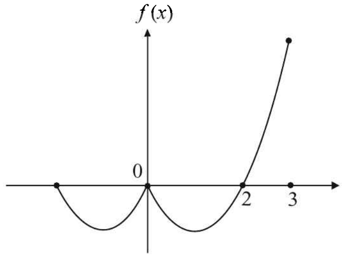
(b) At how many points is $g(x)$ discontinuous in its domain?

(A) 0 (B) 1 (C) 2 (D) 3 (E) None of these

(c) At how many points is $g(x)$ non-differentiable in its domain?

(A) 0 (B) 1 (C) 2 (D) 3 (E) None of these

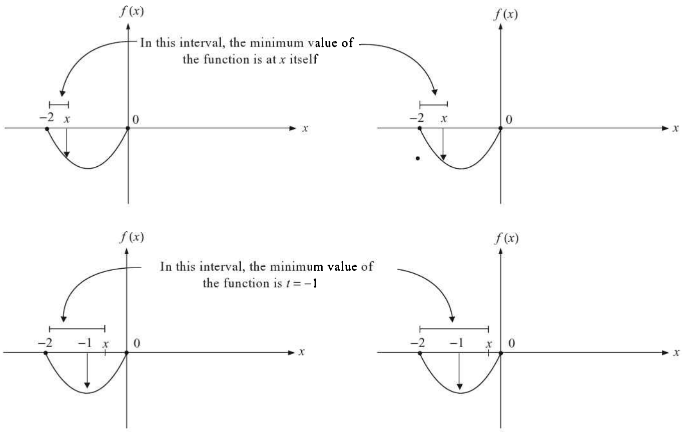
Solution: The graph of $f(x)$ is sketched below (in the relevant domain):



Now we must understand what the definition of $g(x)$ means. Consider the upper definition of $g(x)$:

$$g(x) = \text{Min}\{f(t): -2 \le t \le x, \ -2 \le x < 0\}$$

To evaluate $g(x)$ at any x, we scan the entire interval from -2 to x (this is what the variable t is for), and select that value of $f(t)$ which is minimum in this interval; this minimum value of $f(t)$ becomes the value of the function at x. The figure below illustrates this graphically for four different values of x (we are considering the interval $-2 \le x < 0$, corresponding to the upper definition of $g(x)$):

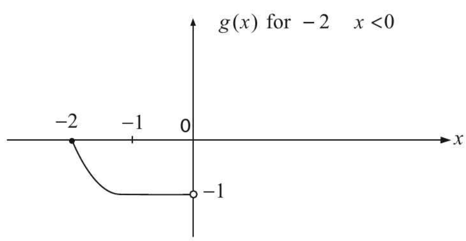


Notice that when x crosses -1 (when $-1 < x < 0$), the minimum value of the function in the interval $[-2, x]$ becomes fixed at $t = -1$. When $-2 < x < -1$, the minimum value of the function in the interval $[-2, x]$ is at $t = x$. So, how do we draw the graph of $g(x)$? We note the following:

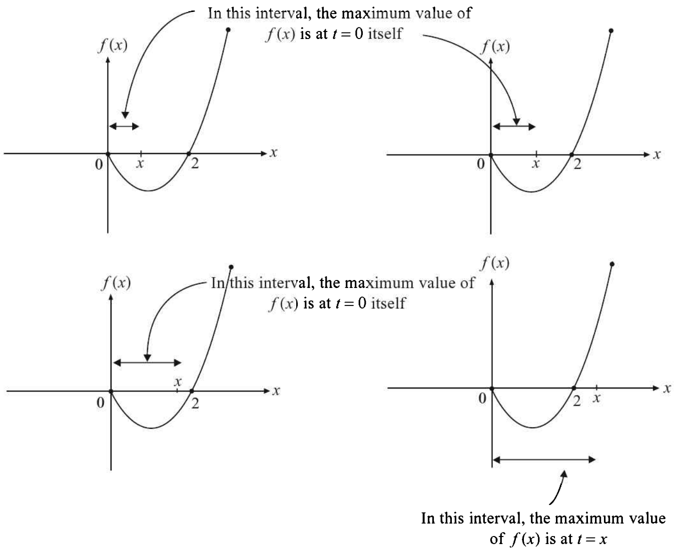
(i) In $[-2, -1]$, the graph of $g(x)$ will the same as $f(x)$ (because the minimum value of $f(x)$ is at $t = x$ itself, as described above).

(ii) In $[-1, 0)$, the minimum value becomes fixed at $t = -1$, equal to -1, so that in this interval the graph of $g(x)$ is constant; $g(x) = -1$ for $x \in [-1, 0)$.

The graph of $g(x)$ for $x \in [-2, 0)$ is sketched below:

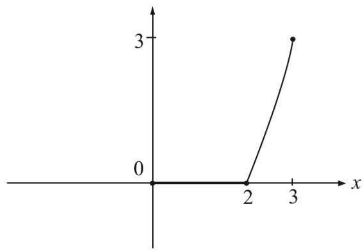


For $0 \le x \le 3$, the definition is $g(x) = \text{Max}\{f(t); 0 \le t \le x, 0 \le x \le 3\}$. The figure below illustrates how to obtain $g(x)$ in this case for four different values of x:

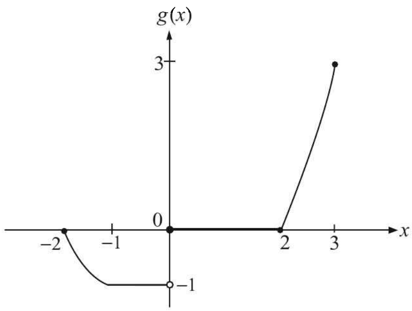


Notice then as long as x lies in the interval [0, 2], the maximum value of $f(x)$ in the interval $[0, x]$ is at $t = 0$, equal to 0. As soon as x becomes greater than 2, the maximum value of $f(x)$ in the interval $[0, x]$ is now at $t = x$. The graph of $g(x)$ for the interval [0, 3] is sketched below:

$g(x)$ for $0 \le x \le 3$



(a) The overall graph for $g(x)$ is therefore:



We see that $g(x)$ is discontinuous at $x = 0$ and non-differentiable at $x = 0, 2$.

(b) $g(x)$ is discontinuous at one point, so the correct option is (B).

(c) $g(x)$ is non-differentiable at two points, so the correct option is (C).

As an exercise, try plotting the graphs of the following functions:

(i) $f(x) = \max\{2x - x^2; 0 \le t \le x, 0 \le x \le 2\}$

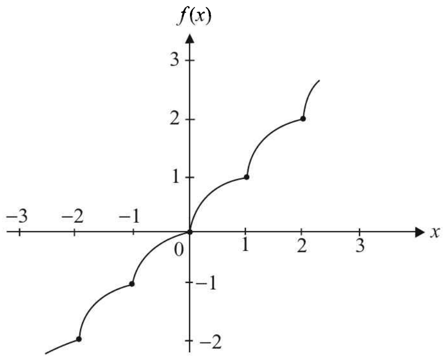
(ii) $f(x) = \min\{|x^2 - 1|; -2 \le t \le x, -2 \le x \le 2\}$ ■

Example 18

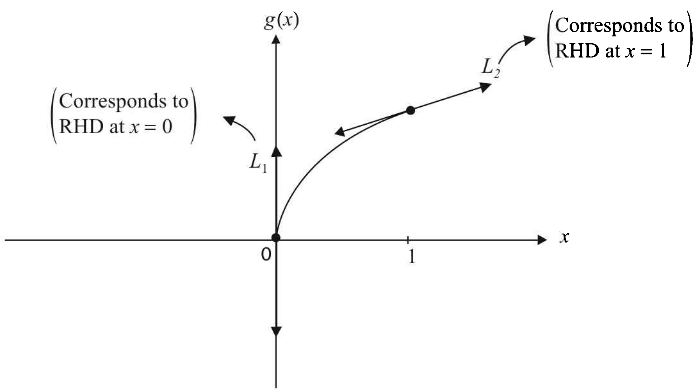
Consider the function $f(x) = [x] + \sqrt{\{x\}}$. This function is

(A) continuous at all integer points.
(B) discontinuous at all integer points.
(C) differentiable at all integer points.
(D) non-differentiable at all integer points.

Solution: The correct options are (A) and (D). The graph of $f(x)$ is drawn below:



$f(x)$ is obviously continuous everywhere but non-differentiable at all integer points. Let us evaluate the LHD and RHD at each integer. For that, we analyse the segment of the curve between any two adjacent integers. Lets pick up the segment between 0 and 1; this segment is part of the segment $f(x) = \sqrt{x}$:



The lines L_1 and L_2 correspond to the RHD at $x = 0$ and LHD at $x = 1$ respectively.

$$\text{RHD (at } x=0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\left\{\frac{\sqrt{h}-0}{h}\right\} = \lim_{h\to 0}\frac{1}{\sqrt{h}}$$

$= \infty$ (This tangent is vertical)

$$\text{LHD (at } x=1) = \lim_{h\to 0} \frac{g(1-h)-g(1)}{h} = \lim_{h\to 0}\frac{\sqrt{1-h}-1}{-h}$$

$$= \lim_{h\to 0}\frac{-h}{-h(\sqrt{1-h}+1)} \text{ (By rationalization)}$$

$$= \frac{1}{2}$$

Therefore, the LHD at each integer point is $\frac{1}{2}$ and the RHD at each integer point is ∞. The function is continuous but non-differentiable at every integer point. ■

Example 19

Let $f(\frac{x+y}{3}) = \frac{f(x)+f(y)+2}{3} \quad \forall x, y \in \mathbb{R}$. Also, it is given that $f'(0) = 2$. Which of the following statements is / are true?

(A) $f'(x) = f'(0) \quad \forall x \in \mathbb{R}$

(B) $f'(x) = f(x) \quad \forall x \in \mathbb{R}$

(C) $f(x)$ is a linear function

(D) none of these

Solution: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+f(3h)+2}{3} - \frac{f(3x)+f(0)+2}{3}}{h}$$

$$= \lim_{h\to 0} \frac{f(3h) - f(0)}{3h} = f'(0) = 2$$

$\Rightarrow \quad f(x) = 2x + c$, which is a linear function.

The correct options are (A) and (C). ■

SUBJECTIVE TYPE EXAMPLES

Example 20

Evaluate the following limits

(a) $\lim\limits_{x\to 2}\left(\dfrac{2}{x(x-2)}-\dfrac{1}{x^2-3x+2}\right)$ (b) $\lim\limits_{n\to\infty}\left[(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})\right], |x|<1$

(c) $\lim\limits_{x\to 3}\dfrac{x^3-7x^2+15x-9}{x^4-5x^2+27x-27}$

Solution: (a) This limit is of the indeterminate form $\infty - \infty$. Combining the two fractions in this limit should lead to a cancellation of the factor giving rise to this indeterminacy, *i.e.*, $(x-2)$:

$$\lim_{x\to 2}\left(\frac{2}{x(x-2)}-\frac{1}{x^2-3x+2}\right)=\lim_{x\to 2}\frac{2(x-1)-x}{x(x-2)(x-1)}=\lim_{x\to 2}\frac{\cancel{(x-2)}}{x(x-1)\cancel{(x-2)}}$$

$$=\lim_{x\to 2}\frac{1}{x(x-1)}=\frac{1}{2}$$

(b) Before trying to solve this, try to intuitively observe that this expression will have a finite limit even though the number of factors being multiplied tends to infinity. This is because the successive factors become closer and closer to 1 and their ‘contribution’ to the final product becomes smaller and smaller. Now, to simplify this product, we multiply it by $\frac{1-x}{1-x}$. This is what happens:

$$\lim_{n\to\infty}\left[\frac{(1-x).(1+x)(1+x^2)(1+x^4)......(1+x^{2^n})}{(1-x).}\right]$$

$$=\lim_{n\to\infty}\left[\frac{(1-x^2)(1+x^2)(1+x^4)......(1+x^{2^n})}{(1-x)}\right]$$

$$=\lim_{n\to 0}\left[\frac{(1-x^4)(1+x^4)......(1+x^{2^n})}{(1-x)}\right]$$

$$=\lim_{n\to\infty}\frac{1-x^{2^{n+1}}}{1+x}$$

Since $|x|<1$, $\lim\limits_{n\to\infty}x^{2^{n+1}}=0$. Hence, the value of the limit is $\frac{1}{1-x}$.

(c) The numerator and denominator both tend to 0 as $x\to 3$ because of the common factor $(x-3)$. Hence, factorization leads to:

$$\lim_{x\to 3}\frac{\cancel{(x-3)}(x^2-4x+3)}{\cancel{(x-3)}(x^2-2x^2-6x+9)}=\lim_{x\to 3}\frac{x^2-4x+3}{x^3-2x^2-6x+9}$$

There is still another common $(x-3)$ left in both the numerator and the denominator. Factorization again leads to

$$\lim_{x\to 3}\frac{(x-1)\cancel{(x-3)}}{\cancel{(x-3)}(x^2+x-3)}=\lim_{x\to 3}\frac{x-1}{x^2+x-3}=\frac{2}{9}$$

■

Example 21

Evaluate the following limits:

(a) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x - \sin x}{(\frac{\pi}{4} - x)(\cos x + \sin x)}$ (b) $\lim\limits_{x\to a} \dfrac{\cos\sqrt{x} - \cos\sqrt{a}}{x - a}$

(c) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$ (d) $\lim\limits_{n\to\infty} \left\{\cos\dfrac{x}{2}\cdot\cos\dfrac{x}{2^2}\cdots\cos\dfrac{x}{2^n}\right\}$

Solution: (a) We have

$$\lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{(\frac{\pi}{4} - x)(\cos x + \sin x)} = \lim_{x\to\pi/4} \frac{1 - \tan x}{(\frac{\pi}{4} - x)(1 + \tan x)} \text{ (Dividing Num and Den by } \cos x)$$

Let $x = \frac{\pi}{4} + h$, so that $\tan x = \frac{1+\tan h}{1-\tan h}$ and the limit above becomes

$$\lim_{h\to 0}\left\{\frac{1 - \frac{1+\tan h}{1-\tan h}}{-h}\right\}\cdot\lim_{x\to\frac{\pi}{4}}\left\{\frac{1}{1+\tan x}\right\} = \lim_{h\to 0}\left\{\frac{-2\tan h}{-h(1-\tan h)}\right\}\cdot\frac{1}{2}$$

$$= \lim_{h\to 0}\left\{\frac{\tan \text{h}}{h}.\frac{1}{1-\tan h}\right\} = 1$$

Alternatively, we could proceed by multiplying the numerator and denominator of the original limit by ($\cos x + \sin x$):

$$\lim_{x\to\frac{\pi}{4}}\left\{\frac{\cos x - \sin x}{(\frac{\pi}{4} - x)(\cos x + \sin x)}\cdot\frac{\cos x + \sin x}{\cos x + \sin x}\right\}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{(\frac{\pi}{4} - x)(\cos x + \sin x)^2} = \lim_{x\to\frac{\pi}{4}} \frac{\cos 2x}{(\frac{\pi}{4} - x)}\cdot\lim_{x\to\frac{\pi}{4}}\frac{1}{(\cos x + \sin x)^2}$$

$$= \lim_{h\to 0}\frac{-\sin 2h}{-h}\cdot\frac{1}{2} \quad \text{(we let } x = \tfrac{\pi}{4} + h \text{ so that } \cos 2x = \cos(\tfrac{\pi}{2} + 2h) = -\sin 2\, h)$$

$$= \lim_{h\to 0}\frac{\sin 2h}{2h} = 1$$

(b) $$\lim_{x\to a}\frac{\cos\sqrt{x} - \cos\sqrt{a}}{x - a} = \lim_{x\to a}\left\{\frac{-2\sin(\frac{\sqrt{x}+\sqrt{a}}{2})\sin(\frac{\sqrt{x}-\sqrt{a}}{2})}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})}\right\}$$

$$= -\frac{\sin\sqrt{a}}{2\sqrt{a}}\cdot\lim_{x\to a}\left\{\frac{\sin(\frac{\sqrt{x}-\sqrt{a}}{2})}{(\frac{\sqrt{x}-\sqrt{a}}{2})}\right\} = -\frac{\sin\sqrt{a}}{2\sqrt{a}}$$

(c) This limit can be solved very easily by using the expansion series for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\infty$$

Therefore, $$\lim_{x\to 0}\frac{x - \sin x}{x^3} = \lim_{x\to 0}\left\{\frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\infty}{x^3}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\infty\right\} = \frac{1}{3!} = \frac{1}{6}$$

Alternatively, we know that $\sin 3x = 3\sin x - 4\sin^3 x$. Hence, $3x - 3\sin x = 3x - \sin 3x - 4\sin^3 x$:

$$L = \lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{3}\lim_{x\to 0}\frac{3x-3\sin x}{x^3} = \frac{1}{3}\lim_{x\to 0}\left\{\frac{3x-\sin 3x}{x^3}-\frac{4\sin^3 x}{x^3}\right\}$$

$$= \frac{1}{3}\lim_{x\to 0}\left\{27\cdot\frac{(3x)-\sin(3x)}{(3x)^3}-\frac{4\sin^3 x}{x^3}\right\}$$

$$= \frac{1}{3}\cdot\left\{27\cdot\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}-4\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3\right\} = 9L-\frac{4}{3}$$

Hence, $L = 9L - \frac{4}{3}$ or $L = \frac{1}{6}$.

(d) We proceed by reducing this expression into a closed form by multiplying it with an appropriate factor as follows:

$$\lim_{n\to\infty}\left\{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}\cdot\frac{\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^{n-1}}\cdot\left(\overbrace{2\cos\tfrac{x}{2^n}\cdot\sin\tfrac{x}{2^n}}^{\text{combine}:\sin 2\theta=2\sin\theta\cos\theta}\right)}{2\sin\frac{x}{2^n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdots\overbrace{\cos\tfrac{x}{2^{n-1}}\cdot\sin\tfrac{x}{2^{n-1}}}^{\text{combine}}}{2\sin\frac{x}{2^n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^{n-2}}\cdot\sin\frac{x}{2^{n-2}}}{2^2\cdot\sin\frac{x}{2^n}}\right\}$$

$$\vdots$$

$$= \lim_{n\to\infty}\frac{\sin x}{2^n\sin(\frac{x}{2^n})} = \lim_{n\to\infty}\frac{\sin x}{\frac{x\cdot(\sin(\frac{x}{2^n}))}{(\frac{x}{2^n})}} = \frac{\sin x}{x}\cdot\frac{1}{\lim_{n\to\infty}\left\{\frac{\sin(\frac{x}{2^n})}{(\frac{x}{2^n})}\right\}}$$

Now, as $n\to\infty, \frac{x}{2^n}\to 0$ and hence

$$\lim_{n\to\infty}\left\{\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\right\} = 1$$

Therefore, our final result is $\frac{\sin x}{x}$. We note that the limit was on n and not on x; therefore, it is not surprising that we obtain x in our final answer. ■

Example 22

Evaluate the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$ (b) $\lim\limits_{x\to 0}\dfrac{x-\ln(1+x)}{x^2}$ (c) $\lim\limits_{x\to\infty}\dfrac{(1+x)^{1/x}-e}{x}$ (d) $\lim\limits_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

Solution: (a) The limit is of the indeterminate form $\frac{0}{0}$, but can be reduced into a combination of two standard limits as follows:

$$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim_{x\to 0}\frac{e^{x^2}-1+1-\cos x}{x^2}$$

$$=\lim_{x\to 0}\left\{\frac{e^{x^2}-1}{x^2}+\frac{1-\cos x}{x^2}\right\}=\lim_{x\to 0}\left\{\frac{e^{x^2}-1}{x^2}+\frac{2\sin^2 x/2}{4\left(x/2\right)^2}\right\}=1+\frac{1}{2}=\frac{3}{2}$$

(b) $\ln(1+x)$ can be expanded as $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$. Hence,

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\lim_{x\to 0}\left\{\frac{\frac{x^2}{2}-\frac{x^2}{3}+\cdots}{x^2}\right\}=\frac{1}{2}$$

(c) The numerator in this limits tends to 0 as $x\to 0$ because $\lim\limits_{x\to 0}(1+x)^{1/x}=e$. Evaluating this limit will require a little artifice in the following manner:

$$\lim_{x\to 0}\left\{\frac{(1+x)^{1/x}-e}{x}\right\}=\lim_{x\to 0}\left\{\frac{e^{\frac{\ln(1+x)}{x}}-e}{x}\right\}$$

$$=e\lim_{x\to 0}\left\{\frac{e^{\frac{\ln(1+x)}{x}-1}-1}{x}\right\}\text{ (Taking } e \text{ common out of the numerator)}$$

Now, as $x\to 0, \left\{\frac{\ln(1+x)}{x}-1\right\}\to 0$ so that the numerator in the limit above is of the form e^h-1 where $h\to 0$. What should we do now? Multiply and divide by h (where $h=\frac{\ln(1+x)}{x}-1$). Our limit will then become

$$e\lim_{x\to 0}\left[\frac{e^{\frac{\ln(1+x)}{x}-1}-1}{\{\frac{\ln(1+x)}{x}-1\}}\right]\cdot\left\{\frac{\ln(1+x)}{x}-1\right\}\cdot\frac{1}{x}$$

$$=e\lim_{h\to 0}\left[\frac{e^h-1}{h}\right]\cdot\lim_{x\to 0}\left\{\frac{\ln(1+x)-x}{x^2}\right\}$$

From part (b), it follows that the second limit has the value $-\frac{1}{2}$. Hence, the overall value for this limit is $-\frac{e}{2}$.

(d) Factoring the numerator and rationalising the denominator gives:

$$\lim_{x\to 0}\left\{\frac{(9^x-1)(3^x-1)}{\sqrt{2}-\sqrt{1+\cos x}}\cdot\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}\right\}$$

$$=\lim_{x\to 0}\frac{(9^x-1)}{x}\cdot\frac{(3^x-1)}{x}\cdot\frac{\left(\sqrt{2}+\sqrt{1+\cos x}\right)}{1-\cos x}\cdot x^2$$

$$= \lim_{x\to 0}\frac{(9^x-1)}{x}\cdot\lim_{x\to 0}\frac{(3^x-1)}{x}\cdot\lim_{x\to 0}(\sqrt{2}+\sqrt{1+\cos x})\cdot\lim_{x\to 0}\frac{2}{\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}}$$

$$= \ln 9.\ \ln 3.\ 2\sqrt{2}.\ 2$$

$$= 8\sqrt{2}\left(\ln 3\right)^2$$ ■

Example 23

Evaluate the following limits:

(a) $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ (b) $\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$ (c) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

(d) $\lim_{x\to 0}(\cos x)^{1/\sin x}$ (e) $\lim_{x\to 0}(\cos x + a\sin b\,x)^{\frac{1}{x}}$

Solution: Notice that all the limits above are of the form $(f(x))^{g(x)}$ where $f(x)\to 1$ and $g(x)\to\infty$, that is, these limits are of the indeterminate form 1^∞. To evaluate such limits, we write $f(x)$ as $(1 + h(x))$, reducing this limit to $e^{\lim_{x\to a} g(x)\cdot h(x)}$, where $h(x) = f(x) - 1$. We will now directly apply this result to evaluate the limits in the current problem.

(a) $$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\cdot\left\{\frac{a^x+b^x+c^x}{3}\right.} = e^{\lim_{x\to 0}\frac{1}{3}\cdot\left\{\frac{(a^x-1)}{x}+\frac{(b^x-1)}{x}+\frac{(c^x-1)}{x}\right\}}$$

$$= e^{\frac{1}{3}((\ln a)+(\ln b)+(\ln c))} = abc^{1/3}$$

(b) $$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\cdot\left\{\tan\left(\frac{\pi}{4}+x\right)-1\right\}}$$

$$= e^{\lim_{x\to 0}\left\{\frac{1}{x}\cdot\frac{2\tan x}{1-\tan x}\right\}} \quad \left(\text{Using}\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}\right)$$

$$= e^{2\lim_{x\to 0}\left\{\frac{\tan x}{x}\cdot\frac{1}{1-\tan x}\right\}} = e^2$$

(c) $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\cdot\left\{\frac{\sin x}{x}-1\right\}} = e^{\lim_{x\to 0}\left\{\frac{\sin x - x}{x^3}\right\}}$$

We can show that the value of the limit in the exponent is $-\frac{1}{6}$, and so the final answer is $e^{-1/6}$.

(d) $$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}} = e^{\lim_{x\to 0}\left\{\frac{1}{\sin x}.(\cos x-1)\right\}} = e^{\lim_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)} = e^{\lim_{x\to 0}(-\tan x/2)} = e^0 = 1$$

(e) $$\lim_{x\to 0}(\cos x + a\sin bx)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\cdot\{\cos x + a\sin bx - 1\}} = e^{\lim_{x\to 0}\left\{\frac{\cos x-1}{x}+ab\frac{\sin bx}{bx}\right\}}$$

$$= e^{\lim_{x\to 0}\left\{\frac{-2\sin^2 x/2}{4(x/2)^2}\cdot x+ab\frac{\sin bx}{bx}\right\}} = e^{\lim_{x\to 0}\left\{-\frac{1}{2}\frac{\sin^2(x/2)}{(x/2)^2}\cdot x+ab\frac{\sin bx}{bx}\right\}} = e^{ab}$$ ■

Example 24

Let $f(x)=\frac{1}{2}\left(x+\frac{a}{x}\right), a \neq 0$. If $\lim\limits_{x\to\sqrt{a}} \frac{f(x)-\sqrt{a}}{\left(x-\sqrt{a}\right)^p}=m(\neq 0)$, find p and m.

Solution: $$\lim_{x\to\sqrt{a}} \frac{f(x)-\sqrt{a}}{(x-\sqrt{a})^p}=m$$

Applying the LH rule twice on this yields

$$\lim_{x\to\sqrt{a}} \frac{\frac{a}{x^3}}{p(p-1)(x-\sqrt{a})^{p-2}}=m$$

$$\Rightarrow \quad p=2,\ p(p-1)=\frac{1}{m\sqrt{a}} \Rightarrow \quad m=\frac{1}{2\sqrt{a}}$$

■

Example 25

Find the condition on $f(x)$ and $g(x)$ which makes the function

$$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

continuous everywhere.

Solution: Note from the definition of $F(x)$ that we have a variable n present. Lets first try to make $F(x)$ independent of n. The way to do that is to consider three separate cases for x:

$$|x|=1,\ |x|>1,\ |x|<1$$

$\boxed{|x|=1}$: $$F(x)=\frac{f(x)+(1)\cdot g(x)}{1+1}=\frac{f(x)+g(x)}{2}$$

$\boxed{|x|>1}$: $$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}=\lim_{n\to\infty}\frac{x^{-2n}f(x)+g(x)}{x^{-2n}+1}$$

$$=g(x)\ \text{(because } x^{-2n}\to 0 \text{ as } n\to\infty \text{ for } |x|>1)$$

$\boxed{|x|<1}$: $$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

$$=f(x)\ \left(\text{because } x^{2n}\to 0 \text{ as } n\to\infty \text{ for } |x|<1\right)$$

Therefore, we can now rewrite $F(x)$ independently of n in the following manner

$$F(x)=\begin{cases} f(x) & \text{when } |x|<1 \quad \text{or} \quad -1<x<1 \\ \dfrac{f(x)+g(x)}{2} & \text{when } |x|=1 \quad \text{or} \quad x=1,-1 \\ g(x) & \text{when } |x|>1 \quad \text{or} \quad x<-1, x>1 \end{cases}$$

$F(x)$ could be discontinuous at only 2 points, $x=1$ or $x=-1$. To ensure continuity at these points, we have

(i) LHL (at $x=-1$) = RHL(at $x=-1$) = $F(-1)$

$$g(-1) = f(-1) = \frac{f(-1)+g(-1)}{2}$$

$$\Rightarrow \quad g(-1) = f(-1)$$

(ii) LHL (at $x = 1$) = RHL(at $x = 1$) = $F(1)$

$$f(1) = g(1) = \frac{f(1)+g(1)}{2}$$

$$\Rightarrow \quad f(1) = g(1)$$

Therefore, for continuity of $F(x)$, the required conditions are

$$f(1) = g(1) \text{ and } f(-1) = g(-1)$$

■

Example 26

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If the function $f(x)$ is continuous at $x = 0$, show that it is continuous for all $x \in \mathbb{R}$.

Solution: In questions that have such functional equations, we should try to substitute certain trial values for the variables to gain useful information; in this particular question, since we are given some condition about $f(x)$ at $x = 0$, we should try to find $f(0)$. Putting $x = y = 0$, we get

$$f(0) = 2f(0) \quad \Rightarrow \quad f(0) = 0$$

Now, since $f(x)$ is continuous at $x = 0$,

$$\text{LHL} = \text{RHL} = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

or equivalently

$$\lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = f(0) = 0$$

For continuity at an arbitrary value of x, the LHL, RHL and $f(x)$ should be equal.

$$\text{LHL(at } x) = \lim_{h\to 0} f(x-h) = \lim_{h\to 0}\{f(x)+f(-h)\} = \lim_{h\to 0} f(x) + \lim_{h\to 0} f(-h)$$

$$= f(x)$$

Similarly,

$$\text{RHL (at } x) = f(x)$$

Therefore, $f(x)$ is continuous for all values of x.

■

Example 27

Show that the function $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.

Solution: We have

$$\text{LHL(at } x = 0) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

Now, as $h \to 0, \frac{-1}{h} \to -\infty$ so $e^{-1/h} \to 0$. Therefore, LHL $= -1$. Similarly,

$$\text{RHL} = \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0}\frac{1-e^{-1/h}}{1+e^{-1/h}} = 1$$

Since $\text{LHL} \neq \text{RHL} \neq f(0), f(x)$ is discontinuous at $x = 0$. ■

Example 28

Let $f(x) = \begin{pmatrix} \frac{a^{2[x]+\{x\}}-1}{2[x]+\{x\}} & x \neq 0 \\ \ln a & x = 0 \end{pmatrix}$. Evaluate the continuity of $f(x)$ at $x = 0$.

Solution: The LHL and RHL might differ due to the discontinuous nature of $[x]$ and $\{x\}$. Lets determine their values:

$$\text{LHL(at } x = 0) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\frac{a^{2[-h]+\{-h\}}-1}{2[-h]+\{-h\}}$$

For $h > 0$, it is obvious that $[-h] = -1$ and $\{-h\} = 1-h$. Therefore,

$$\text{LHL} = \lim_{h\to 0}\frac{a^{-2+1-h}-1}{-2+1-h} = \lim_{h\to 0}\frac{a^{-1-h}-1}{-1-h} = \frac{a^{-1}-1}{-1} = 1-a^{-1}$$

Similarly,

$$\text{RHL} = \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{a^{2[h]+\{h\}}-1}{2[h]+\{h\}} = \lim_{h\to 0}\frac{a^{h}-1}{h} = \ln a$$

Since $\text{LHL} \neq \text{RHL}, f(x)$ is discontinuous at $x = 0$. ■

Example 29

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Solution: Since $f(x)$ is continuous at $x = 0$,

$$\lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = f(0) \qquad (1)$$

To evaluate $f(0)$, we substitute $x = y = 0$ in the given functional relation to get

$$f(0) = f(0)^2 \Rightarrow \quad f(0) = 0 \text{ or } f(0) = 1$$

(i) If $f(0) = 0$, then $f(x) = f(x+0) = f(x)\cdot f(0) = 0$, *i.e.*, $f(x) = 0$ for all values of x so that $f(x)$ is continuous everywhere.

(ii) We now assume $f(0) = 1$ We have,

$$\text{LHL at any } x = \lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x)\cdot f(-h) = f(x)\lim_{h\to 0} f(-h)$$
$$= f(x)\cdot f(0) \text{ (From (1))}$$
$$= f(x)$$

Similarly,

$$\text{RHL at any } x = \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x)f(h) = f(x)$$

Hence, $f(x)$ is continuous for all $x \in \mathbb{R}$. ■

Example 30

If $f(x+y)=f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x)=1+g(x)G(x)$, where $\lim\limits_{x\to 0} g(x)=0$ and $\lim\limits_{x\to 0} G(x)$ exists, prove that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Solution: We have

$$\lim_{h\to 0} g(h)=\lim_{h\to 0} g(-h)=0$$

$$\text{and } \lim_{h\to 0} G(h)=\lim_{h\to 0} G(-h)=k\,\{\text{some finite number}\}$$

Now, consider the LHL of $f(x)$ (at any x):

$$\text{LHL}=\lim_{h\to 0} f(x-h)=\lim_{h\to 0} f(x)f(-h) = f(x)\lim_{h\to 0}\left(1+g(-h)G(-h)\right) = f(x)$$

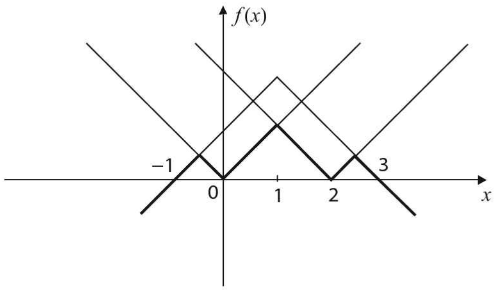
Similarly, the RHL (at any x) of $f(x)$ will be equal to $f(x)$.

$$\Rightarrow \quad f(x) \text{ is continuous for all } x.$$ ■

Example 31

If $f(x)=\min\{|x|, |x-2|, 2-|x-1|\}$, draw the graph of $f(x)$ and discuss its continuity and differentiablity.

Solution: To drawn the graph of $f(x)$, we plot the three 'mod' functions on the same axes, and then take the minimum:



It is immediately obvious that $f(x)$ is continuous everywhere but non-differentiable at $x=-\frac{1}{2}, 0, 1, 2, \frac{5}{2}$. ■

Example 32

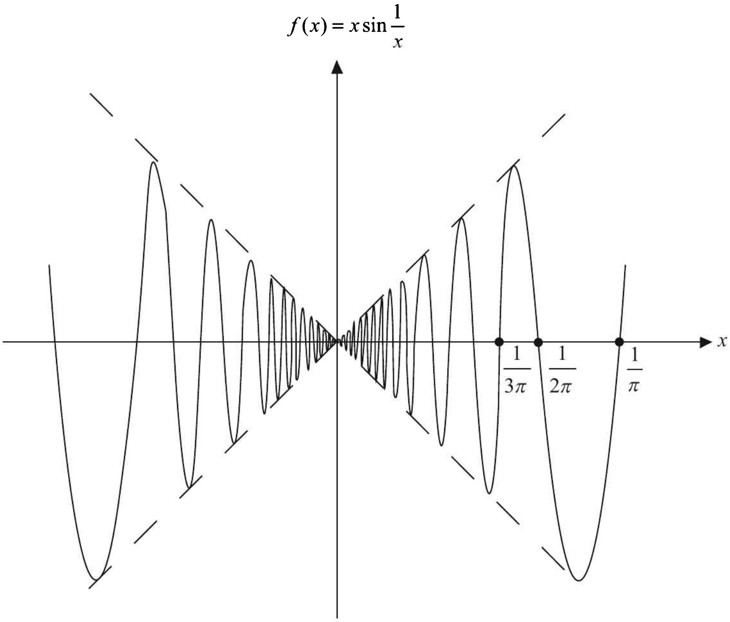
Let $f(x)=\left\{\begin{smallmatrix} x\sin\frac{1}{x}, & x\neq 0 \\ 0 & , x=0\end{smallmatrix}\right\}$ and $g(x)=\left\{\begin{smallmatrix} x^2\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0\end{smallmatrix}\right\}$. Evaluate the continuity and differentiability of $f(x)$ and $g(x)$.

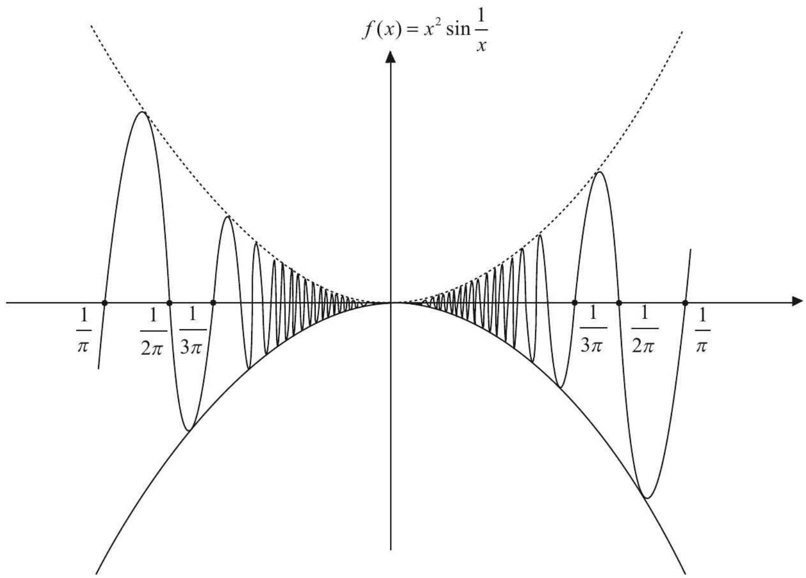
Solution: We will first try to graphically understand the behaviour of these two functions and then verify our results analytically. Notice that no matter what the argument of the sin function is, its magnitude will always remain between –1 and 1. Therefore,

$$\left|x\sin\frac{1}{x}\right| \le |x| \text{ and } \left|x^2\sin\frac{1}{x}\right| \le |x|^2$$

This means that the graph of $x\sin\frac{1}{x}$ will always lie between the lines $y=\pm x$ and the graph of $x^2\sin\frac{1}{x}$ will always lie between the two curves $y=\pm x^2$. Also, notice that as $|x|$ increases, $\frac{1}{x}$ decreases in a progressively slower manner while when $|x|$ is close to 0, the increase in $\frac{1}{x}$ is very fast (as $|x|$ decreases, visualise the graph of $y=\frac{1}{x}$). This means that near the origin, the variation in the graph of $\sin\frac{1}{x}$ will be extremely rapid because the successive zeroes of the function will become closer and closer. As we keep on increasing x, the variation will become

slower and slower and the graph will 'spread out'. For example, for $x > \frac{1}{\pi}$ there will be no finite zero of the function. Only when $x \to \infty$ will $\sin\frac{1}{x}$ again approach 0.





The two figures above show the approximation variation we should expect for these two functions (these graphs are highly exaggerated to emphasize our point; to see an accurate plot, you can use a number of plotting tools available online). Notice how the lines $y = \pm x$ *envelope* the graph of the function in the first case and the curves $y = \pm x^2$ *envelope* the graph of the function in the second case. The envelopes shrink to zero vertical width at the origin in both cases. Therefore, we must have:

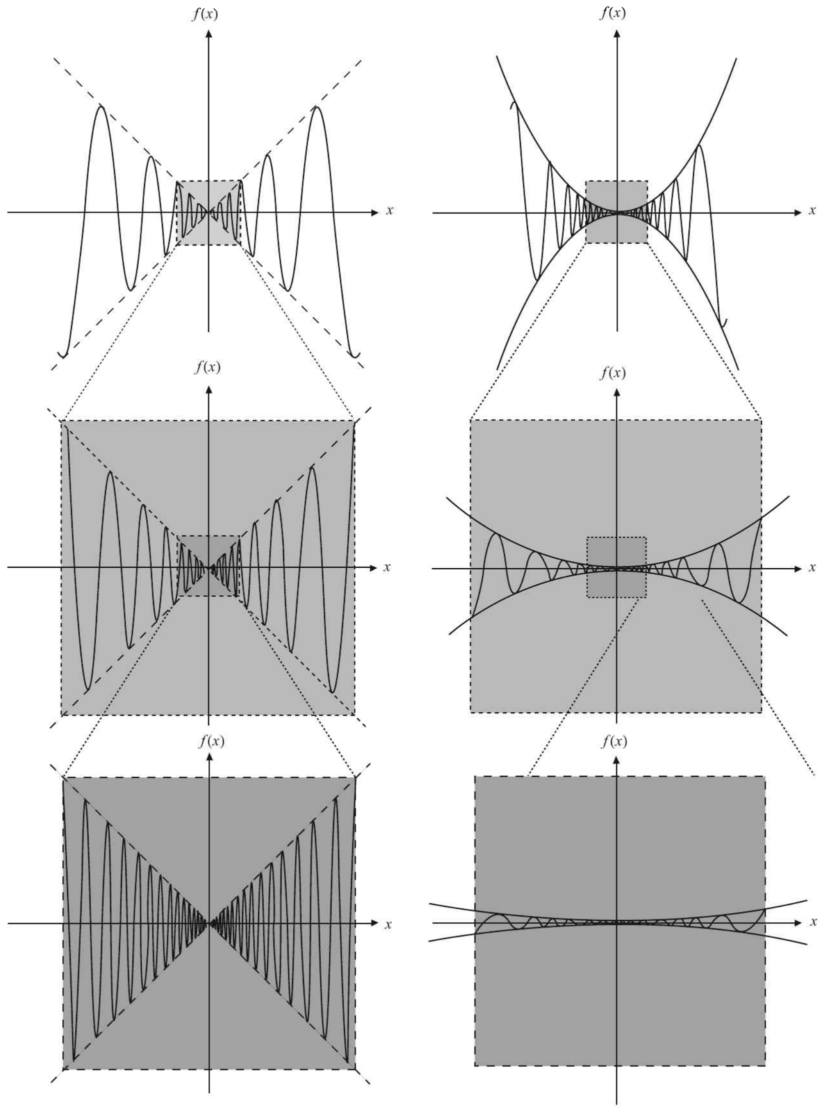
$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 \text{ and } \lim_{x\to 0} x^2\sin\frac{1}{x} = 0$$

This is also analytically obvious; $\sin\frac{1}{x}$ is a finite number between –1 and 1; when it gets multiplied by x (where $x \to 0$), the whole product gets infinitesmally small.

Now lets try to get an intuitive understanding on what will happen to the derivatives of these two functions at the origin. For $f(x) = x\sin\frac{1}{x}$, the slope of the envelope is constant (± 1). Thus, the sinusoidal function inside the envelope will keep on oscillating as we approach the origin, while shrinking in width due to the shrinking envelope. The slope of the curve also keeps on changing and does not approach a fixed value.

However, for $g(x) = x^2\sin\frac{1}{x}$, the slope of the envelope is itself decreasing as we approach the origin, apart from shrinking in width. This envelope will *compress* or *hammer out* or *flatten* the sin oscillations near the origin. What should therefore happen to the derivative? It should become 0 at the origin!

Let us 'zoom in' on the graphs of both the functions around the origin, to get more insight into the situation:



These graphs are not very accurate and are only of an approximate nature; but they do give us a lot of insight on the behaviour of these two functions near the origin. The $x\sin\frac{1}{x}$ graph keeps 'continuing' in the same manner no matter how much we zoom in; however, in the $x^2\sin\frac{1}{x}$ graph, the decreasing slope of the envelope itself tends to flatten out the curve and make its slope tend to 0. Hence, the derivative of $x\sin\frac{1}{x}$ at the origin will not have any definite value, while the derivative of $x^2\sin\frac{1}{x}$ will be 0 at the origin . Lets verify this analytically:

(i) $\boxed{f(x) = x\sin\frac{1}{x} \text{ at } x = 0}$

$$\text{LHD} = \text{RHD} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin\frac{1}{h}}{h} = \lim_{h\to 0}\left(\sin\frac{1}{h}\right)$$

This limit, as we know, does not exist; hence, the derivative for $f(x)$ does not exist at $x = 0$.

(ii) $\boxed{f(x) = x^2\sin\frac{1}{x} \text{ at } x = 0}$

$$\text{LHD=RHD} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$$ ■

Example 33

Let $\alpha \in \mathbb{R}$. Prove that a function $f:\mathbb{R}\to\mathbb{R}$ is differentiable at α if and only if there exists a function $g:\mathbb{R}\to\mathbb{R}$ which is continuous at α and satisfies $f(x)-f(\alpha) = g(x)(x-\alpha)$ for all $x\in\mathbb{R}$.

Solution: The proof that we seek is two-way that is, when we say that

(P is true) if and only if (Q is true)

we mean that

(P is true) implies (Q is true)

and(Q is true) implies (P is true)

For this question, we first assume the existence of a function $g(x)$ where $g(x)$ satisfies

$$f(x)-f(\alpha) = g(x)(x-\alpha)$$

and $g(x)$ is continuous at $x = \alpha$. Due to this continuity, $\lim_{x\to\alpha} g(x)$ exists

But, $$\lim_{x\to\alpha} g(x) = \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha} = \lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h} = f'(\alpha)$$

Hence, $f'(\alpha)$ exists or $f(x)$ is differentiable at $x = \alpha$. The other way proof is left as an exercise to the reader. ■

Example 34

If $x^2+y^2 = t+\frac{1}{t}$ and $x^4+y^4 = t^2+\frac{1}{t^2}$, then prove that $\frac{dy}{dx} = \frac{-1}{x^3y}$.

Solution: We first try to use the two given relations to get rid of the parameter t, so that we obtain a (implicit) relation between x and y.

$$x^2+y^2=t+\frac{1}{t}$$

Squaring, we get

$$x^4+y^4+2x^2y^2=t^2+\frac{1}{t^2}+2 \tag{1}$$

Using the second relation in (1), we get

$$2x^2y^2=2 \Rightarrow \quad y^2=\frac{1}{x^2}$$

Differentiating both sides w.r.t x, we get

$$2y\frac{dy}{dx}=\frac{-2}{x^3} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{-1}{x^3y}$$

■

Example 35

If $y^2=a^2\cos^2 x+b^2\sin^2 x,$ then prove that

$$\frac{d^2y}{dx^2}+y=\frac{a^2b^2}{y^3}$$

Solution: The final relation that we need to obtain is independent of sin x and cos x; this gives us a hint that using the given relation, we must first get rid of sin x and cos x:

$$\begin{aligned} y^2 &= a^2\cos^2 x+b^2\sin^2 x \\ &= \frac{1}{2}\{a^2(2\cos^2 x)+b^2(2\sin^2 x)\} \\ &= \frac{1}{2}\{a^2(1+\cos 2x)+b^2(1-\cos 2x)\} \\ &= \frac{1}{2}\{(a^2+b^2)+(a^2-b^2)\cos 2x\} \end{aligned}$$

$$\Rightarrow \quad 2y^2-(a^2+b^2)=(a^2-b^2)\cos 2x \tag{1}$$

Differentiating both sides of (1) with respect to x, we get

$$4y\frac{dy}{dx}=-2(a^2-b^2)\sin 2x$$

$$\Rightarrow \quad -2y\frac{dy}{dx}=(a^2-b^2)\sin 2x \tag{2}$$

We see now that squaring (1) and (2) and adding them will lead to an expression independent of the trigonometric terms:

$$(2y^2-(a^2+b^2))^2+4y^2\left(\frac{dy}{dx}\right)^2=(a^2-b^2)^2$$

A slight rearrangement gives:

$$\left(\frac{dy}{dx}\right)^2+y^2-(a^2+b^2)=-\frac{a^2b^2}{y^2} \tag{3}$$

Differentiating both sides of (3) with respect to x, we have

$$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+2y\frac{dy}{dx}=\frac{2a^2b^2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}+y=\frac{a^2b^2}{y^3}$$

■

Limits, Continuity, Differentiability and Differentiation

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. If $\lim\limits_{x\to 0} \frac{a\sin^2 x + b\log\cos x}{x^4} = \frac{1}{2}$, the value of $a-b$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) None of these

P2. The value of $\lim\limits_{x\to 1} \frac{(\log(1+x)-\log 2)(3.4^{x-1}-3x)}{((7+x)^{1/3}-(1+3x)^{1/2})\sin \pi x}$ is

(A) $\frac{2}{3\pi}\left(\log 4 - \frac{1}{2}\right)$ (B) $\frac{4}{3\pi}\left(\log 4 - \frac{2}{3}\right)$ (C) $\frac{9}{4\pi}(\log 4 - 1)$ (D) None of these

P3. Let $f(x)$ and $g(x)$ be two functions such that

$$f(x) = x^2 g(x), \quad \lim_{x\to 0} g(x)\operatorname{cosec} x = \frac{f'''(0)}{\lambda}$$

The value of λ is

(A) 2 (B) 3 (C) 4 (D) 6 (E) None of these

P4. Let $f(x)$ be a function such that

$$\lim_{t\to x} \frac{t^k f(x) - x^k f(t)}{t^k - x^k} = x^k, \quad f(1) = 1$$

The function $f(x)$ is given by

(A) $x^k \ln\left(\frac{e}{x^k}\right)$ (B) $x^{k-1} \ln\left(\frac{e}{x^{k+1}}\right)$ (C) $x^{k+1} \ln\left(\frac{e}{x^{k-1}}\right)$ (D) none of these

P5. If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in \mathbb{R}$ and $xy \neq 1$, and $\lim\limits_{x\to 0} \frac{f(x)}{x} = 2$, the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$ (E) none of these

P6. If $\lim\limits_{h\to 0} \frac{\int_0^h f(x)dx - h(af(0) + bf(\frac{h}{3}) + cf(h))}{h^4}$ is a finite non-zero number, the value of $a + 2b + 3c$ is

(A) $\frac{5}{4}$ (B) $\frac{7}{4}$ (C) $\frac{9}{4}$ (D) $\frac{11}{4}$

P7. Consider a function $f(x)$ which has exactly two roots at $x = a$. If

$$\lim_{x \to a} \left(\frac{\alpha f'(x)}{f(x)} - \frac{1}{x-a} \right) = m(\neq 0)$$

the value of α is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

P8. If $f'(1) = f''(1) \neq 0$, the value of $\lim\limits_{h \to 0} \frac{f(1+2h+h^2)-f(1)-2hf'(1)}{f(1+h)-f(1)-hf'(1)}$ is

(A) 3 (B) 4 (C) 5 (D) 6

P9. The value of $\lim\limits_{x \to a} \frac{\int_0^x f(t)\,dt - \frac{x-a}{6}\left(f(a)+4f\left(\frac{a+x}{2}\right)+f(x)\right)}{(x-a)^5}$ is

(A) $-\frac{f'''(a)}{4!5!}$ (B) $-\frac{2f'''(a)}{4!5!}$ (C) $\frac{f'''(a)}{4!5!}$ (D) $\frac{2f'''(a)}{4!5!}$

P10. (a) The value of $\lim\limits_{x \to 0^+} \left[\frac{x^2 \ln(\cos x)}{(\cos x - 1)^2}\right]$ is

(A) -1 (B) -2 (C) -3 (D) -4

(b) $\lim\limits_{x \to +\infty} \left[(x+2) \ln\left(\frac{x}{1+x} \right) \right]$

(A) -1 (B) -2 (C) -3 (D) 0

In both the problems, [ ] represents the greatest integer function.

P11. A function $f:(-1,1) \to \mathbb{R}$ is given by

$$f(x) = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots \infty$$

Let i be the number of solutions of the system

$$y = f(x)$$

$$(x-1)^2 + y^2 = 2$$

Let α_r be the roots of this system. Find the value of $\sum_{r=1}^{i} [\alpha_r]$.

P12. In terms of a parameter t and a constant k, a curve is specified as follows:

$$x = t + \frac{1}{t}, \quad y = t^k + \frac{1}{t^k}$$

If $\left(\frac{dy}{dx}\right)^2 = \lambda\left(\frac{y^2-4}{x^2-4}\right)$, the value of λ in terms of k is

(A) k^2 (B) $2k^2$ (C) $\frac{k^2}{2}$ (D) None of these

P13. Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} ae^x \sin x, & x < 0 \\ x^2 + b, & x \geq 0 \end{cases}$$

Suppose that $f'(0)$ exists.
The value of $a + b + f'(0)$ is

(A) 1 (B) 2 (C) 3 (D) None of these

SUBJECTIVE TYPE EXAMPLES

P14. (a) Find $\lim\limits_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+\cdots+a_n^{1/x}}{n}\right)^{nx}$ where all the a_i's are positive.

(b) Evaluate $\lim\limits_{x\to 0^+}\frac{\log_{\sin x}\cos x}{\log_{\sin x/2}\cos(x/2)}$

P15. Let $f(x)$ be defined as $f(x)=\lim\limits_{n\to\infty}\frac{e^x-x^{2n}\sin x}{1+x^{2n}}$. By the intermediate value theorem, since $f(0)=1$, $f(\frac{\pi}{2})=-1$, this means that $f(x)$ must have a root in $(0,\frac{\pi}{2})$. Is this correct?

P16. Let $f:[-1,1]\to R$, $f'(0)=\lim\limits_{n\to\infty} nf\,(\frac{1}{n})$ and $f(0)=0$. Find the value of $\lim\limits_{n\to\infty}(\frac{2}{\pi}(n+1)\cos^{-1}(\frac{1}{n})-n)$.

P17. Evaluate $\lim\limits_{x\to 0}\frac{1}{x^{k+1}}\int_0^x(x^k+x^{k-1}y+x^{k-2}y^2+\cdots+y^k)f(y)\,dy$.

P18. A function $f(x)$ is defined as:

$$f(x)=\begin{cases}-x^2, & x\le 0\\ 5x-4, & 0<x\le 1\\ 4x^2-3x, & 1<x\le 2\\ 3x+5, & x>2\end{cases}.$$

Discuss the continuity and differentiability of $f(x)$.

P19. Let $f(x)=\{\begin{smallmatrix}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{smallmatrix}\}$ and $g(x)=f(|x|)+|f(x)|$. Evaluate the continuity and differentiability of $g(x)$ in the interval $[-2, 2]$ by drawing the graph.

P20. Let $f(x)=\{\begin{smallmatrix}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0\\ 0, & x=0\end{smallmatrix}\}$. Evaluate the continuity and differentiability of $f(x)$

P21. Let a function $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)\ne 0$ for any $x\in\mathbb{R}$. If the function $f(x)$ is differentiable at $x=0$, show that $f'(x)=f'(0)f(x)$ for all $x\in\mathbb{R}$.

P22. If $f(xy)=f(x)+f(y)$ for all $x,y>0$ and $f(x)$ is differentiable at $x=1$, then prove that $f(x)$ is differentiable for all $x>0$.

P23. Let $f(x)=\{\begin{smallmatrix}x+a, & x<0\\ |x-1|, & x\ge 0\end{smallmatrix}\}$ and $g(x)=\{\begin{smallmatrix}x+1, & x<0\\ (x-1)^2+b, & x\ge 0\end{smallmatrix}\}$, where a and b are non-negative real numbers.

(a) Determine the composite function $g(f(x))$.
(b) If $g(f(x))$ is continuous for all real x, determine the values of a and b.
(c) For these values of a and b, will $g(f(x))$ be differentiable at $x=0$?

P24. Let $f(x)$ be the function

$$f(x)=\begin{cases}2x^2+12x+16, & -4\le x\le -2\\ 2-|x|, & -2<x\le 1\\ 4x-x^2-2, & 1<x\le 3\end{cases}$$

Plot $[f(x)]$ (where $[\sqrt{\ }]$ represents the greatest integer function) and discuss its continuity.

P25. Let $f(x)$ be defined as

$$f(x)=\begin{cases}x^{\frac{k^2-3k+2}{k^2-5k+4}}\cos\left(\dfrac{1}{x}\right), & x\ne 0\\ 0, x=0\end{cases}$$

For what values of k is $f(x)$ differentiable at $x = 0$?

P26. Let $f(x)$ be a continuous and differentiable function around $x = a$.

(a) Find $\lim\limits_{x \to a}\{\frac{f'(x)}{f(x)-f(a)} - \frac{1}{x-a}\}$.

(b) Find $\lim\limits_{x \to a}\{\frac{f''(x)}{2\{f(x)-f(a)-(x-a)f'(a)\}} - \frac{1}{(x-a)^2}\}$.

P27. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ f or all $x \geq 1$, then prove that $P(x) > 0$ for all $x > 1$.

P28. If $f'(x) = g(x)$, find the derivative of $f^{-1}(x)$.

P29. If $ax^2 + 2hxy + by^2 = 1$, then find h such that

$$\frac{d^2y}{dx^2} = 0 \;\forall\; x, y$$

Limits, Continuity, Differentiability and Differentiation

PART-D: Solutions to Advanced Problems

OBJECTIVE TYPE EXAMPLES

S1. Using the LH rule, we have

$$\lim_{x\to 0} \frac{a\sin 2x - b\tan x}{4x^3} = \frac{1}{2}$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{a(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots) - b(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots)}{4x^3} = \frac{1}{2}$$

$$\Rightarrow \quad a = -1, \; b = -2$$

$$\Rightarrow \quad a - b = 1$$

The correct option is (B).

S2. The limit L is the form $\frac{0}{0}$; we let $x = 1 + h$, so that as $x \to 1$, $h \to 0$, and in terms of h, L can be expressed as follows:

$$L = \lim_{h\to 0} \frac{(\log(2+h) - \log 2)(3\cdot 4^h - 3 - 3h)}{((8+h)^{\frac{1}{3}} - (4+3h)^{\frac{1}{2}})(-\sin \pi h)}$$

$$= \lim_{h\to 0} \frac{3\left\{\frac{1}{2}\frac{\log(1+\frac{h}{2})}{\frac{h}{2}}\right\}\left\{\frac{4^h - 1}{h} - 1\right\}}{\left\{\frac{(8+h)^{\frac{1}{3}} - 8^{\frac{1}{3}}}{h} - 3\frac{(4+3h)^{\frac{1}{2}} - 4^{\frac{1}{2}}}{3h}\right\}\left\{-\pi\frac{\sin \pi h}{\pi h}\right\}}$$

$$= \frac{3\cdot\frac{1}{2}\cdot(\log 4 - 1)}{(\frac{1}{3}\cdot\frac{1}{4} - 3\cdot\frac{1}{2}\cdot\frac{1}{2})(-\pi)} = \frac{\frac{3}{2}(\log 4 - 1)}{\frac{2\pi}{3}} \qquad = \frac{9}{4\pi}(\log 4 - 1)$$

The correct option is (C).

S3. We have

$$\lim_{x\to 0} g(x)\operatorname{cosec} x = \lim_{x\to 0}\frac{g(x)}{\sin x} = \lim_{x\to 0}\frac{f(x)}{x^2 \sin x} = \lim_{x\to 0}\frac{f(x)}{x^3}$$

Using the LH rule thrice, this equals $\frac{f'''(0)}{6} \Rightarrow \lambda = 6.$ The correct option is (D).

S4. Using the LH rule, we have

$$\lim_{t\to x}\frac{k t^{k-1} f(x) - x^k f'(t)}{k t^{k-1}} = x^k$$

$$\Rightarrow \quad f(x) - \frac{x}{k} f'(x) = x^k$$

This is a first order linear differential equation, which can be easily solved, to obtain $y = x^k \ln(\frac{e}{x^k})$. The correct option is (A). Although this question involves solving a differential equation, it was included here as a demonstration of the application of the LH rule.

S5. We note that $f(0) = 0$ and $f(x) = -f(-x)$. We now write the expression for the derivative at any point x:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x+h) + f(-x)}{h}$$

$$= \lim_{h\to 0}\frac{f(\frac{h}{1+x(x+h)})}{h} \qquad \left\{\begin{array}{l}\text{Using the given relation}\\ \text{satisfied by } f(x)\end{array}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{1}{1+x(x+h)}\cdot\left(\frac{f(\frac{h}{1+x(x+h)})}{\frac{h}{1+x(x+h)}}\right)\right\}$$

$$= \frac{1}{1+x^2}\cdot\lim_{y\to 0}\frac{f(y)}{y} \qquad \left\{\text{where } y = \frac{h}{1+x(x+h)}\right\}$$

$$= \frac{2}{1+x^2} \qquad \left\{\text{it is give that } \lim_{x\to 0}\frac{f(x)}{x} = 2\right\}$$

Thus,

$$\frac{df}{dx} = \frac{2}{1+x^2} \Rightarrow \quad f(x) = 2\tan^{-1} x + C$$

But $f(0) = 0$ gives $C = 0$:

$$\Rightarrow \quad f(x) = 2\tan^{-1} x \qquad \Rightarrow \quad f\left(\frac{1}{\sqrt{3}}\right) =$$

The correct option is (C).

S6. Using the LH rule successively, we have

$$L = \lim_{h\to 0} \frac{f(h) - h(\frac{b}{3} f'\left(\frac{h}{3}\right) + cf'(h)) - (af(0) + bf(\frac{h}{3}) + cf(h))}{4h^3} \tag{1}$$

$$= \lim_{h\to 0} \frac{f'(h) - h(\frac{b}{9} f''(\frac{h}{3}) + cf''(h)) - 2(\frac{b}{3} f'(\frac{h}{3}) + cf'(h))}{12h^2} \tag{2}$$

$$= \lim_{h\to 0} \frac{f''(h) - h(\frac{b}{27} f'''\left(\frac{h}{3}\right) + cf'''(h)) - 3(\frac{b}{9} f''(\frac{h}{3}) + cf''(h))}{24h} \tag{3}$$

$$= \lim_{h\to 0} \frac{(f'''(h) - h(\frac{b}{81} f''''(\frac{h}{3}) + cf''''(h)) - 4(\frac{b}{27} f'''\left(\frac{h}{3}\right) + cf'''(h)))}{24} \tag{4}$$

In each of (1), (2) and (3), the numerator must tend to 0 for the limit to be finite. This gives:

$$a + b + c = 1, \quad 1 - \frac{2b}{3} - 2c = 0, \quad 1 - \frac{b}{3} - 3c = 0$$

$$\Rightarrow \quad a = 0, \quad b = \frac{3}{4}, \quad c = \frac{1}{4} \quad \Rightarrow \quad a + 2b + 3c = \frac{9}{4}$$

The correct option is (C).

S7. Since $f(x)$ has exactly two roots at $x = a$, it can be written as $(x-a)^2 g(x)$ for some polynomial $g(x)$. Thus,

$$f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{2}{x-a} + \frac{g'(x)}{g(x)}$$

Now, $\frac{\alpha f'(x)}{f(x)} - \frac{1}{x-a} = \frac{2\alpha - 1}{x-a} + \alpha \frac{g'(x)}{g(x)}$. For the limit of this expression to exist at $x = a$, we must have $2\alpha - 1 = 0$, that is, $\alpha = \frac{1}{2}$. The correct option is (B).

S8. Using the LH rule, we have

$$L = \lim_{h\to 0} \frac{(2+2h)f'(1+2h+h^2) - 2f'(1)}{f'(1+h) - f'(1)}$$

$$= \lim_{h\to 0} \frac{(2+2h)^2 f''(1+2h+h^2) + 2f'(1+2h+h^2)}{f''(1+h)} = \frac{4f''(1) + 2f'(1)}{f''(1)} = 6$$

The correct option is (D).

S9: Using the LH rule successively, we have

$$L = \lim_{x\to a} \frac{f(x) - \frac{x-a}{6}(2f'(\frac{a+x}{2}) + f'(x)) - \frac{1}{6}(f(a) + 4f(\frac{a+x}{2}) + f(x))}{5(x-a)^4}$$

$$= \lim_{x\to a} \frac{f'(x) - \frac{x-a}{6}(f''(\frac{a+x}{2}) + f''(x)) - \frac{2}{6}(2f'(\frac{a+x}{2}) + f'(x))}{20(x-a)^3}$$

Continuining this way, we reach

$$L = \lim_{x\to a} \frac{f'''(x) - \frac{x-a}{6}\left(\frac{1}{8}f'''\left(\frac{a+x}{2}\right) + f'''(x)\right) - \frac{5}{6}\left(\frac{1}{4}f'''\left(\frac{a+x}{2}\right) + f'''(x)\right)}{120}$$

$$= \frac{f'''(a) - \frac{5}{6}\left(\frac{1}{4}f'''(a) + f'''(a)\right)}{120} = -\frac{f'''(a)}{4!5!}$$

The correct option is (A).

S10. Since the greatest integer function is involved, we'll need to find the direction of approach of the function (which is the argument of the greatest integer function) to its limiting value, in addition to the limiting value itself.

(a) Noting that $1-\cos x = 2\sin^2\frac{x}{2}$, we have

$$L = \lim_{x\to 0^+}\left[\frac{x^2 \ln(1-2\sin^2\frac{x}{2})}{4\sin^4\frac{x}{2}}\right]$$

$$= \lim_{x\to 0^+}\left[2\underbrace{\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2}_{L_1}\underbrace{\frac{\ln(1-2\sin^2\frac{x}{2})}{2\sin^2\frac{x}{2}}}_{L_2}\right]$$

If we apply the limit $(x\to 0^+)$ to L_2, we have

$$\lim_{x\to 0^+} L_2 = \lim_{y\to 0^+}\frac{\ln(1-y)}{y} \qquad \left(y = 2\sin^2\frac{x}{2}\right)$$

$$= \lim_{y\to 0^+}\left(-1-\frac{y}{2}-\frac{y^2}{3}-\cdots\right)$$

$$= -1^-$$

Also, as $x\to 0^+$, $L_1\to 1^+$. Thus,

$$2L_1L_2 \to -2^- \quad \text{as} \quad x\to 0^+ \quad \Rightarrow \quad L = \lim_{x\to 0^+}[2L_1L_2] = -3$$

The correct option is (C).

(b) We have

$$L = \lim_{x\to\infty}\left[\underbrace{\left(\frac{x+2}{x+1}\right)}_{L_1}\underbrace{\left(\frac{\ln(1-\frac{1}{x+1})}{\frac{1}{x+1}}\right)}_{L_2}\right]$$

Once again, as $x\to\infty$, $L_1\to 1^+$ while $\frac{1}{x+1}\to 0^+$, so $L_2\to -1^-$. Thus,

$$L_1L_2 \to -1^- \text{ as } x\to 0^+ \quad \Rightarrow \quad L = \lim_{x\to\infty}[L_1L_2] = -2$$

The correct option is (B).

S11. We observe that

$$(1-x)\,(1+x)\,(1+x^2)\,(1+x^4)\ldots\infty = \lim_{n\to\infty}(1-x^{2^n}) = 1 \quad (\text{since } |x|<1)$$

Thus,

$$(1+x)\,(1+x^2)\,(1+x^4)\ldots\infty = \frac{1}{1-x}$$

We now take the log of both sides and differentiate:

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots\infty = \frac{1}{1-x}$$

Thus, $f(x)=\frac{1}{1-x}$. This is a hyperbola, which touches the given circle at $(2,-1)$ and $(0,1)$ (verify).

$$\Rightarrow \quad \sum_{r=1}^{i}[\alpha_r] = 0+2 = 2$$

S12. We have

$$\frac{dx}{dt} = 1-t^{-2} = t^{-1}(t-t^{-1}) = t^{-1}\sqrt{(t-t^{-1})^2-4} = t^{-1}\sqrt{x^2-4}$$

$$\frac{dy}{dt} = kt^{k-1}-k\,t^{-k-1} = k\,t^{-1}\,(t^k-t^{-k}) = k\,t^{-1}\,\sqrt{y^2-4}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = k^2\left(\frac{y^2-4}{x^2-4}\right) \Rightarrow \quad \lambda = k^2$$

The correct option is (A).

S13. Since $f'(0)$ exists, the function $f(x)$ is both continuous and differentiable at $x = 0$. This implies that

$$f(0^-) = f(0^+) \quad \Rightarrow \quad 0 = b$$

$$f'(0^-) = f'(0^+) \quad \Rightarrow \quad a = 0$$

Thus, $f(x)=0$ for $x<0$ and x^2 for $x\ge 0$. Also, we have $a+b=0$ and $f'(0)=0$. Thus, the value of $a+b+f'(0)$ is equal to 0, so none of the first three options is correct. The correct choice is therefore (D).

SUBJECTIVE TYPE EXAMPLES

S14. **(a)** The given limit L is of the form 1^∞. Thus,

$$L = e^{\lim\limits_{x\to\infty} nx\left\{\frac{a_1^{1/x}+a_2^{1/x}+\ldots+a_n^{1/x}}{n}-1\right\}} = e^{\lim\limits_{x\to\infty}\left\{\frac{\left(a_1^{\frac{1}{x}}-1\right)+\left(a_2^{\frac{1}{x}}-1\right)+\cdots+\left(a_n^{\frac{1}{x}}-1\right)}{\frac{1}{x}}\right\}}$$

$$= e^{(\ln a_1+\ln a_2+\ldots+\ln a_n)} = a_1\, a_2\, a_3 \ldots a_n$$

(b) The given limit L is of the form $\frac{0}{0}$:

$$L = \lim_{x\to 0^+} \frac{\ln\cos x}{\ln\cos(\frac{x}{2})}\cdot\frac{\ln\sin(\frac{x}{2})}{\ln\sin x}$$

$$= \lim_{x\to 0^+} \frac{2\tan x}{\tan(\frac{x}{2})}\cdot\frac{\cot(\frac{x}{2})}{2\cot x} \quad \{\text{Using the LH rule on the two limits separately}\}$$

$$= \lim_{x\to 0^+} \frac{\tan^2 x}{\tan^2\frac{x}{2}} = 4$$

S15. $f(x)$ is discontinuous at $x = 1$, so the intermediate value theorem does not apply. Proving that $f(x)$ is discontinuous is left to the reader as an exercise.

S16. If we denote the limit we wish to evaluate by L, then

$$L = \lim_{n\to\infty} n\left(\frac{2}{\pi}(1+\tfrac{1}{n})\cos^{-1}(\tfrac{1}{n})-1\right)$$

$$= \lim_{n\to\infty} nf\left(\frac{1}{n}\right) \quad \{\text{where } f(x) = \frac{2}{\pi}(1+x)\cos^{-1}x-1\}$$

Since $f(0) = 0$, we have $L = f'(0) = 1-\frac{2}{\pi}$.

S17. Using the substitution $y = tx$, the limit reduces to

$$\lim_{x\to 0}\frac{1}{x^{k+1}}\int_0^1\left(\sum_{i=0}^{k}(x^i(tx)^{k-i})\right)f(tx)(x\,dt) = \lim_{x\to 0}\int_0^1\left(\sum_{i=0}^{k}t^{k-i}\right)f(tx)\,dt$$

$$= f(0)\int_0^1(1+t+t^2+\cdots+t^k)\,dt$$

$$= f(0)\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right)$$

S18. The critical points for this function are $x = 0, 1, 2$. Lets analyse $f(x)$ for each of these critical points separately.

(i) $\boxed{x=0 \quad \{f(0)=0\}}$

$$\text{LHL} = \lim_{x\to 0^-}(-x^2) = 0 \qquad \text{LHD} = \lim_{h\to 0}\frac{-(-h)^2-0}{-h} = 0$$

$$\text{RHL} = \lim_{h\to 0^+}(5x-4) = -4 \qquad \text{RHD} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{5h-4}{h} = -\infty$$

Therefore, this function is non-continuous (and non-differentiable) at $x = 0$.

(ii) $\boxed{x=1 \;\; \{f(1)=1\}}$

$$\text{LHL} = \lim_{x\to 1^-}(5x-4) = 1 \qquad \text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{5(1-h)-4-1}{-h} = 5$$

$$\text{RHL} = \lim_{h\to 1^+}(4x^2-3x) = 1 \qquad \text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{4(1+h)^2-3(1+h)-1}{h} = 5$$

Since LHL = RHL and LHD = RHD, $f(x)$ is continuous and differentiable at $x = 1$.

(iii) $\boxed{x=2\{f(2)=10\}}$

$$\text{LHL} = \lim_{x\to 2^-}(4x^2-3x) = 10 \qquad \text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{h}$$

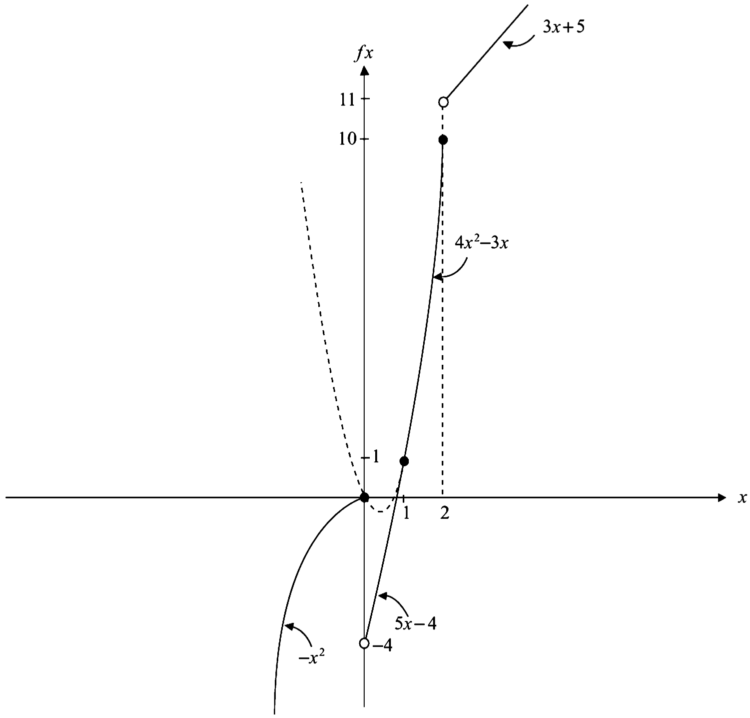
$$= \lim_{h\to 0}\frac{\{4(2-h)^2-3(2-h)\}-10}{-h}$$

$$= 13$$

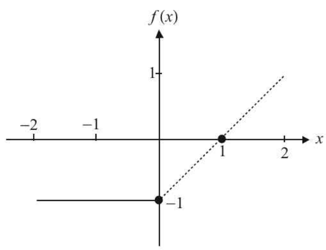
$$\text{RHL} = \lim_{x\to 2^+}(3x+5) = 11 \qquad \text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

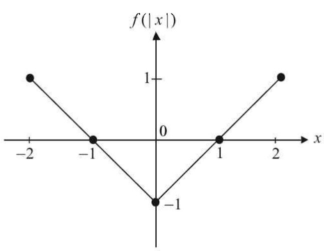
$$= \lim_{h\to 0}\frac{\{3(2+h)+5\}-10}{h} = \infty$$

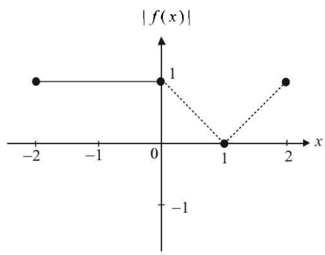
We see that $f(x)$ is non-continuous (and non-differentiable) at $x = 0$ and $x = 2$. The graph is plotted below:

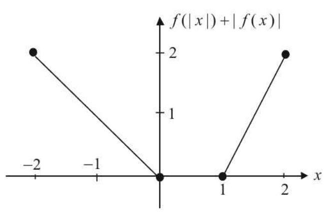


S19. From the graph of $f(x)$, we can easily derive the graphs of $f(|x|)$ and $|f(x)|$, and add them point by point to get the graph of $g(x)$.









Verify for yourself the result of the addition of the two graphs. It is obvious from the resultant graph that $g(x)$ is continuous everywhere but non-differentiable at $x = 0$ and $x = 1$.

S20. We should first of all write $f(x)$ separately for $x > 0$ and $x < 0$; using $|x| = \left\{\begin{smallmatrix} x\ ,\ x>0 \\ -x, x<0 \end{smallmatrix}\right\}$, we have

$$f(x) = \left\{\begin{matrix} xe^{-\frac{2}{x}} & x > 0 \\ 0 & x = 0 \\ x & x < 0 \end{matrix}\right\}$$

At this juncture, we do not have sufficient knowledge to accurately plot the graph for $x > 0$ (that involves more knowledge of derivatives and graph-plotting - which will be covered in the chapter on *Applications of Derivatives*); we will hence follow an analytical approach. The critical point is only $x = 0$:

$$\text{LHL(at } x = 0) = \lim_{x \to 0^-} (x) = 0 \qquad \text{RHL(at } x = 0) = \lim_{x \to 0^+} (xe^{\frac{-2}{x}}) = 0$$

$$\text{LHD(at } x = 0) = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = 1 \qquad \text{RHD(at } x = 0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{he^{-2/h}}{h} = 0$$

Therefore, this function is continuous at $x = 0$ (and everywhere else), but not differentiable at $x = 0$.

S21. Substituting $x = y = 0$ in the given relation, we get

$$f(0) = f(0)^2 \Rightarrow f(0) = 1 \; \{\text{since } f(x) \neq 0 \text{ for any } x\}$$

It is given that $f(x)$ is differentiable at $x = 0$, *i.e.*, $f'(0)$ exists.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - 1}{h} \qquad (1)$$

Now we write down the expression for $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$

$$= f(x). \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(x) \cdot f'(0) \; \{\text{from (1)}\}$$

$$\Rightarrow \quad f'(x) = f'(0) \cdot f(x) \text{ for all } x$$

S22. Substituting $x = y = 1$, we get $f(1) = 0$. Since $f(x)$ is differentiable at $x = 1$, $f'(1)$ exists:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad = \lim_{h \to 0} \frac{f(1+h)}{h} \qquad (1)$$

Now,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

To evaluate $f'(x)$, we have to somehow manipulate its expression so that we are able to use the expression for $f'(1)$ we evaluated in (1). We do this as follows:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\{x(1+\frac{h}{x})\}-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left\{f(x)+f\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}\qquad\left\{\begin{matrix}\text{Using the relation}\\ \text{given in the question}\end{matrix}\right\}$$

$$=\lim_{h\to 0}\frac{f(1+h/x)}{h}$$

$$=\lim_{h\to 0}\frac{f(1+\frac{h}{x})}{x\cdot\frac{h}{x}}\qquad\{\text{Introduction of } x \text{ in the denominator}\}$$

$$=\lim_{\theta\to 0}\frac{f(1+\theta)}{x\cdot\theta}\qquad\left\{\theta=\frac{h}{x}\right\}$$

$$=\frac{f'(1)}{x}\qquad\{\text{Using (1)}\}$$

Therefore, $f'(x)$ has a finite value for all $x>0$:

$$\Rightarrow\quad f(x)\ \text{is differentiable everywhere.}$$

S23. (a) Evaluation of the composition of piecewise defined functions can be tricky; hence follow the solution to this problem carefully.

$$g(f(x))=\begin{cases}f(x)+1 & f(x)<0\\ (f(x)-1)^2+b & f(x)\geq 0\end{cases}$$

The conditions '$f(x)<0$' and '$f(x)\geq 0$' have to be written in terms of x. Notice from the definition of $f(x)$ that $f(x)<0$ only when $x+a<0$, *i.e.*,

$$f(x)<0\Rightarrow x<-a.$$

So, $f(x)\geq 0$ when $x\geq -a$. But also notice that the definition of $f(x)$ changes at $x=0$. Hence, $g(f(x))$ can be rewritten as

$$g(f(x))=\begin{cases}f(x)+1 & x<-a\\ (f(x)-1)^2+b & x\geq -a\end{cases}=\begin{cases}x+a+1 & x<-a\\ (x+a-1)^2+b & -a\leq x<0\\ (|x-1|-1)^2+b & x\geq 0\end{cases}$$

$$=\begin{cases}x+a+1 & x<-a\\ (x+a-1)^2+b & -a\leq x<0\\ x^2+b & 0\leq x<1\\ (x-2)^2+b & 1\leq x\end{cases}$$

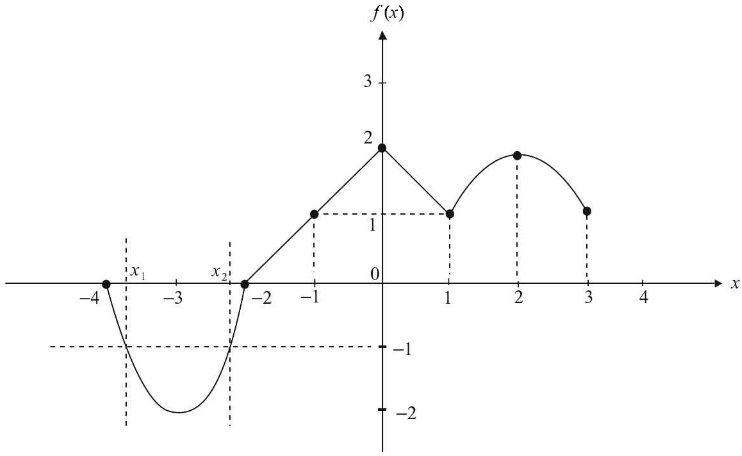
This is the 'simplified' definition of $g(f(x))$.

(b) Now our task is easy. We just need to equate LHL and RHL at each of the critical points $x = -a, 0, 1$ to find out a and b. The remaining part of this question is left as an exercise to you. The answers are:

$$a = 1 \quad \text{and} \quad b = 0$$

(c) For these values, $f(x)$ is non differentiable at $x = \pm 1$.

S24. We first draw the graph of $f(x)$ as accurately as possible:



To draw the required graph, notice that the value of $[f(x)]$ will change every time (the value of) $f(x)$ crosses an integer. For example, notice from the graph the following facts:

$$\text{When } x \in (-4, x_1], \quad 0 < f(x) \le -1 \Rightarrow [f(x)] = -1$$

$$\text{When } x \in (x_1, x_2), \quad -2 \le f(x) < -1 \Rightarrow [f(x)] = -2$$

$$\text{When } x = 0 \text{ or } x = 2, \quad f(x) = 2 \Rightarrow [f(x)] = 2$$

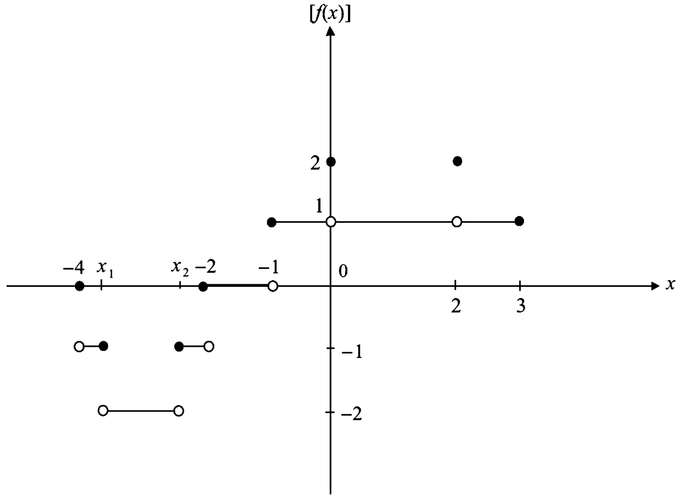
and so on. x_1 and x_2 can be evaluated by solving

$$f(x) = 2x^2 + 12x + 16 = -1 \quad \Rightarrow \quad 2x^2 + 12x + 17 = 0 \quad \Rightarrow \quad x = -3 \pm \frac{\sqrt{10}}{4}$$

$$\Rightarrow \quad x_1 = -3 - \frac{\sqrt{10}}{4}, x_2 = -3 + \frac{\sqrt{10}}{4}$$

The complete definition of $[f(x)]$, and its graph, is given below:

$$[f(x)] = \begin{cases} 0 & x = -4 \\ -1 & -4 < x \le x_1 \\ -2 & x_1 < x < x_2 \\ -1 & x_2 \le x < -2 \\ 0 & -2 \le x < -1 \\ 1 & -1 \le x < 0 \\ 2 & x = 0 \\ 1 & 0 < x < 2 \\ 2 & x = 2 \\ 1 & 2 < x \le 3 \end{cases}$$



This function is discontinuous at the following points:

$$x = -4,\ x_1,\ x_2,\ -2,\ -1,\ 0,\ 2$$

S25. For the function $x^{\lambda}\cos\left(\frac{1}{x}\right)$ to be differentiable at $x = 0$, λ must be greater than 1. This can be easily proven and is left to the reader as an exercise. Thus,

$$\frac{k^2-3k+2}{k^2-5k+4} > 1 \quad \Rightarrow \quad k > 4$$

S26. (a) By combining the two terms into one, the given limit becomes

$$\lim_{x\to a}\frac{f'(x)\,(x-a)-(f(x)-f(a))}{(x-a)(f(x)-f(a))} = \lim_{x\to a}\frac{f''(x)(x-a)}{f(x)-f(a)+(x-a)f'(x)} \text{ (LH rule)}$$

$$= \lim_{x\to a}\frac{f''(x)}{\frac{f(x)-f(a)}{x-a}+f'(x)} = \frac{f''(a)}{2f'(a)}$$

(b) Using a similar approach, we can obtain the limit in this case as $\frac{f'''(a)}{3f'(a)}$.

S27. If you think about this problem graphically, it is very straightforward to understand. Since $P(1) = 0$ and $P'(1) > 0$, $P(x)$ must be increasing at $x = 1$. To a little right of $x = 1$, $P(x)$ is positive, but $P'(x) > P(x)$, so $P(x)$ is still increasing. Continuining this reasoning, we see that $P(x)$ must be positive for all $x > 1$.

Now, we construct a rigorous argument:

$$P'(x)-P(x) > 0 \quad \Rightarrow \quad e^{-x}P'(x)-e^{-x}P(x) > 0 \quad \Rightarrow \quad (e^{-x}P(x))' > 0$$

Thus, $e^{-x}P(x)$ is increasing on $(1,\infty)$:

$$e^{-x}P(x) > e^{-1}P(1) = 0 \quad \Rightarrow \quad P(x) > 0$$

S28. To evaluate the required derivative, we can apply the chain rule on the following relation:

$$f(f^{-1}(x)) = x \quad \Rightarrow \quad \frac{d}{dx}\{f(f^{-1}(x))\} = \frac{d}{dx}(x) = 1$$

$$\Rightarrow \quad f'(f^{-1}(x))\frac{d}{dx}(f^{-1}(x)) = 1$$

$$\Rightarrow \quad \frac{d}{dx}(f^{-1}(x))\frac{1}{f'(f^{-1}(x))} = \frac{1}{g(f^{-1}(x))}$$

For example, we know that $\frac{d}{dx}(\sin x) = \cos x$:

$$\Rightarrow \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\cos(\cos^{-1}\sqrt{1-x^2})} = \frac{1}{\sqrt{1-x^2}}$$

In this way, we can evaluate the derivative of any inverse function, given the derivative of the original function.

S29. Differentiating twice, we obtain:

$$2a + 2h(2y' + xy'') + 2b((y')^2 + yy'') = 0$$

Using $y'' = 0$,

$$2a + 4hy' + 2b(y')^2 = 0$$

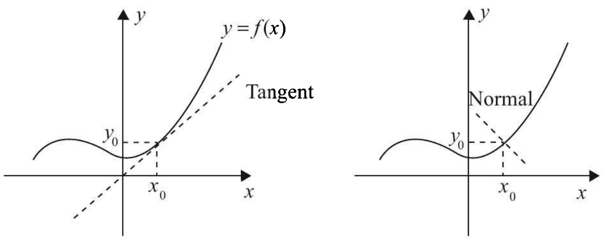
Since this is true for every x, y, and y' is a constant, $h^2 = ab$. In that case, a single solution for y' exists. The reader is urged to carefully understand this line of argument.

Applications of Derivatives

PART A: Summary of Important Concepts

1. Tangents and Normals

Consider a function $f(x)$ for which a tangent and a normal need to be drawn at $x = x_0$:



If we denote the slopes of the tangent and normal (at $x = x_0$) respectively as m_T and m_N, we have

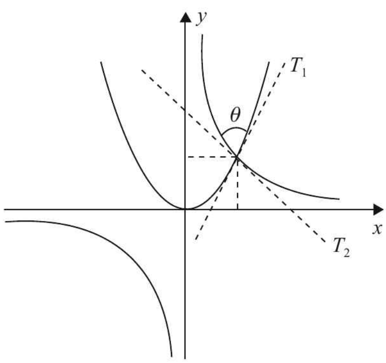
$$m_T = \left.\frac{dy}{dx}\right|_{x = x_0}, \quad m_N = -\left.\frac{dx}{dy}\right|_{x = x_0},$$

and the required equations are

$$\text{Tangent: } y - y_0 = m_T(x - x_0)$$

$$\text{Normal: } y - y_0 = m_N(x - x_0)$$

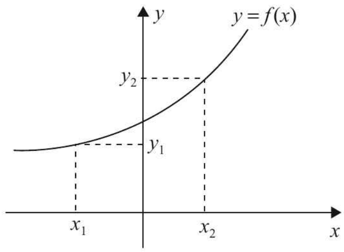
To find the angle of intersection of two curves, we simply evaluate the slopes of the tangents to the two curves at the point of intersection, and use them to find the angle θ at which the tangents intersect:



$$\theta = \tan^{-1}\left|\frac{m_{T_1} - m_{T_2}}{1 + m_{T_1} m_{T_2}}\right|$$

2. Monotonicity

A functions $y = f(x)$ is *monotonically increasing* (or *strictly increasing*) if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$:



$x_1 < x_2$
$y_1 < y_2$

If $x_1 < x_2$ does not imply the strict inequality $f(x_1) < f(x_2)$ but does imply the inequality $f(x_1) \le f(x_2)$, then it will be termed *increasing* (but not monotonically increasing). For example, $f(x) = [x]$ is increasing but not monotonically increasing. Analogously, we have *monotonically decreasing* functions and *decreasing functions*. If a function is continuous and differentiable on some domain D, then the following will hold true:

$$f'(x) > 0 \ \ \forall x \in D: f(x) \text{ is monotonically increasing on } D$$

$$f'(x) < 0 \ \forall x \in D: f(x) \text{ is monotonically decreasing on } D$$

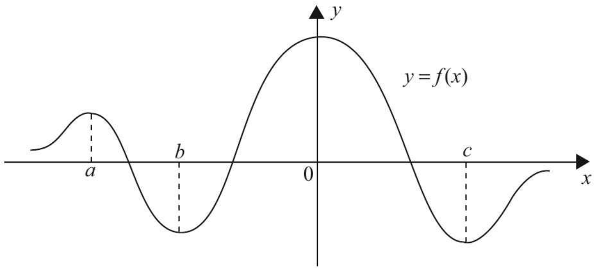
If $f'(x) = 0$ for some points, then also the function may show monotonic behaviour. For example, think of $f(x) = x^3$; even though $f'(x) = 0$ for $x = 0$, the function is still monotonically increasing. This is because f' takes the value of 0 only for *one point*, and not for an *extended interval*. Here are some more facts about monotonicity:

(a) If $f(x)$ is strictly increasing, then $f^{-1}(x)$ is also strictly increasing. Similarly, if $f(x)$ is strictly decreasing,then $f^{-1}(x)$ is also strictly decreasing.
(b) If $f(x)$ and $g(x)$ have the same monotonicity (both increasing or decreasing) on $[a, b]$, then $f(g(x))$ and $g(f(x))$ are monotonically increasing on $[a, b]$.
(c) If $f(x)$ and $g(x)$ have opposite monotonicity on $[a, b]$, then $f(g(x))$ and $g(f(x))$ are strictly decreasing on $[a, b]$.
(d) If $f'(x) > 0 \ \ \forall x \in (a, b)$ except for a finite (or an infinitely countable) number of points where $f'(x) = 0$, $f(x)$ is still strictly increasing on (a, b). This is why $f'(x) > 0 \ \forall x \in D$ is not a necessary condition for strict increase. For example, consider the function $f(x) = x + \cos x$ Note that $f'(x) = 1 - \sin x$ is not always positive (at $x = 2n\pi + \frac{\pi}{2}, (n \in \mathbb{Z}), f'(x) = 0)$; even then, $f(x)$ increases strictly, because the points at which $f'(x) = 0$ are countable. In other words, f' is never equal to 0 for an extended interval.
(e) Similarly, if $f'(x) < 0 \ \ \forall x \in (a, b)$ except for a finite (or an infinitely countable) number of points where $f'(x) = 0$, $f(x)$ is still strictly decreasing on (a, b).

3. Maxima and Minima

3.1 Extrema Points

Consider an arbitrary function $f(x)$ with the following graph:



The concept of *Maxima and Minima* is a way to characterize the peaks and troughs of $f(x)$. For example, we see that there is a peak at $x = a$; this point is therefore a *local maximum*; similarly, $x = 0$ is another local maximum; however, since $f(0)$ has the largest value on the entire domain, $x = 0$ is also a *global maximum*. Analogously, $x = b$ and $x = c$ are *local minimum* points; $x = c$ is also a *global minimum*. The rigorous definitions of extrema points are as follows:

(i) Local Maximum: A point $x = a$ is a local maximum for $f(x)$ if in the *neighbourhood* of a, *i.e.*, in $(a-\delta, a+\delta)$ where δ can be made arbitrarily small, $f(x) < f(a)$ for all $x \in (a-\delta, a+\delta)\setminus\{a\}$. This simply means that if we consider a small region (interval) around $x = a$, $f(a)$ should be the maximum in that interval.

(ii) Global Maximum: A point $x = a$ is a global maximum for $f(x)$ if $f(x) \le f(a)$ for all $x \in D$ the domain of $f(x)$).

Analogous definitions will exist for *local minimum* and *global minimum* points.

3.2 Evaluating Extrema Points

There exist a number of techniques to evaluate the extrema points for an arbitrary function which is continuous and differentiable:

(a) **First Derivative Test**

Suppose that $f'(a) = 0$. How do we decide whether the point $x = a$ is a local maximum, a local minimum, or neither? The first derivative test tells us how:

(i) $x = a$ is a local maximum for $f(x)$ if
$f'(x)$ changes from $(+\text{ve}) \to (-\text{ve})$ as x crosses a (from left to right). This means that $f' > 0$ just to the left of $x = a$, and $f' < 0$ just to the right of $x = a$.

(ii) $x = a$ is a local minimum for $f(x)$ if
$f'(x)$ changes from $(-\text{ve}) \to (+\text{ve})$ as x crosses a.

(iii) $x = a$ is not an extremum point for $f(x)$ if
$f'(x)$ does not change sign as x crosses a.

(b) **Second Derivative Test**

$x = a$ is a local maximum for $f(x)$ if
$f'(a) = 0$ and $f''(a) < 0$
$x = a$ is a local minimum for $f(x)$ if
$f'(a) = 0$ and $f''(a) > 0$
What happens if $f''(a)$ is also 0? The following test is applied in such cases:

(c) **Higher Order Derivative Test**

If $f'(\text{a}) = f''(a) = f'''(\text{a}) = \cdots = f^{n-1}(\text{a}) = 0$ and $f^n(a) \ne 0$, then
If n is even and

$$f^n(a) > 0 \;\Rightarrow\; x = a \text{ is a point of local minimum.}$$

$$f^n(a) < 0 \;\Rightarrow\; x = a \text{ is a point of local maximum.}$$

otherwise

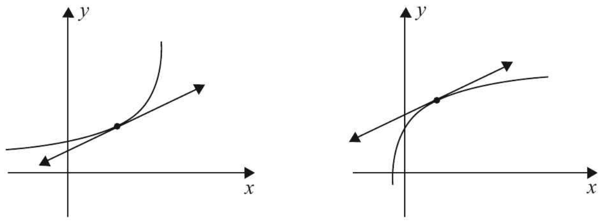
If n is odd

$$\Rightarrow \; x = a \text{ is neither a local maximum nor a local minimum.}$$

n is basically the number of times you have to differentiate $f(x)$ so that $f^n(a)$ becomes non- zero with all the lower derivatives being 0 at $x = a$.

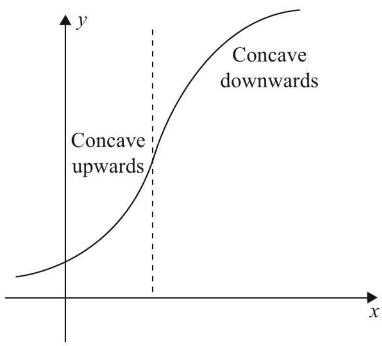
4. Convexity and Concavity

Observe the following two graphs:



(A) (B)

On graph A, if you draw a tangent anywhere, the entire curve will lie *above* this tangent. Such a curve is called a *concave upwards* curve . For graph B, the entire curve will lie *below* any tangent drawn to itself. Such a curve is called a *concave downwards curve*. The concavity's nature can of course be restricted to particular intervals. For example, a graph might be concave upwards in some interval while concave downwards in another.



A graph is concave upwards in an interval if $\frac{d^2y}{dx^2} > 0$ for that interval, and concave downwards in aninterval if $\frac{d^2y}{dx^2} < 0$ for that interval. If $\frac{d^2y}{dx^2} = 0$ at some point $x = a$ and the value of $\frac{d^2y}{dx^2}$ changes sign as x crosses a (that is, the values of $\frac{d^2y}{dx^2}$ just to the left of $x = a$ and just to the right of $x = a$ have opposite signs), then $x = a$ is a *point of inflexion* - that is, the concavity of the graph changes at $x = a$.

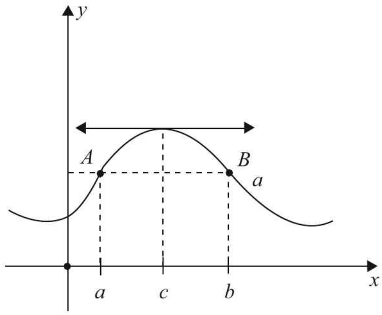
5. Mean Value Theorems and other Applications

5.1 Rolle's Theorem

Let $f(x)$ be a function defined on $[a, b]$ such that

(i) it is continuous on $[a, b]$
(ii) it is differentiable on (a, b)
(iii) $f(a) = f(b)$

Then, there exists a real number $c \in (a, b)$ such that $f'(c) = 0$. The geometrical interpretation of this theorem is quite straightforward. Consider an arbitrary curve $y = f(x)$ and two points $x = a$ and $x = b$ such that $f(a) = f(b)$.



Since A and B are joined by a continuous and differentiable curve, at least one point $x = c$ will always exist in (a, b) where the tangent drawn is horizontal, or equivalently, $f'(c) = 0$. Convince yourself that no matter what curve joins A and B, as long as it is continuous and differentiable, one such c will always exist. From Rolle's theorem, it follows that between any two zeroes of a polynomial $f(x)$ will lie a zero of the polynomial $f'(x)$.

5.2 Lagrange's Mean Value Theorem

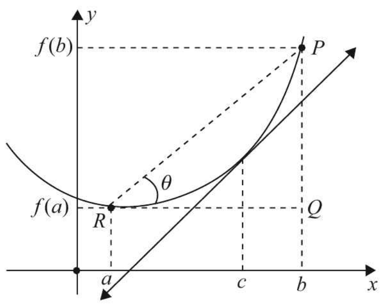
Let $f(x)$ be a function defined on $[a, b]$ such that

(i) it is continuous on $[a, b]$ (ii) it is differentiable on (a, b)

Then, there exists a real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

To interpret this theorem geometrically, we take an arbitrary function $y = f(x)$ and two arbitrary points $x = a$ and $x = b$ on it:



$$\frac{f(b) - f(a)}{b - a} = \frac{PQ}{QR} = \tan\theta$$

We see that no matter what the curve between R and P is like, as long as it is continuous and differentiable, there will exist a $c \in (a, b)$ such that the tangent drawn at $x = c$ will have a slope equal to $\tan\theta$, *i.e.*, the average slope from $x = a$ to $x = b$.

5.3 Errors and Approximations

We can use differentials to calculate small changes in the dependent variable of a function corresponding to small changes in the independent variable. The theory behind it is quite simple: We know that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

For small Δx, we can therefore approximate Δy as $f'(x)\Delta x$. Suppose we have to calculate $(4.016)^2$.

We let $y = x^2$, $x_0 = 4$ and $y_0 = 16$. Now,

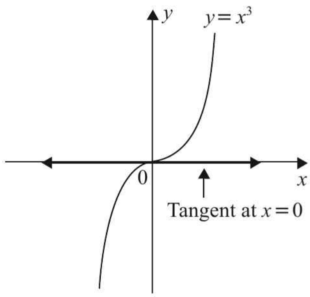
$$y' = 2x,\ \Delta x = 0.016$$

$$\Rightarrow \quad \Delta y \approx f'(x)\cdot\Delta x = 2x\Big|_{x_0=4} \times 0.016 = 8\times 0.016 = 0.128$$

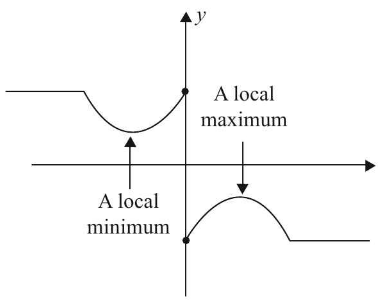
$$\Rightarrow \quad y = y_0 + \Delta y \approx 16.128$$

IMPORTANT IDEAS AND TIPS

1. *Drawing Tangents and Normals*. Tangents and normals can only be drawn to a curve for those points where the curve is continuous and differentiable. This is an important fact which is overlooked many times. For example, for the function $f(x) = |x|$, tangents and normals can be drawn at every point except the origin, since the function is not differentiable for $x = 0$.
2. *Can a Tangent Cross the Curve?* A tangent to a curve will not cross the curve in most cases, but it is possible. For example, consider the function $y = x^3$. The tangent at $x = 0$ has slope 0 and crosses the curve, as shown below. If you consider this figure carefully, the reason that the tangent crosses the curve is because the concavity of the graph is changing (from concave downwards to concave upwards) at $x = 0$. Technically, $x = 0$ is a point of inflexion. At any point of inflexion for a given curve, the tangent will cross the curve.



3. *Derivative Tests*. The derivative tests to determine the extrema points of a function can only be applied if the function is continuous and differentiable around those points. For example, you cannot apply the derivative tests to piecewise defined functions as in general they may not be differentiable at points where their definition changes. However, in any sub-interval where the function is continuous and differentiable, the derivative tests can be applied.
4. *Relative Magnitudes of Maxima and Minima.* There is generally no relation between the relative magnitudes of maxima and minima values of a function. In the following figure, for example, a local minimum value has a larger magnitude than a local maximum value.

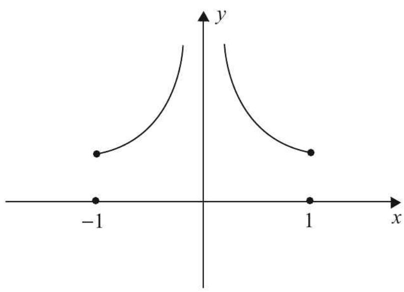


5. *Deciding a Point as a Maximum or Minimum Point.* In problems which involve finding maxima and minima of functions, many students tend to stop after finding out the critical points, *i.e.*, those points where the function's derivative vanishes. For example, when asked to find the maxima values of a function, a student will evaluate the points at which the derivative vanishes, suppose that one such point is obtained. The student might assume that this point will be the maxima point whereas it might not be the case at all-the point may be a minima point. To be absolutely certain whether a given point is a point of maximum or minimum, you must evaluate either the change in the sign of $f'(x)$, or the sign of $f''(x)$ at that point. You must not stop at finding only the critical points.

6. *Can $f(x)$ be Monotonic if $f'(x)=0$ for Some Points?* A function $f(x)$ can be strictly monotonic on a given interval even if $f'(x)=0$ for some points in the interval. For example, suppose you are given a function $y=f(x)$ and told that $f'(x)>0$ for all points in an interval D, except some points where $f'(x)=0$. Can $f(x)$ still be strictly increasing? The answer is yes. If the points where $f'(x)=0$ are finite in number (or infinitely countable), then $f(x)$ will be strictly increasing. On the other hand, if the points where $f'(x)=0$ form an extended interval, then $f(x)$ will stay constant for that entire interval, and so it cannot be strictly increasing.

7. *Points of Inflexion.* For a point to qualify as a point of inflexion for a function $y=f(x)$, it is not necessary for $f'(x)$ to be 0 at that point. What is required is that $f''(x)$ should be 0 at that point and $f''(x)$ should change sign across that point, that is, $f''(x)$ should have opposite signs just to the immediate left and immediate right of that point.

8. *Mean Value Theorems.* Rolle's Theorem and Lagrange's Mean Value Theorem can only be applied to functions on intervals where the functions are continuous and differentiable. In addition, for Rolle's theorem to be applicable, the end-point values of the function must be equal. Consider as an example the function $f(x)=\frac{1}{|x|}$ on the interval $[-1,1]$. Even though the end-point values are the same (equal to 1), there is no point c in $[-1,1]$ where $f'(c)=0$. This is because the function is non-continuous (and hence non-differentiable) at $x=0$, so Rolle's theorem is not applicable.



9. *The Point $x=c$ in the Mean Value Theorems.* In both Rolle's theorem and Lagrange's theorem, there is a point (generally denoted by c) for which

$$f'(c)=0 \text{ (for Rolle's theorem) or}$$
$$f'(c)=(f(b)-f(a))/(b-a) \text{ (for Lagrange's theorem)}$$

Note that there is at least one such c and there may be more, for which these conditions hold. Also, one of the most important facts to remember is that these theorems tell us absolutely nothing about where exactly such c-values might be in the original interval $[a,b]$. All they tell is that there is at least one c. A really good example to illustrate this pitfall is the problem in Example 22 in the next section of the chapter (refer to it), where many students would have proceeded as follows. Instead of constructing a new function $h(x)=f(x)-2g(x)$, they would have applied the Rolle's theorem separately on $f(x)$ and $g(x)$, saying that there exists at least one c in $(0,1)$ such that $f'(c)=4$, and there exists at least one c in $(0,1)$ such that $g'(c)=2$, and so the assertion that there exists at least one c in $(0,1)$ such that $f'(c)=2g'(c)$ is true. However, this argument is totally flawed, because the c-value for which $f'(c)=4$ will in general not be the same as the c-value for which $g'(c)=2$. Nothing in the Rolle's theorem tells you where exactly the c-value is. All it tells you is that there exists at least one such c-value.

Applications of Derivatives

PART-B: Illustrative Examples

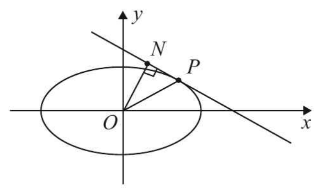
Example 1

P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose centre is O and N is the foot of the perpendicular from O upon the tangent at P. Let $A_{\max}$ be the maximum area of ΔOPN. The value of $\frac{a^2-b^2}{A_{\max}}$ is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: Let P have the co-ordinates $(a\cos\theta,\, b\sin\theta)$:

$$\left.\frac{dy}{dx}\right|_P = \frac{-b}{a}\cot\theta$$



Equation of tangent: $bx\cos\theta + ay\sin\theta = ab$

The distance of this tangent from the origin O is:

$$ON = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

To evaluate the area A of ΔOPN, we need PN apart from ON:

$$PN^2 = OP^2 - ON^2 = (a^2\cos^2\theta + b^2\sin^2\theta) - \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{(a^4+b^4)\sin^2\theta\cos^2\theta + a^2b^2(\sin^4\theta+\cos^4\theta) - a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{(a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\Rightarrow \quad PN = \frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\Rightarrow \quad A = \frac{1}{2}\times PN\times ON = \frac{ab(a^2-b^2)\sin\theta\cos\theta}{2(a^2\sin^2\theta + b^2\cos^2\theta)} = \frac{ab(a^2-b^2)}{2(a^2\tan\theta + b^2\cot\theta)} \qquad (1)$$

For A to be maximum, the denominator in (1) should be minimum, *i.e.*, we can minimize:

$$D(\theta) = a^2\tan\theta + b^2\cot\theta$$

For $D(\theta)$ to be minimum:

$$\frac{d(D(\theta))}{d\theta} = 0 \quad \text{and} \quad \frac{d^2(D(\theta))}{d\theta^2} > 0$$

$$\Rightarrow \quad \frac{d(D(\theta))}{d\theta} = a^2 \sec^2\theta - b^2 \text{cosec}^2\theta = 0 \quad \Rightarrow \quad \tan\theta = \pm\frac{b}{a}$$

Verify that $\left.\dfrac{d^2(D(\theta))}{d\theta^2}\right|_{\tan\theta=\frac{b}{a}} > 0$

$$\Rightarrow \quad D(\theta) \text{ has a local minimum at } \tan\theta = \frac{b}{a}$$

$$\Rightarrow \quad D_{\min}(\theta) = a^2\tan\theta + b^2\cot\theta\Big|_{\tan\theta=\frac{b}{a}} = 2ab$$

$$\Rightarrow \quad A_{\max} = \frac{ab(a^2-b^2)}{2D_{\min}} = \frac{a^2-b^2}{4} \quad \Rightarrow \quad \frac{a^2-b^2}{A_{\max}} = 4$$

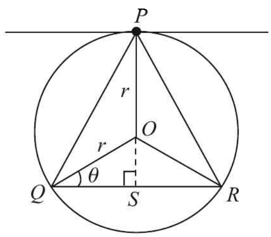
The correct option is (C). ■

Example 2

A point P is given on the circumference of a circle of radius r. The chord QR is parallel to the tangent line at P. Let $A_{\max}$ be the maximum possible area of ΔPQR. The value of $\frac{4A_{\max}}{\sqrt{3}\,r^2}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Observe that since QR is parallel to the tangent at P, the triangle PQR must be isosceles. This will become more clear upon carefully observing the following figure:



We can assume either $\angle OQS$ or the length OS as the variable on which the area A will be dependent. We let the variable be $\angle OQS = \theta$.

$$\Rightarrow \quad OS = r\sin\theta \quad \Rightarrow \quad PS = r + r\sin\theta$$

$$\text{and} \quad QR = 2r\cos\theta \;\Rightarrow\; A = \frac{1}{2}\times QR\times PS = r^2\cos\theta(1+\sin\theta)$$

For maximum area, $\frac{dA}{d\theta} = 0$ and $\frac{d^2A}{d\theta^2} < 0$:

$$\frac{dA}{d\theta} = r^2\{-\sin\theta(1+\sin\theta) + \cos\theta.\cos\theta\} = r^2\{\cos^2\theta - \sin^2\theta - \sin\theta\}$$

$$= r^2\{1 - \sin\theta - 2\sin^2\theta\} = r^2(1+\sin\theta)(1-2\sin\theta)$$

This is 0 when $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\sin\theta = -1$ (not a possible case)
You can verify that $\frac{d^2A}{d\theta^2} < 0$ for $\theta = \frac{\pi}{6}$. Therefore, the area is maximum for $\theta = \frac{\pi}{6}$.

$$A_{\max} = r^2\cos\theta(1+\sin\theta)\Big|_{\theta=\frac{\pi}{6}} = \frac{3\sqrt{3}\,r^2}{4} \quad \Rightarrow \quad \frac{4A_{\max}}{\sqrt{3}r^2} = 3$$

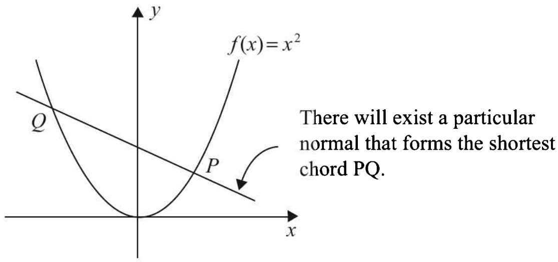
The correct option is (C). ■

Example 3

What is the shortest length of the chord intercepted on a normal to the curve $y = x^2$?

(A) $\sqrt{5}$ (B) $\sqrt{6}$ (C) $\frac{5}{2}$ (D) $\frac{\sqrt{13}}{2}$ (E) None of these

Solution: Notice that there will exist a particular normal for which the chord intercepted by the parabola is the shortest. The maximum length of this chord is of course unbounded (infinity).



Let us assume P to have the coordinates (t, t^2). We will write the equation of the normal at P, find the other intersection point (the point Q) of this normal with the parabola, and then find PQ in terms of t. Then we will find t for which PQ is minimum.

$$y = x^2 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_P = \left.2x\right|_P = 2t \quad \Rightarrow \quad m_N = \frac{-1}{2t}$$

Equation of normal: $y - t^2 = \frac{-1}{2t}(x - t)$ (1)

Let the point Q be (t', t'^2). Since Q lies on the normal at P, the co-ordinates of Q must satisfy (1):

$$\Rightarrow \quad t'^2 - t^2 = \frac{-1}{2t}(t' - t) \quad \Rightarrow \quad (t' + t)(t' - t) = \frac{-1}{2t}(t' - t)$$

$$\Rightarrow \quad t' + t = \frac{-1}{2t} \quad (\because t' - t \neq 0) \Rightarrow \quad t' = -t - \frac{1}{2t}$$

The length PQ is given by:

$$PQ^2 = (t' - t)^2 + (t'^2 - t^2)^2 = (t' - t)^2\{1 + (t' + t)^2\}$$

$$= \left(2t + \frac{1}{2t}\right)^2 \left\{1 + \frac{1}{4t^2}\right\} = 4t^2\left(1 + \frac{1}{4t^2}\right)^3$$

To minimize PQ, we can equivalently minimize PQ^2.

$$\Rightarrow \quad \frac{d(PQ)^2}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^3 + 4t^2 \times 3\left(1 + \frac{1}{4t^2}\right)^2 \times \frac{-2}{4t^3} = 4\left(1 + \frac{1}{4t^2}\right)\left(\frac{2t^2 - 1}{t}\right)$$

This is 0 when $t^2 = \frac{1}{2}$ or $t = \pm\frac{1}{\sqrt{2}}$. Verify that $\left.\frac{d^2(PQ^2)}{dt^2}\right|_{t=\pm\frac{1}{\sqrt{2}}} > 0$

$$\Rightarrow \quad PQ \text{ is minimum for } t = \pm\frac{1}{\sqrt{2}}$$

For these values of t, we have $PQ^2 = \frac{27}{4}$, so that $PQ_{\min} = \frac{3\sqrt{3}}{2}$. Thus, none of the given options is correct, so the answer is (E). ■

Example 4

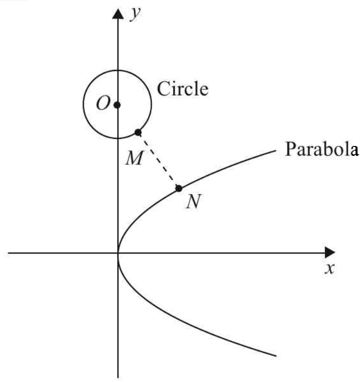
The shortest distance between two points, one of which lies on the curve $y^2 = 4ax$, and the other on the circle $x^2 + y^2 - 24ay + 128a^2 = 0$, is

(A) $2\sqrt{5}\,a$ (B) $(\sqrt{5}-1)a$ (C) $(\sqrt{5}+1)a$ (D) $4(\sqrt{5}-1)a$ (E) $4(\sqrt{5}+1)a$

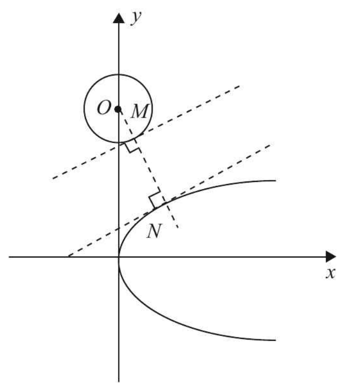
Solution: Notice that the circle's equation can be written equivalently as

$$(x-0)^2 + (y-12a)^2 = (4a)^2$$

so that its centre is (0, 12*a*) and radius is 4*a*. Let *MN* represent the shortest distance between the circle and the parabola. Since the point *N* on the parabola is nearest to the circle, it will also be nearest to the centre of the circle *O*, from amongst all the other points on the parabola. Hence, to determine *MN*, we may equivalently find the shortest distance between the circle's centre and any point on the parabola.



Now, from the geometry of the figure above, notice a very important fact. The tangent drawn at *M* must be perpendicular to *ON*, or equivalently, *ON must be a normal to the parabola.* Only then will *N* be the closest point on the parabola from *O*. Convince yourself that this should be true.



We take an arbitrary point on the parabola as $(at^2, 2at)$. We will write the normal to the parabola at this point and make this normal pass through the point *O*.

$$y^2 = 4ax$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y} \quad \Rightarrow \quad m_N(\text{at}(at^2, 2at)) = \left.\frac{-dx}{dy}\right|_{(at^2, 2at)} = -t$$

Equation of normal: $y - 2at = -t(x - at^2) \quad \Rightarrow \quad tx + y = 2at + at^3$ (1)

So that this normal passes through O, the coordinates of $O(0, 12a)$ must satisfy (1):

$$\Rightarrow \quad 12a = 2at + at^3 \quad \Rightarrow \quad 2t + t^3 = 12 \quad \Rightarrow \quad t = 2$$

Note that we obtained a cubic in t, but it was simple enough so that we could write the answer for t by simple observation. We therefore get the co-ordinates of N as $(4a, 4a)$. Hence,

$$ON = \sqrt{(4a-0)^2 + (4a-12a)^2} = \sqrt{80a^2} = 4\sqrt{5}a$$

$$\Rightarrow \quad MN = 4\sqrt{5}a - 4a = 4(\sqrt{5}-1)a$$

The correct option is (D). ■

Example 5

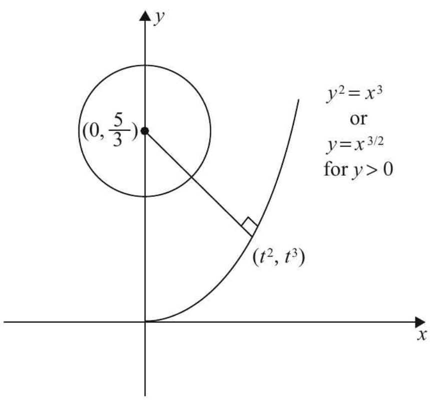
The shortest distance between the curves $y^2 = x^3$ and $9x^2 + 9y^2 - 30y + 16 = 0$ is

(A) $\frac{\sqrt{13}}{3}$ (B) $\frac{2\sqrt{13}}{3}$ (C) $\frac{\sqrt{17}}{3}$ (D) $\frac{2\sqrt{17}}{3}$ (E) None of these

Solution: Note that the equation of the second curve can be rearranged as:

$$9x^2 + 9\left(y - \frac{5}{3}\right)^2 - 9 = 0 \quad \Rightarrow \quad x^2 + \left(y - \frac{5}{3}\right)^2 = 1$$

This is a circle of radius 1 centred at $(0, \frac{5}{3})$. To find the shortest distance between the two curves, we can equivalently find the shortest distance between the curve $y^2 = x^3$ and the centre of this circle, *i.e.*, $(0, \frac{5}{3})$.



A general point on the curve $y^2 = x^3$ can be taken as (t^2, t^3). Its distance from the circle's centre is given by:

$$l^2 = (t^2 - 0)^2 + \left(t^3 - \frac{5}{3}\right)^2 \quad = t^4 + t^6 - \frac{10t^3}{3} + \frac{25}{9}$$

Now, we minimize l^2 with respect to t:

$$\frac{d(l^2)}{dt} = 4t^3 + 6t^5 - 10t^2 = 2t^2(3t^3 + 2t - 5) = 2t^2(t-1)(3t^2 + 5t + 5)$$

This is 0 when $t = 0, 1$ (note that $3t^2 + 5t + 5 > 0\ \forall t \in \mathbb{R}$). Now, you may verify that

$$\left.\frac{d^2(l^2)}{dt^2}\right|_{t=1} > 0,$$

so that $t = 1$ is a point of local minimum (what about $t = 0$?):

$$\Rightarrow \quad l^2_{\min} = 1 + 1 - \frac{10}{3} + \frac{25}{9} = \frac{13}{9} \quad \Rightarrow \quad l_{\min} = \frac{\sqrt{13}}{3}$$

The correct option is (A). ■

Example 6

Let $f(x) = \left\{\begin{smallmatrix} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{smallmatrix}\right\}$, where a is a positive constant. Let $L(a)$ denote the total length of all the interval(s) on which $f'(x)$ is increasing. Note that L is a function of a. For $a = 3$, the value of $\frac{1}{L'(a)}$ is

(A) 3 (B) 6 (C) 9 (D) 12

Solution: Notice that we are required to find the intervals of increase of $f'(x)$ and not $f(x)$. Therefore, we need to first determine $f'(x)$ from $f(x)$, and then check the sign of the derivative of $f'(x)$ in different intervals, *i.e.*, the sign of $f''(x)$. Observe that $f(x)$ is continuous and differentiable at $x = 0$ so that $f'(x)$ is defined at $x = 0$. Therefore,

$$f'(x) = \begin{Bmatrix} (1 + ax)e^{ax}, & x \le 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{Bmatrix}$$

Notice again that $f'(x)$ is also continuous and differentiable at $x = 0$ so that $f''(x)$ is also defined at $x = 0$:

$$f''(x) = \begin{Bmatrix} (2 + ax)ae^{ax}, & x \le 0 \\ 2a - 6x, & x > 0 \end{Bmatrix}$$

Interval(s) of strict increase for $f'(x)$: $f''(x) > 0$

$$\Rightarrow \quad 2 + ax > 0 (\text{if } x \le 0) \text{ and } 2a - 6x > 0 (\text{if } x > 0)$$

$$\Rightarrow \quad x > \frac{-2}{a} (\text{if } x \le 0) \text{ and } x < \frac{a}{3} (\text{if } x > 0)$$

$$\Rightarrow \quad \frac{-2}{a} < x \le 0 \text{ and } 0 < x < \frac{a}{3} \quad \Rightarrow \quad \frac{-2}{a} < x < \frac{a}{3}$$

Therefore, $f'(x)$ is strictly increasing on the interval $\left(\frac{-2}{a}, \frac{a}{3}\right)$. We see that $L(a) = \frac{a}{3} + \frac{2}{a}$ which gives $L'(a) = \frac{1}{3} - \frac{2}{a^2}$. Thus $L'(3) = \frac{1}{9}$ so that $\frac{1}{L'(3)} = 9$. The correct option is (C). ■

Example 7

Which of the following gives the values of a for which the function $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$ increases on $\mathbb{R}$?

(A) $[0, \infty)$ (B) $(0, \infty)$ (C) $\left[\frac{1}{2}, \infty\right)$ (D) $[1, \infty)$ (E) None of these

Solution: We want $f'(x) \ge 0\ \forall x \in \mathbb{R}$. Now,

$$f'(x) = \cos x - 2a\cos 2x - \cos 3x + 2a$$

$$= \cos x - 2a(2\cos^2 x - 1) - (4\cos^3 x - 3\cos x) + 2a$$

$$= 4a + 4\cos x - 4a\cos^2 x - 4\cos^3 x = 4a\sin^2 x + 4\cos x \sin^2 x$$

$$= 4\sin^2 x(a + \cos x)$$

This is always non-negative if $a \geq 1$ (since the minimum value of cos x is -1). Therefore, the required values of a are:

$$a \in [1, \infty)$$

The correct option is (D). ■

Example 8

For which of the following values of the parameter a does the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum?

(A) $a < -3$ (B) $-1 < a < 1$ (C) $3 < a < 4$ (D) $5 < a < 6$

Solution: We have

$$f'(x) = 3x^2 + 6(a-7)x + 3(a^2 - 9)$$

For $f(x)$ to have a maximum at some point, $f'(x) = 0$ and $f''(x) < 0$ for that point. Now,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 + 6(a-7)x + 3(a^2-9) = 0$$

$$\Rightarrow \quad x^2 + 2(a-7)x + (a^2-9) = 0$$

$$\Rightarrow \quad x = -(a-7) \pm \sqrt{58 - 14a} \tag{1}$$

For $f'(x)$ to have real roots,

$$58 - 14a > 0 \quad \Rightarrow \quad a < \frac{29}{7} \tag{2}$$

Now we determine which of the roots of $f'(x)$ in (1) will give a local maximum and which will give a local minimum.

$$f''(x) = 6x + 6(a-7) \quad = 6(x + a - 7)$$

$$\text{At} \quad x_1 = -(a-7) + \sqrt{58-14a} \Rightarrow f''(x) = 6\sqrt{58-14a} > 0$$

$$\text{At} \quad x_2 = -(a-7) - \sqrt{58-14a} \Rightarrow f''(x) = -6\sqrt{58-14a} < 0$$

Therefore, $x = x_2$ is a point of local maximum. Since we want this to be positive, we have

$$-(a-7) - \sqrt{58-14a} > 0$$

$$\Rightarrow \quad 7 - a > \sqrt{58-14a}$$

Upon squaring, we get

$$a^2 - 9 > 0$$

$$\Rightarrow \quad a < -3 \text{ or } a > 3 \tag{3}$$

From (2) and (3),

$$a < -3 \text{ or } 3 < a < \frac{29}{7}$$

Therefore, both (A) and (C) are correct. ■

Example 9

Let $f(x) = \sin^3 x + \lambda \sin^2 x$ for $\frac{-\pi}{2} < x < \frac{\pi}{2}$. In which of the following intervals can λ lie so that $f(x)$ has exactly one minimum and one maximum?

(A) $\left(-\frac{3}{2}, -1\right)$ (B) $(-1, 0)$ (C) $(0,1)$ (D) $\left(1, \frac{3}{2}\right)$ (E) $\left(\frac{3}{2}, 2\right)$

Solution: We have to equivalently find those values of λ for which $f'(x)$ has two roots; $f''(x)$ should be positive for one root and negative for the other.

$$f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x = \sin 2x\left(\lambda + \frac{3}{2}\sin x\right)$$

For $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$, $f'(x)$ is 0 when: $x = 0$, $\sin^{-1}(\frac{-2\lambda}{3})$. Note that if $\lambda = 0$, $\sin^{-1}(\frac{-2\lambda}{3}) = 0$ so that the two roots of $f'(x)$ will no longer remain distinct and $f(x)$ will not have two extremum points as required. Hence, λ should not be 0.

For $\sin^{-1}(\frac{-2\lambda}{3})$ to be defined, we have

$$-1 < \frac{-2\lambda}{3} < 1 \quad \Rightarrow \quad \frac{-3}{2} < \lambda < \frac{3}{2}$$

Therefore, $\lambda \in \left(\frac{-3}{2}, \frac{3}{2}\right) \setminus \{0\}$

To verify that $f(x)$ will satisfy the required condition for these values of λ, let us evaluate $f''(x)$:

$$f''(x) = \frac{3}{2}\cos x \sin 2x + (3\sin x + 2\lambda)\cos 2x$$

$$\Rightarrow \quad f''(0) = 2\lambda \text{ and}$$

$$f''\left(\sin^{-1}\left(\frac{-2\lambda}{3}\right)\right) = \frac{3}{2}\cos x \sin 2x\Big|_{\sin^{-1}\left(\frac{-2\lambda}{3}\right)} \qquad \left\{\begin{matrix}\text{The second term in } f'' \\ \text{becomes 0 at this point}\end{matrix}\right\}$$

Now, if $\lambda \in \left(\frac{-3}{2}, 0\right)$

$$\Rightarrow \quad f''(0) < 0 \text{ and}$$

$$f''\left(\sin^{-1}\left(\frac{-2\lambda}{3}\right)\right) > 0 \qquad \left(\begin{matrix}\text{This is because for } \lambda \in \left(\frac{-3}{2}, 0\right), \frac{-2\lambda}{3} \in (0,1) \\ \text{so that } \sin^{-1}\left(\frac{-2\lambda}{3}\right) \in \left(0, \frac{\pi}{2}\right)\end{matrix}\right)$$

On the other hand, if $\lambda \in (0, \frac{3}{2})$

$$\Rightarrow \quad f''(0) > 0 \text{ and}$$

$$f''\left(\sin^{-1}\left(\frac{-2\lambda}{3}\right)\right) < 0 \qquad \left(\begin{matrix}\text{As above, this is because for } \lambda \in (0, \frac{3}{2}), \\ \frac{-2\lambda}{3} \in (-1, 0) \text{ so that } \sin^{-1}\left(\frac{-2\lambda}{3}\right) \in \left(\frac{-\pi}{2}, 0\right)\end{matrix}\right)$$

We see that for $\lambda \in (\frac{-3}{2}, \frac{3}{2}) \setminus \{0\}$, either of 0 or $\sin^{-1}(\frac{2\lambda}{3})$ is a local maximum and the other is a local minimum. The required values of λ are therefore: $\lambda \in (\frac{-3}{2}, \frac{3}{2}) \setminus \{0\}$, which means that all the options except (E) are correct. ■

Example 10

The point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$, which is farthest from the point (0, –2) is
(A) (0, 1) (B) (0, –1) (C) (0, 2) (D) (0, –2) (E) dependent on the value of a.

Solution: Upon rearrangement, the equation of the curve becomes $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ which is the equation of an ellipse. A general variable point P on this ellipse can be taken as (a cosθ, 2 sinθ). Let r represent the distance of P from (0, –2). We have

$$r^2 = (a\cos\theta - 0)^2 + (2\sin\theta + 2)^2 = a^2\cos^2\theta + 4(1 + \sin\theta)^2$$

For r^2 to be maximum, $\frac{d(r^2)}{d\theta} = 0$ and $\frac{d^2(r^2)}{d\theta^2} < 0$. Now,

$$\frac{d(r^2)}{d\theta} = -2a^2\sin\theta\cos\theta + 8(1+\sin\theta)\cos\theta \quad = \cos\theta\{(8-2a^2)\sin\theta + 8\}$$

This is 0 when $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $(8 - 2a^2)\sin\theta = -8 \Rightarrow \sin\theta = \frac{4}{a^2-4}$
But $\frac{4}{a^2-4} > 1$ (verify) and sinθ cannot be greater than 1. Hence, this case does not give any valid value of θ. This means that $\theta = \frac{\pi}{2}$. Verify that $\left.\frac{d^2(r^2)}{d\theta^2}\right|_{\theta=\pi/2} < 0$:

$$\Rightarrow \quad \theta = \frac{\pi}{2} \text{ is a local maximum for } r^2$$

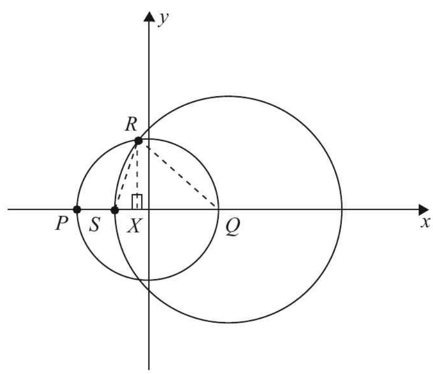
The required point is $(a\cos\theta, 2\sin\theta)\big|_{\theta=\pi/2}$ or (0, 2). The correct option is (C). ■

Example 11

The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Let r denote the radius of the variable circle, and let A denote the area of ΔQSR. Let $r_{\text{area-max}}$ denote that value of r for which A is maximum, and let $A_{\max}$ be the maximum value of A. Which of the following are true?

(A) $A = \frac{1}{2}r^2\sqrt{4-r^2}$ (B) $\frac{dA}{dr} = \frac{8r-3r^3}{4\sqrt{4-r^2}}$ (C) $r_{\text{area-max}} = \sqrt{\frac{8}{3}}$ (D) $A_{\max} = \frac{4\sqrt{3}}{9}$

Solution: The situation described in the question has been translated into the diagram below; observe it carefully:



The variable here is the radius of the circle centred at Q, which is r. We need to express the area A of ΔQSR in terms of r. Observe ΔQSR carefully in the diagram. QR and QS are known (both are equal to r). We need to find RX (the height) in terms of r. We can do this by finding the co-ordinates of R:

$$x^2 + y^2 = 1$$

$$(x-1)^2 + y^2 = r^2 \quad \{\text{circle centred at } Q\}$$

Solving these two equations, the co-ordinates of R turn out to be:

$$\left(1-\frac{r^2}{2}, \frac{r\sqrt{4-r^2}}{2}\right)$$

Therefore,

$$RX = \frac{r\sqrt{4-r^2}}{2} \quad \Rightarrow \quad A = \frac{1}{2}\times QS \times RX = \frac{1}{4}r^2\sqrt{4-r^2}$$

For maximum area, $\dfrac{dA}{dr} = 0$ and $\dfrac{d^2A}{dr^2} < 0$:

$$\frac{dA}{dr} = \frac{1}{4}\left\{2r\sqrt{4-r^2} - \frac{r^3}{\sqrt{4-r^2}}\right\} = \frac{8r-3r^3}{4\sqrt{4-r^2}}$$

This is 0 when $8r - 3r^3 = 0$ or $r = \sqrt{\frac{8}{3}}$. Verify that $\left.\frac{d^2A}{dr^2}\right|_{r=\sqrt{\frac{8}{3}}} < 0$

Therefore, $r_{\text{area-max}} = \sqrt{\frac{8}{3}}$ is a local maximum for A.

$$A_{\max} = \left.\frac{1}{4}r^2\sqrt{4-r^2}\right|_{r=r_{\text{area-max}}} = \frac{4\sqrt{3}}{9}$$

Thus, options (B), (C) and (D) are correct. ■

Example 12

Let $g(x) = f(x) + f(1-x)$ and $f''(x) < 0$ for all $x \in [0,1]$. Which of the following statements are true?

(A) $f'(x)$ is decreasing on[0, 1]

(C) $g(x)$ is decreasing in $\left(\frac{1}{3}, \frac{2}{3}\right)$.

(B) $g(x)$ is increasing in $\left[0, \frac{1}{3}\right)$.

(D) $g(x)$ is decreasing in $\left(\frac{2}{3}, 1\right]$.

Solution: We will consider two intervls: $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$, and show that $g'(x) > 0\ \forall x \in [0, \frac{1}{2})$ and $g'(x) < 0\ \forall\, x \in (\frac{1}{2}, 1]$. From the given functional relation between $f(x)$ and $g(x)$, we have:

$$g'(x) = f'(x) - f'(1-x)$$

Therefore, we have to show that

$$f'(x) > f'(1-x) \qquad \forall x \in \left[0, \frac{1}{2}\right) \tag{1}$$

$$\text{and} \quad f'(x) < f'(1-x) \qquad \forall x \in \left(\frac{1}{2}, 1\right] \tag{2}$$

Since $f''(x) < 0 \ \forall x \in [0,1]$, $f'(x)$ is decreasing on [0, 1]. This means that if we take any x value in $[0, \frac{1}{2})$, $(1-x)$ will be greater than x so that $f'(1-x)$ will be less than $f'(x)$. In other words, (1) is satisfied by virtue of the fact that $f'(x)$ is decreasing. On similar lines, when we assume any x value in $(\frac{1}{2}, 1]$, we will see that (2) is also satisfied for the same reason (that $f'(x)$ is decreasing). Thus,

$$g'(x) > 0 \ \forall \ x \in \left[0, \frac{1}{2}\right) \Rightarrow g(x) \text{ is increasing on } \left[0, \frac{1}{2}\right)$$

$$g'(x) < 0 \ \forall \ x \in \left[\frac{1}{2}, 1\right) \Rightarrow g(x) \text{ is decreasing on } \left[\frac{1}{2}, 1\right)$$

We see that given this information, only options (A), (B) and (D) are correct. ■

Example 13

The function $y = f(x)$ is represented parametrically as:

$$\begin{aligned} x &= g(t) = t^5 - 5t^3 - 20t + 7 \\ y &= h(t) = 4t^3 - 3t^2 - 18t + 3 \end{aligned} \quad (-2 < t < 2)$$

Which of the following statements are true?

(A) $f(x)$ has exactly one point of maximum for $t \in (-2, -\frac{3}{2})$.
(B) $f(x)$ has exactly one point of minimum for $t \in (0, 1)$.
(C) $f(x)$ has exactly one point of minimum for $t \in (1, 2)$.
(D) $t = -1$ is a point of maximum for $f(x)$.
(E) $x = \frac{3}{2}$ is a point of minimum for $f(x)$.

Solution: We need to employ parametric differentiation here to determine the points where $\frac{dy}{dx} = 0$:

$$g'(t) = \frac{dx}{dt} = 5t^4 - 15t^2 - 20 = 5(t^4 - 3t^2 - 4) = 5(t^2 - 4)(t^2 + 1)$$

For $t \in (-2, 2)$, $t^2 - 4 < 0$ so that $g'(t) < 0 \ \forall t \in (-2, 2)$, $g'(t) \neq 0$ for any t in $(-2, 2)$. We have

$$h'(t) = \frac{dy}{dt} = 12t^2 - 6t - 18 = 6(2t^2 - t - 3) = 6(2t - 3)(t + 1)$$

$$h'(t) = 0 \text{ when } t = -1, 3/2$$

$$\Rightarrow \frac{dy}{dx} = \frac{h'(t)}{g'(t)} = 0 \text{ when } t = -1, \frac{3}{2}$$

Now, $$\frac{d^2y}{dx^2} = \frac{g'(t)h''(t) - h'(t)g''(t)}{(g'(t))^3} = \left.\frac{h''(t)}{(g'(t))^2}\right|_{t=-1,\frac{3}{2}} \quad (h'(t) = 0 \text{ for } t = -1, 3/2)$$

Also, we have $$h''(t)\big|_{t=-1} = 24t - 6\big|_{t=-1} = -30 < 0$$

$$h''(t)\big|_{t=3/2} = 24t - 6\big|_{t=3/2} = 30 > 0$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1} < 0 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{t=3/2} > 0$$

$\Rightarrow$ $t = -1$ is a maximum and $t = 3/2$ is a minimum for $y = f(x)$.

We see that (A) and (B) are incorrect, while (C) and (D) are correct. (E) is incorrect because the point of minimum is $t = \frac{3}{2}$ and not $x = \frac{3}{2}$. ■

Example 14

If α, β and γ are the roots of $x^3 + x^2 - 5x - 1 = 0$, the magnitude of $[\alpha]+[\beta]+[\gamma]$ is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: It is obvious that the required value can be found out if for each root we can find out the two (successive) integers between which that root lies. For example, if α lies between I and I +1, then $[\alpha] = \text{I}$ and so on.

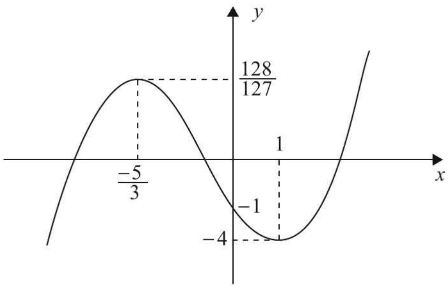
Let $f(x) = x^3 + x^2 - 5x - 1$

$$\Rightarrow \quad f'(x) = 3x^2 + 2x - 5 = (x-1)(3x+5)$$

$$\Rightarrow \quad f'(x) > 0 \text{ for } x \in (-\infty, -\tfrac{5}{3}) \cup (1, \infty) \text{ and } f'(x) < 0 \text{ for } x \in (-\tfrac{5}{3}, 1)$$

$$\Rightarrow \quad f(x) \text{increases in} (-\infty, \tfrac{-5}{3}), \text{decreases in } (\tfrac{-5}{3}, 1) \text{ and again increases in } (1, \infty).$$

Also, $f(\frac{-5}{3}) = \frac{128}{27}$ and $f(1) = -4$. Additionally, $f(0) = -1$:



The approximate graph for $f(x)$ is drawn above. We see that the three roots of $f(x)$ lie not far from the origin and their approximate locations can be found out by evaluating $f(x)$ for different integers close to 0.

$$f(0) = 0 + 0 - 0 - 1 = -1 < 0$$

$$\left.\begin{array}{l} f(1) = 1 + 1 - 5 - 1 = -4 < 0 \\ f(2) = 8 + 4 - 10 - 1 = 1 > 0 \end{array}\right\} \quad \begin{array}{l} \text{One root } \alpha \text{ lies between 1 and 2} \\ \Rightarrow \quad [\alpha] = 1 \end{array}$$

$$f(-1) = -1 + 1 + 5 - 1 = 4 > 0\} \quad \begin{array}{l} \text{One root } \beta \text{ lies between 0 and } -1 \\ \Rightarrow \quad [\beta] = -1 \end{array}$$

$$\left.\begin{array}{l} f(-2) = -8 + 4 + 10 - 1 = 5 > 0 \\ f(-3) = -27 + 9 + 15 - 1 = -4 < 0 \end{array}\right\} \quad \begin{array}{l} \text{One root } \gamma \text{ lies between } -2 \text{ and } -3 \\ \Rightarrow \quad [\gamma] = -3 \end{array}$$

$$\Rightarrow \quad [\alpha]+[\beta]+[\gamma] = -3$$

The magnitude of this sum is 3. The correct option is (B). ■

SUBJECTIVE TYPE EXAMPLES

Example 15

Let $f(x)=\frac{\ln(\pi+x)}{\ln(e+x)}$. Prove that $f(x)$ is decreasing on $[0,\infty)$.

Solution: We have

$$f'(x)=\frac{\frac{\ln(e+x)}{\pi+x}-\frac{\ln(e+x)}{e+x}}{(\ln(e+x))^2}=\frac{(e+x)\ln(e+x)-(\pi+x)\ln(\pi+x)}{(e+x)(\pi+x)(\ln(e+x))^2}=\frac{g(x)}{h(x)}$$

The substitution in the last step was done for convenience. To determine the sign of $f'(x)$ in $[0,\infty)$, we first note that $h(x)>0\ \forall\, x\in[0,\infty)$, so that we need to only worry about the sign of $g(x)$. The form of $g(x)$ suggests that we can construct a new function $G(x)=x\ln x$ to determine the sign of $g(x)$, as follows:

$$G(x)=x\ln x \quad\Rightarrow\quad G'(x)=1+\ln x$$

$$\Rightarrow\quad G'(x)>0\ \forall x\in\left(\frac{1}{e},\infty\right) \text{ and } G'(x)<0\ \forall\, x\in\left(0,\frac{1}{e}\right)$$

$$\Rightarrow\quad G(x) \text{ is increasing on} \left(\frac{1}{e},\infty\right)$$

$$\Rightarrow\quad x\ln x \text{ increases on} \left(\frac{1}{e},\infty\right)$$

$$\Rightarrow\quad (\pi+x)\ln(\pi+x)>(e+x)\ln(e+x)\quad \forall x\in[0,\infty)\qquad \left\{\begin{array}{l}\text{since}(\pi+x)>(e+x)\\ >\frac{1}{e}\ \ \forall\, x\in[0,\infty)\end{array}\right\}$$

$$\Rightarrow\quad g(x)<0\quad \forall\, x\in[0,\infty)\quad\Rightarrow\quad f'(x)<0\quad \forall\, x\in[0,\infty)$$

$$\Rightarrow\quad f(x) \text{ is decreasing on } [0,\infty)$$ ■

Example 16

Let $f(x)$ be a real function and $g(x)$ be a function given by

$$g(x)=f(x)-(f(x))^2+(f(x))^3 \text{ for all } x\in\mathbb{R}.$$

Prove that $f(x)$ and $g(x)$ increase or decrease together.

Solution: To prove the stated assertion, we must show that for any x, $f'(x)$ and $g'(x)$ have the same sign. Differentiating the given functional relation in the question, we get:

$$g'(x)=f'(x)-2f(x)f'(x)+3(f(x))^2f'(x)=f'(x)\{1-2f(x)+3(f(x))^2\}$$
$$=f'(x)\{1-2y+3y^2\}\quad (f(x) \text{ has been substituted by } y \text{ for convenience})$$

To show that $f'(x)$ and $g'(x)$ have the same sign, we must show that $(3y^2-2y+1)$ is always positive, no matter what the value of y (or $f(x)$) is. Let $h(y)=3y^2-2y+1$. We have

Discriminant of $h(y)=4-12=-8<0$

$\Rightarrow$ The parabola for $h(y)$ will not intersect the horizontal axis.

$\Rightarrow$ $h(y)>0$ for all values of y. $\Rightarrow$ $3y^2-2y+1>0$ for all values of y.

$\Rightarrow$ $f'(x)$ and $g'(x)$ have the same sign. $\Rightarrow$ $f(x)$ and $g(x)$ increase or decrease together. ■

Example 17

Show that $\cos(\sin x) > \sin(\cos x)\ \ \forall\, x \in (0, \frac{\pi}{2})$.

Solution: Instead of considering the expressions $\cos(\sin x)$ and $\sin(\cos x)$, we can consider $\sin(\frac{\pi}{2} - \sin x)$ and $\sin(\cos x)$. This is because 'sin' is a monotonically increasing function in $(0, \frac{\pi}{2})$, so that to determine the larger of the two values above, we just need to compare their arguments, *i.e.*, $(\frac{\pi}{2} - \sin x)$ and $\cos x$. For $(0, \frac{\pi}{2})$,

$$\sin x + \cos x < \frac{\pi}{2}$$

because the maximum value of LHS is $\sqrt{2}$ while the RHS = 1.57. Therefore,

$$\cos x < \frac{\pi}{2} - \sin x \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \sin(\cos x) < \sin\left(\frac{\pi}{2} - \sin x\right) \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \sin(\cos x) < \cos(\sin x) \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

■

Example 18

If $f'(\sin x) < 0$ and $f''(\sin x) > 0\ \ \forall x \in \mathbb{R}$, then find the intervals of monotonicity of the function $g(x) = f(\sin x) + f(\cos x),\ x \in [0, \pi/2]$.

Solution: To obtain the intervals of monotonicity of $g(x)$, we need to analyse $g'(x)$.

$$g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$$

This is 0 when

$$f'(\sin x)\cos x = f'(\cos x)\sin x$$

$$\Rightarrow \quad f'(\sin x)\cos x = f'\left(\sin\left(\frac{\pi}{2} - x\right)\right)\cos\left(\frac{\pi}{2} - x\right)$$

One of the possible roots of this equation is given by:

$$x = \frac{\pi}{2} - x \quad \Rightarrow \quad x = \frac{\pi}{4}$$

Now, we analyze $g''(x)$:

$$g''(x) = f''(\sin x)\cos^2 x + f''(\cos x)\sin^2 x - \{f'(\sin x)\sin x + f'(\cos x)\cos x\} \quad (1)$$

It is given that $f'(\sin x) < 0\ \ \forall x \in \mathbb{R}$:

$$\Rightarrow \quad f'(\cos x) < 0 \quad \forall x \in \mathbb{R} \qquad \left\{\because \cos x = \sin\left(\frac{\pi}{2} - x\right)\right\}$$

Similarly,

$$f''(\sin x) > 0 \quad \forall x \in \mathbb{R} \quad \Rightarrow \quad f''(\cos x) > 0 \quad \forall x \in \mathbb{R}$$

From (1), observe carefully that these two conditions above imply:

$$g''(x) > 0\ \forall x \in \mathbb{R}$$

Since $g''(x)$ is always positive, the graph for $g(x)$ is a concave upwards curve, so that $g(x)$ has only one extremum point (a minimum). Also, we have already deduced one minimum point as $x = \frac{\pi}{4}$. That must be the *only* extremum point. Therefore:

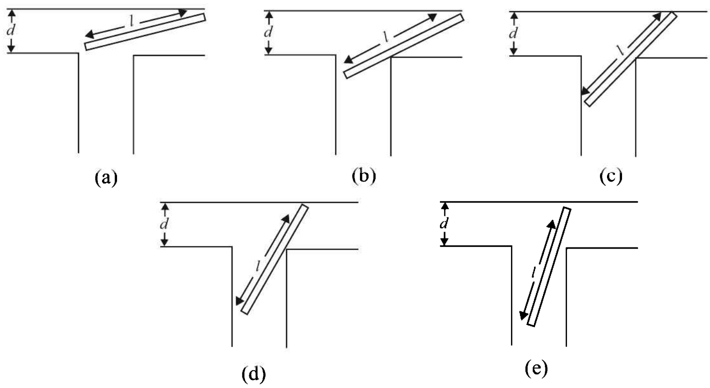
$g(x)$ decreases on $(0, \frac{\pi}{4})$, $g(x)$ increases on $(\frac{\pi}{4}, \frac{\pi}{2})$ and

$x = \frac{\pi}{4}$ is a minimum for $g(x)$. ■

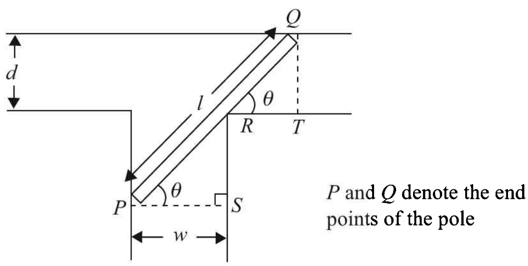
Example 19

A pole l feet long is to be carried horizontally from one corridor to another corridor perpendicular to each other. If the first corridor is d meters wide, find the minimum required width of the other corridor.

Solution: The situation described in the question is depicted in the sequence of figures below, as the pole is being moved from one corridor to the other.



Let the width of the second corridor be w. Convince yourself that if w is below a particular value (if the second corridor is too narrow), the pole will never be able to be moved into the second corridor. Now, consider the case when w has a value such that the pole is *just* able to move into the second corridor. This situation is depicted in the figure above. Observe (and visualise in your mind) carefully that an instant will come when the ends of the pole touch the walls of both the corridors and the pole touches the corner (turning). This is depicted in the figure above (part (c)) and reproduced below in more detail:



For the entire movement of the rod from one corridor to the other (while all the time touching the point R), the instant depicted above will be the one when the horizontal component of PR, *i.e.*, PS, will be maximum. Before and after this instant, PS will have a lower value. Therefore, the width w must be at least greater than the maximum value of PS; only then will the pole pass:

$$w_{\min} = PS_{\max}$$

Therefore, our aim is to find $PS_{\max}$. From the geometry of the figure, we can write:

$$d\,\text{cosec}\theta + (PS)\sec\theta = l$$

$$\Rightarrow \quad PS = \frac{\ell - d\,\text{cosec}\theta}{\sec\theta} = \ell\cos\theta - d\cot\theta$$

For maximum PS, $\dfrac{d(PS)}{d\theta} = 0$ and $\dfrac{d^2(PS)}{d\theta^2} < 0$. Now,

$$\frac{d(PS)}{d\theta} = -\ell\sin\theta + d\,\text{cosec}^2\theta$$

This is 0 when

$$\ell\sin\theta = d\,\text{cosec}^2\theta \quad \Rightarrow \quad \sin\theta = \left(\frac{d}{\ell}\right)^{\frac{1}{3}}$$

Verify that $\left.\dfrac{d^2(PS)}{d\theta^2}\right|_{\theta=\sin^{-1}\left(\frac{d}{\ell}\right)^{\frac{1}{3}}} < 0$

$$\Rightarrow \quad \sin\theta = (\tfrac{d}{\ell})^{\frac{1}{3}} \text{ is a point of maximum for } PS.$$

$$\Rightarrow \quad PS_{\max} = \ell\cos\theta - d\cot\theta\Big|_{\theta=\sin^{-1}\left(\frac{d}{\ell}\right)^{\frac{1}{3}}} = (\ell^{\frac{2}{3}} - d^{\frac{2}{3}})^{\frac{3}{2}}$$

$$\Rightarrow \quad w_{\min} = (\ell^{\frac{2}{3}} - d^{\frac{2}{3}})^{\frac{3}{2}}$$

■

Example 20

Find the bigger of the two numbers e^{π} and π^{e}.

Solution: To determine the bigger of the two numbers e^{π} and π^{e}, we can equivalently determine the bigger of the two numbers $e^{\frac{1}{e}}$ and $\pi^{1/\pi}$ (why?). This latter alternative is helpful because we can now construct a function $f(x) = x^{\frac{1}{x}}$ and analyse this for monotonicity. We can then find which of the two numbers $f(e)$ and $f(\pi)$ is larger.

$$f(x) = x^{1/x};\, x > 0 \quad \Rightarrow \quad \ln f(x) = \frac{1}{x}\ln x$$

Differentiating both sides, we get

$$\frac{1}{f(x)}.f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} \quad \Rightarrow \quad f'(x) = x^{\frac{1}{x}}\frac{(1-\ln x)}{x^2}$$

$$\Rightarrow \quad f'(x) > 0 \text{ if } 1 - \ln x > 0 \quad \text{or} \quad x < e$$

$$\text{and} \quad f'(x) < 0 \text{ if } 1 - \ln x < 0 \quad \text{or} \quad x > e$$

Therefore, $f(x)$ increases on $(0, e)$ and decreases on (e, ∞).

$$\Rightarrow \quad f(e) > f(\pi) \quad (\text{since } f \text{ decreases for } x > e \text{ and } \pi \text{ is greater than } e)$$

$$\Rightarrow \quad e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}} \quad \Rightarrow \quad e^{\pi} > \pi^{e}$$

■

Example 21

If $a, b, c \in \mathbb{R}$ and $a + b + c = 0$, show that the equation $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

Solution: Notice that the expression $3ax^2+2bx+c$ can be obtained from the differentiation of $f(x) = ax^3+bx^2+cx+d$, where d is an arbitrary constant. We will now try to use $f(x)$ to prove the stated assertion. Since the question mentions the interval [0, 1], we first find out $f(0)$ and $f(1)$:

$$f(0) = d$$

$$f(1) = a+b+c+d = d \qquad (\text{since } a+b+c=0)$$

Also, since $f(x)$ is a polynomial function, it is differentiable and hence Rolle's theorem can be applied to it for the interval [0, 1]:

There exists at least one $p \in [0, 1]$ such that $f'(p) = 0$, *i.e.*,

$3ax^2+2bx+c=0$ for at least one $p \in [0,1]$.

$\Rightarrow$ $3ax^2+2bx+c=0$ has at least one root in [0, 1]. ■

Example 22

If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 2, g(0) = 0, \ f(1) = 6$ and $g(1) = 2$, then show that there exists $c \in (0,1)$ such that $f'(c) = 2g'(c)$.

Solution: The nature of the proof required hints that we have to use one of the Mean Value Theorems. We construct a new function for this purpose:

$$h(x) = f(x) - 2g(x)$$

Now,

$$h(0) = f(0) - 2g(0) = 2 \text{ and } h(1) = f(1) - 2g(1) = 2$$

Also, since $f(x)$ and $g(x)$ are differentiable on [0, 1], $h(x)$ must also be differentiable on [0, 1]. Therefore, Rolle's theorem can be applied on $h(x)$ for the interval [0, 1]:

There exists $c \in [0,1]$ such that $h'(c) = 0$.

$\Rightarrow$ $f'(c) - 2g'(c) = 0$ for some $c \in [0, 1]$.

$\Rightarrow$ $f'(c) = 2g'(c)$ for some $c \in [0, 1]$.

Refer to the last point in the Important Ideas and Tips section to understand an important misconception which arises in the context of this problem. ■

Example 23

Plot the graphs of the following functions:

(a) $f(x) = 2\sin x + \cos 2x$

(b) $f(x) = 3x + \dfrac{3x}{x-1}$

(c) $f(x) = \dfrac{3x}{2}\ln\left(e - \dfrac{1}{3x}\right)$

(d) $f(x) = \sqrt{1+x^2}\sin\dfrac{1}{x}$

(e) $f(x) = \dfrac{2x^3}{x^2-4}$

(f) $f(x) = \dfrac{x}{\ln x}$

Solution: **(a)** Observe that $f(x)$ is periodic with period 2π. Therefore, we can analyse $f(x)$ only for the interval $[0, 2\pi]$, and by virtue of its periodicity, this analysis will remain applicable in all intervals of the form $[2n\pi, (2n+1)\pi]$, $n \in \mathbb{Z}$. Now,

$$f'(x) = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x)$$

This is 0 when

$$\cos x = 0 \ or \ 1 - 2\sin x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore, we have 4 extremum points, namely

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

We now evaluate $f''(x)$ at each of these points. We have $f''(x) = -2\sin x - 4\cos 2x$, and so:

$$f''\left(\frac{\pi}{6}\right) = -2\times\frac{1}{2} - 4\times\frac{1}{2} = -3 < 0$$

$$\Rightarrow \quad x = \frac{\pi}{6} \text{ is a point of local maximum; } f\left(\frac{\pi}{6}\right) = \frac{3}{2}$$

$$f''\left(\frac{\pi}{2}\right) = -2\times 1 - 4\times(-1) = 2 > 0$$

$$\Rightarrow \quad x = \frac{\pi}{2} \text{ is a point of local minimum; } f\left(\frac{\pi}{2}\right) = 1$$

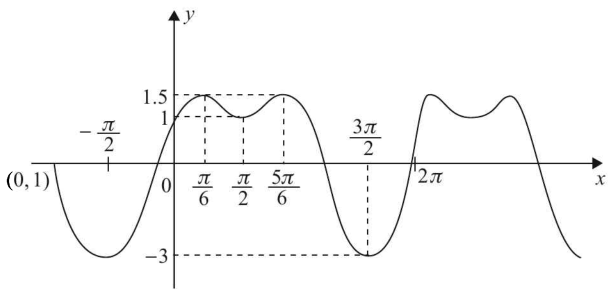
$$f''\left(\frac{5\pi}{6}\right) = -2\times\frac{1}{2} - 4\times\frac{1}{2} = -3 < 0$$

$$\Rightarrow \quad x = \frac{5\pi}{6} \text{ is a point of local maximum; } f\left(\frac{5\pi}{6}\right) = \frac{3}{2}$$

$$f''\left(\frac{3\pi}{2}\right) = (-2\times -1) - 4\times(-1) = 6 > 0$$

$$\Rightarrow \quad x = \frac{3\pi}{2} \text{ is a point of local minimum; } f\left(\frac{3\pi}{2}\right) = -3$$

Additionally, $f(0) = 1$ and $f(2\pi) = 1$. The graph is obtained by plotting these points and joining them by a smooth curve; and finally replicating this graph periodically:



(b) Notice that $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = +\infty$:

$\Rightarrow \quad x = 1$ is an asymptote for the given curve.

Now,

$$f'(x) = 3 + \frac{3(x-1) - 3x}{(x-1)^2} = 3 - \frac{3}{(x-1)^2} = \frac{3(x^2 - 2x)}{(x-1)^2}$$

This is 0 when $x = 0, 2$. Therefore, there are two extremum points:

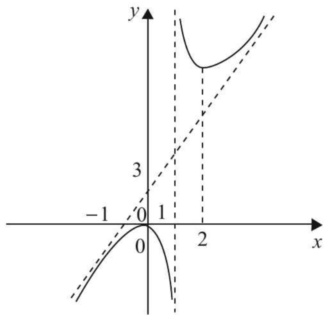
$$f(0) = 0 \text{ and } f(2) = 12$$

We need not determine $f''(x)$ here; we just observe that there is another asymptote to this curve. This is how it can be determined:

$$\lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty}\left(3+\frac{3}{x-1}\right) = 3$$

Also, $$\lim_{x\to\pm\infty}(y-3x) = \lim_{x\to\pm\infty}\left(\frac{3x}{x-1}\right) = 3$$

Therefore, $y - 3x = 3$ is another asymptote to the curve. We now simply draw the two asymptotes (and deduce that $x = 2$ must be a minimum point and $x = 0$ must be a maximum point). The graph is drawn below; observe the details carefully:



(c) The domain of this function is given by:

$$e - \frac{1}{3x} > 0 \quad \Rightarrow x < 0 \text{ or } x > \frac{1}{3e}$$

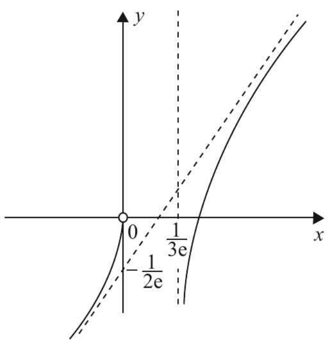
Since $\lim\limits_{x\to\frac{1}{3e}^+} f(x) = -\infty$, $x = \frac{1}{3e}$ is an asymptote to the given curve. Also, $\lim\limits_{x\to 0^-} f(x) = 0$.

Now, $$\lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty}\left(\frac{3}{2}\ln\left(e-\frac{1}{3x}\right)\right) = \frac{3}{2}\ln e = \frac{3}{2}$$

$$\lim_{x\to\pm\infty}\left(y-\frac{3x}{2}\right) = \frac{3}{2}\lim_{x\to\pm\infty} x\left(\ln\left(e-\frac{1}{3x}\right)-1\right) = \frac{-1}{2e} \quad \text{(verify)}$$

$\Rightarrow \quad y = \frac{3x}{2} - \frac{1}{2e}$ is another asymptote to the curve.

We now draw the two asymptotes from which the graphs easily follows (notice that we did not even evaluate $f'(x)$; why?).



(d) In this problem $y = \sqrt{1+x^2}$ will act as an *envelope* for $f(x)$. Around the origin, the oscillations will be extremely fast and will 'spread out' away from the origin.

Now, $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\sqrt{1+x^2}\sin\frac{1}{x}\right) = \lim_{x\to\infty}\left[\frac{\sqrt{1+x^2}}{x}\cdot\left\{\frac{\sin(\frac{1}{x})}{\frac{1}{x}}\right\}\right] = 1$$

$\Rightarrow \quad y = 1$ is an asymptote to the curve.

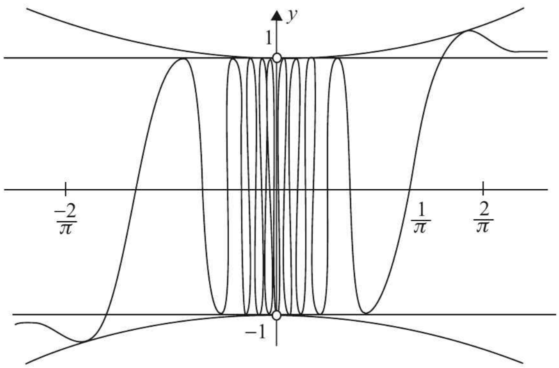
Similarly, $$\lim_{x\to-\infty} f(x) = -1$$

$\Rightarrow \quad y = -1$ is also an asymptote to the curve.

The roots of $f(x) = 0$ are given by:

$$\frac{1}{x} = n\pi \ ; n\in\mathbb{Z} \quad \Rightarrow \quad x = \frac{1}{n\pi}; n\in\mathbb{Z}$$

This information is now sufficient do draw the graph of $f(x)$:



(e) The domain of the given function is

$$D = \mathbb{R}\setminus\{-2, 2\}$$

$\Rightarrow \quad x = -2$ and $x = 2$ are asymptotes to the curve.

Now we evaluate the essentials limits of $f(x)$:

$$\lim_{x\to\infty} f(x) = +\infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

$$\lim_{x\to2^+} f(x) = +\infty \text{ and } \lim_{x\to2^-} f(x) = -\infty$$

$$\lim_{x\to-2^+} f(x) = +\infty \text{ and } \lim_{x\to-2^-} f(x) = -\infty$$

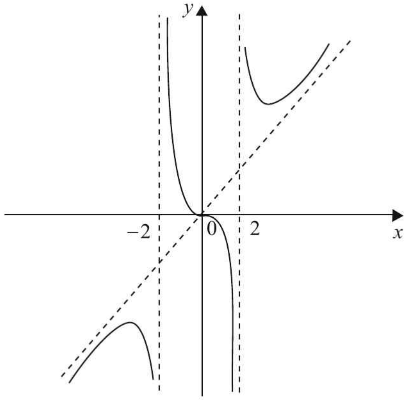
$$\lim_{x\to0} f(x) = 0 \text{ and } f(0) = 0$$

$$\lim_{x\to\pm\infty}\left(\frac{y}{x}\right) = \lim_{x\to\pm\infty}\left(\frac{2x^2}{x^2-4}\right) = 2 \text{ and } \lim_{x\to\pm\infty}(y-2x) = \lim_{x\to\pm\infty}\left(\frac{8x}{x^2-4}\right) = 0$$

$\Rightarrow \quad y = 2x$ is another asymptote to the curve.

Finally, $$f'(x) = \frac{2x^2(x^2-12)}{(x^2-4)^2}.$$

This is 0 when $x = 0, \pm 2\sqrt{3}$. While drawing the graph, it becomes obvious that $x = 2\sqrt{3}$ is a local minimum point while $x = -2\sqrt{3}$ is a local maximum point. The only additional observation we need to make is that $f''(0) = 0$ so that $x = 0$ is neither a local maximum nor a local minimum but a point of inflexion:



(f) The domain of the given function will be $(0, \infty)\setminus\{1\}$.

$$\lim_{x\to 1^+} \frac{x}{\ln x} = +\infty, \; \lim_{x\to 1^-} \frac{x}{\ln x} = -\infty$$

$\Rightarrow$ $x = 1$ will be an asymptote to the given curve.

Also, $\quad \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x}$ (L.H rule) $= \infty$ and $\lim_{x\to 0} \frac{x}{\ln x} = 0$

Now, $\quad f'(x) = \frac{\ln x - 1}{(\ln x)^2}$

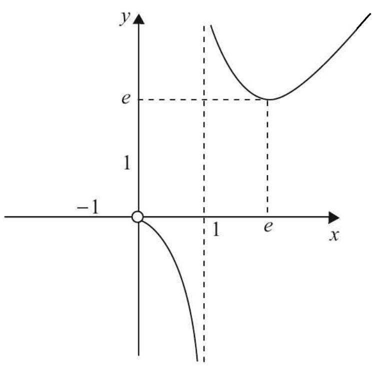
This is 0 when $x = e$:

$\Rightarrow$ $f'(x) > 0$ for $x > e$ or $f(x)$ increases on (e, ∞)

$\Rightarrow$ $f'(x) < 0$ for $x < e$ or $f(x)$ decreases on $(0, e)\setminus\{1\}$

$\Rightarrow$ $f(e) = e$

The graph, based on this information, is drawn below.

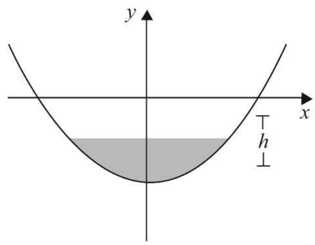


■

Applications of Derivatives

PART-C: Advanced Problems

P1. Consider the parabola $x^2 = 4a(y+a)$, $a > 0$. A person starts to paint the region below the x-axis in such a manner that the height of the painted region h increases at a constant rate of 1 unit per hour.



Let the rate at which the painted area A increases be represented by R. The value of $\frac{R}{\sqrt{ah}}$ is

(A) 1 (B) 2 (C) 4 (D) 8

P2. Two facts are specified about the motion of a particle:

(I) The distance x travelled by the particle depends on time t as

$$x(t) = \sqrt{at^2 + 2bt + c}, \quad a, b, c \text{ are constants}$$

(II) The acceleration of the particle is $\frac{d^2x}{dt^2} = \frac{\lambda}{x^n}$.

(a) The value of λ is

(A) $ac + b^2$ (B) $ac - b^2$ (C) $a^2 + c^2 - b^2$ (D) None of these

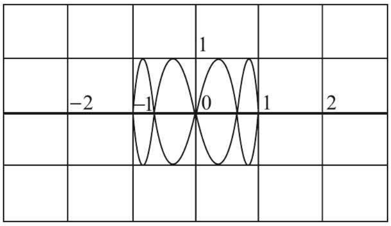
(b) The value of n is

(A) 2 (B) 3 (C) 4 (D) 5

P3. A parametric function is specified as

$$x = \sin t, \; y = \sin \lambda t, \quad t \in [0, 2\pi],$$

and its plot is given as follows:



The value of λ is

(A) 2 (B) 4 (C) 6 (D) 8

P4. Let f be a twice-differentiable function such that

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \; n = 1, 2, 3, \ldots$$

The magnitude of $f''(0)$ is

(A) 1 (B) 2 (C) 3 (D) 4

P5. The tangent at a point P_1 (other than (0, 0)) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Which of the following statements are true?

(A) The x-coordinates of $P_1, P_2, \ldots, P_n$ form an arithmetic progression.

(B) The y-coordinates of $P_1, P_2, \ldots, P_n$ form a geometric progression.

(C) $[\text{area } (\Delta P_1 P_2 P_3)]/[\text{area } \Delta P_2 P_3 P_4] = \dfrac{1}{16}$

(D) $[\text{area } (\Delta P_i P_{i+1} P_{i+2})]/[\text{area } \Delta P_{i+1} P_{i+2} P_{i+3}]$ is fixed.

P6. P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and Q is the corresponding point on the auxiliary circle $x^2 + y^2 = a^2$. Let $\theta_{\max}$ be the greatest angle between the tangents at P and Q. The value of $\tan(\theta_{\max})$ is

(A) $\dfrac{a-b}{\sqrt{ab}}$ (B) $\dfrac{a-b}{2\sqrt{ab}}$ (C) $\dfrac{a+b}{2\sqrt{ab}}$ (D) $\dfrac{a+b}{3\sqrt{ab}}$ (E) $\dfrac{a-b}{3\sqrt{ab}}$

P7. Let $f(x) = \frac{x^2-x+1}{x^2+x+1}$. Let $g(x) = 2f^3(x) - 9f^2(x) + 12f(x) + 5$. In which of the following intervals is $g(x)$ increasing?

(A) $\left(\dfrac{-3-\sqrt{5}}{2}, -1\right)$ (B) $\left(-1, \dfrac{-3+\sqrt{5}}{2}\right)$ (C) $\left(\dfrac{-3+\sqrt{5}}{2}, 0\right)$ (D) $(1, \infty)$

P8. The number of extreme points of $f(x) = \int_0^{x^2} \frac{y^2-10y+9}{2^y+3^y}\, dy$ is

(A) 3 (B) 4 (C) 5 (D) 6 (E) None of these

P9. A circle of radius 1 unit touches the positive x-axis and the positive y-axis at P and Q respectively. A variable line L passing through the origin intersects this circle in M and N. What is the equation of L for which the area of $\Delta\, MNQ$ is maximum?

(A) $x - \sqrt{3}y = 0$ (B) $x + \sqrt{3}y = 0$ (C) $x - 3y = 0$ (D) $x + 3y = 0$ (E) None of these

P10. From a fixed point P on the circumference of a circle of radius a, PN is drawn perpendicular to the tangent at a variable point M of the circle. What is the maximum area of $\Delta\, PMN$?

(A) $\frac{\sqrt{3}}{8}a^2$ (B) $\frac{\sqrt{3}}{4}a^2$ (C) $\frac{3\sqrt{3}}{8}a^2$ (D) $\frac{\sqrt{3}}{2}a^2$ (E) None of these

P11. The minimum value of the function

$$f(x) = |\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$$

is

(A) $\sqrt{2}-1$ (B) $\sqrt{2}+1$ (C) $2\sqrt{2}-1$ (D) $2\sqrt{2}+1$ (E) None of these

P12. The minimum value of $|f(x)|$ where $f(x)$ is given by

$$f(x) = \frac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$$

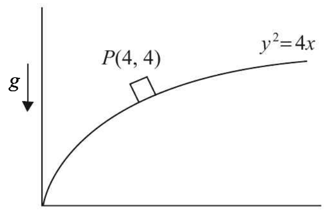
is

(A) 3 (B) 6 (C) 9 (D) 12

P13. Assuming that the petrol burnt in driving a motor boat varies as the cube of its velocity, let s be the most economical speed of the boat with respect to the ground when going against a current of c miles per hour. The value of $\frac{c}{s}$ is

(A) 1 (B) 1.5 (C) 2 (D) 2.5

P14. Consider a frictionless parabolic surface with the equation $y^2 = 4x$ in the vertical plane.
A mass m is placed at the point $P(4, 4)$ and let go.
Let Q be the point where the mass loses contact with the surface. The radius of curvature at any point on $y = f(x)$ is given by $r = \left|\frac{(1+(f'(x)^2)^{\frac{3}{2}}}{f''(x)}\right|$. Assume that all units of length are in meters.



(a) The point Q is given by

(A) $\left(\frac{1}{16},\frac{1}{2}\right)$ (B) $\left(\frac{1}{8},\frac{1}{\sqrt{2}}\right)$ (C) $\left(\frac{1}{4},1\right)$ (D) $(1,2)$

(b) The radius of curvature at Q is

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $4\sqrt{2}$ (D) $5\sqrt{2}$

(c) In m/s, the velocity of the mass m at Q is

(A) $\sqrt{g}$ (B) $2\sqrt{g}$ (C) $3\sqrt{g}$ (D) $4\sqrt{g}$

P15. If $f(x)$ is a twice differentiable function such that $f(a)=0,\ f(b)=2,\ f(c)=-1,\ f(d)=2,\ f(e)=0,$ where $a<b<c<d<e$, then the minimum number of zeroes of $g(x)=\{f'(x)\}^2 + f''(x)\cdot f(x)$ in the interval $[a, e]$ is

(A) 4 (B) 5 (C) 6 (D) 7

SUBJECTIVE TYPE EXAMPLES

P16. Let $f(x)$ and $g(x)$ be two functions such that

$$f''(x) = 2f'(x) - 2f(x)$$
$$g(x) = f'(x) - f(x)$$

Let $F(x)$ be defined as

$$F(x) = [f^2(x) + g^2(x)][f^2(-x) + g^2(-x)]$$

If $F(0) = 1$, find the value of $F(1)$.

P17. Let $f(x) = x^3 + e^{\frac{x}{2}}$, and let $g(x) = f^{-1}(x)$. Find the value of $g'(1)$.

P18. Let $y(x)$ be a function of x such that

$$\sum_{r=0}^{n} x^r y^{n-r} = x + y, \quad x, y \neq 0$$

Find $y'(x)$ when $y(x) = 2x$.

P19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f\left(\frac{1}{n}\right) = \sin\left(\frac{1}{n}\right) \quad \forall \ n \geq 1, n \in \mathbb{Z}$$

Find the values of $f(0)$ and $f'(0)$.

P20. Let $f(x)$ and $g(x)$ be two functions such that $f''(x) = g'(x), f'(x) = g(x), f(0) = 1$ and $g(0) = 0$. Can $f(x) = 0$ have real roots?

P21. Let $f(x) = ax - \frac{x^3}{1+x^2}$, where a is a constant. Show that if $a \geq \frac{9}{8}$, then $f(x)$ is increasing for every x.

P22. Let $f(x) = (x^2 + bx + c)e^x$. For all $x \in \mathbb{R}$,

(a) does $f > 0$ imply $f' > 0$? (b) does $f' > 0$ imply $f > 0$?

P23. (a) For what real values of a does the function $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ increase for all x?

(b) Find the positive values of a for which the point of minimum of the function

$$f(x) = 1 + a^2x - x^3 \text{ satisfies } \frac{x^2 + x + 2}{x^2 + 5x + 6} < 0.$$

P24. Let $g(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$ and $f''(x) < 0 \quad \forall x \in (0,2)$. In(0, 2), find the interval where $g(x)$ increases.

P25. a, b, c are real numbers, and $f(x)$ is given by

$$f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$$

$f(x)$ is decreasing on $(\lambda, 0)$. What is the minimum value of λ?

P26. Let $q \in (0,1)$. Show that the equation $x = q\sin x + a, a \in \mathbb{R}$ has a unique real solution. Find the interval in which that solution lies.

P27. A function $f(x)$ defined for $x \geq 0$ satisfies:

$$f(0) = 0; \quad f'(x) > 0 \quad \forall \quad x > 0; \quad f''(x) < 0 \quad \forall \quad x > 0$$

Define a function $g(x) = \frac{f'(x)}{f(x)}$.

(a) If $g(x) = \frac{a}{x}$, find $f(x)$ and hence the range of values of a.

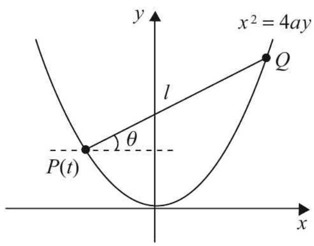
(b) Repeat part-(a) for $g(x) = \frac{b}{x^2}$ and find the range of values of b

P28. Determine the points of maxima and minima of the function

$$f(x) = \frac{1}{8}\ln x - bx + x^2, \ x > 0 \text{ where } b \geq 0 \text{ is a constant.}$$

P29. Let f be a twice-differentiable function such that $f(x) + f''(x) = -xg(x)f'(x)$, where $g(x)$ is non-negative. Prove that $f(x)$ is bounded, that is, there exists a real $C > 0$ such that $|f(x)| < C$ for every x.

P30. Consider the parabola $x^2 = 4ay$, $a > 0$. A rod of length l (greater than the semi-latus-rectum of the parabola) rests on the parabola as shown.



Let θ be the inclination of the rod. Find the value(s) of θ for which the height of the center of the rod is minimum.

P31. A square of side $2a$ always passes through two different points c distance apart in the same horizontal line. Find the minimum distance of the center of the square from the line in the general case and in the special case when $a = c$.

P32. One corner of a long rectangular sheet of paper of width l units is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

P33. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + a_n = 0$, then can we say that the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$$

has at least one root in (0, 1)?

P34. Use Lagrange's mean value theorem (LMVT) to show that if $f''(x) > 0$, then

$$f\left(\frac{x_1 + x_2}{2}\right) < \left(\frac{f(x_1) + f(x_2)}{2}\right)$$

P35. Let f, g, h be monotonically increasing functions defined on $\mathbb{R} \to \mathbb{R}$. Show that

(a) $\frac{f(\xi)-f(a)}{g(b)-g(\xi)} = \frac{f'(\xi)}{g'(\xi)}$ for atleast one $a < \xi < b$

(b) $\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{1}{2}(b-c)\,(c-a)\,(a-b) \begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix}$

for at least one ξ and one $\eta \in \mathbb{R}$.

P36. Find k such that the following equations possess only one real root:

(a) $(1+x^2)e^x - k = 0$
(b) $(e^x - 1) - k\tan^{-1} x = 0$

P37. Taylor's expansion can be used to approximate the value of a function in the vicinity of a particular point, and is given by

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^n(x+h\theta)$$

Note that to terminate the series, the last argument is $x + h\theta$. Find $\lim\limits_{h\to 0} \theta$.

Applications of Derivatives

PART-D: Solutions to Advanced Problems

S1. We evaluate the area of the painted region as a function of h:

$$A = \int_{-2\sqrt{ah}}^{+2\sqrt{ah}} \left((-a+h) - \left(\frac{x^2}{4a} - a \right) \right) dx = \frac{8}{3a}(ah)^{\frac{3}{2}}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{4}{a} \cdot (ah)^{\frac{1}{2}} \cdot a\left(\frac{dh}{dt} \right) = 4\sqrt{ah}$$

Therefore, $\frac{R}{\sqrt{ah}} = 4$. The correct option is (C).

S2. We have $x^2 = at^2 + 2bt + c$:

$$\Rightarrow \quad 2x\frac{dx}{dt} = 2at + 2b$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{at+b}{x} \quad \Rightarrow \quad \frac{d^2x}{dt^2} = \frac{ax^2 - (at+b)^2}{x^3} \qquad \text{(how?)}$$

$$= \frac{a(at^2+2bt+c) - (a^2t^2 + 2abt + b^2)}{x^3} = \frac{ac-b^2}{x^3}$$

The answers to the two parts are (B) and (B).

S3. We note that along the line $y = 1$, $\frac{dy}{dx}$ has the value of 0 for *four* distinct points, whereas along the line $x = 1$, $\frac{dy}{dx}$ has the value of infinity (that is, the tangent is vertical) for *one* point. Thus, $\lambda = 4$. The deduction of this value of λ from the observations we have made is left to the reader as an exercise. The correct option is (B).

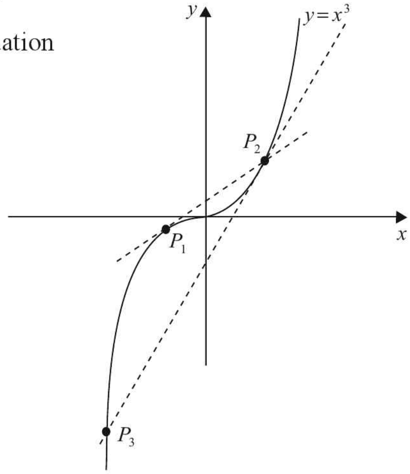
S4. Let $g(x) = f(x) - \frac{1}{1+x^2}$. Note that for $n \in \mathbb{Z}^+$, $g\left(\frac{1}{n}\right) = 0$. Thus, by the continuity of $g(x)$, we have $g(0) = g'(0) = g''(0) = 0$ (how?):

$$\Rightarrow \quad g'(x) = f'(x) + \frac{2x}{(1+x^2)^2} \quad \Rightarrow \quad f'(0) = 0$$

$$\Rightarrow \quad g''(x) = f''(x) + \frac{2}{(1+x^2)^2} + 2x\left(\frac{-4x}{(1+x^2)^3} \right) \quad \Rightarrow \quad f''(0) = -2$$

Therefore, the required magnitude is 2. The correct option is (B).

S5. We assume that P_1 is the point (t_1, t_1^3), and the point P_j is (t_j, t_j^3).



The tangent at the point P_j will have the slope $3t_j^2$, and the equation

$$y - t_j^3 = 3t_j^2(x - t_j)$$

If this meets the curve again at $P_{j+1}(t_{j+1}, t_{j+1}^3)$, then

$$t_{j+1}^3 - t_j^3 = 3t_j^2\ (t_{j+1} - t_j)$$

$$\Rightarrow\quad t_{j+1}^2 + t_j t_{j+1} + t_j^2 = 3t_j^{\ 2}$$

$$\Rightarrow\quad (t_{j+1} - t_j)(t_{j+1} + 2t_j) = 0$$

Since $t_{j+1} \neq t_j$, we have $t_{j+1} = -2t_j$.

Therefore, the x-coordinates of $P_1, P_2, ..., P_n$ form a GP, the common ratio of the GP being –2. The y-coordinates also form a GP, with the common ratio being –8. Now,

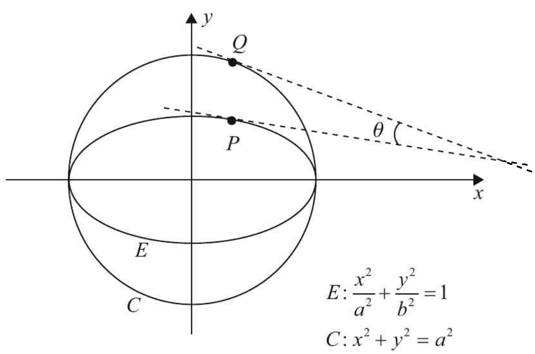
$$\text{area}(\Delta P_1P_2P_3) = \begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix} = \begin{vmatrix} t_1 & t_1^3 & 1 \\ -2t_1 & -8t_1^3 & 1 \\ 4t_1 & 64t_1^3 & 1 \end{vmatrix}$$

whereas

$$\text{area}\left(\Delta P_2P_3P_4\right) = \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \begin{vmatrix} -2t_1 & -8t_1^3 & 1 \\ 4t_1 & 64t_1^3 & 1 \\ -8t_1 & 512t_1^3 & 1 \end{vmatrix} = \frac{1}{16}\begin{vmatrix} t_1 & t_1^3 & 1 \\ -2t_1 & -8t_1^3 & 1 \\ 4t_1 & 64t_1^3 & 1 \end{vmatrix}$$

The ratio of the two areas is thus $\frac{1}{16}$. Also, we see that a similar calculation when applied to [area $(\Delta P_i\ P_{i+1}\ P_{i+2})] / [\text{area } \Delta P_{i+1}\ P_{i+2}\ P_{i+3}]$ will give its value as $\frac{1}{16}$ again. Thus, the correct options are (B), (C) and (D).

S6. We assume P to be the point $(a\cos\phi, b\sin\phi)$, so that Q is the point $(a\cos\phi, a\sin\phi)$. The tangents at P and Q respectively have these slopes:



$$m_P = -\frac{b}{a}\cot\phi$$

$$m_Q = -\cot\phi$$

Thus,

$$\tan\theta = \left|\frac{m_P - m_Q}{1 + m_P m_Q}\right| = \left|\frac{(a-b)\cot\phi}{a + b\cot^2\phi}\right| = \left|\frac{a-b}{a\tan\phi + b\cot\phi}\right|$$

To find the maximum value of θ, *i.e.*, $\theta_{\max}$, we find the minimum value of $f(\phi) = a\tan\phi + b\cot\phi$:

$$f'(\phi) = a\sec^2\phi - b\,\mathrm{cosec}^2\phi = 0$$

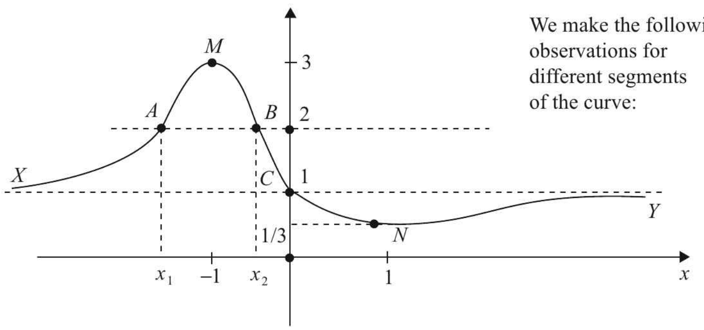
$$\Rightarrow \quad \tan^2\phi = \frac{b}{a} \quad \Rightarrow \quad f_{\min}(\phi) = 2\sqrt{ab} \quad \Rightarrow \quad \tan\theta_{\max} = \frac{a-b}{2\sqrt{ab}}$$

Therefore, the answer is (B).

S7. We observe that $g'(x) = (6f^2(x) - 18f(x) + 12)f'(x)$, *i.e.*,

$$g'(x) = 6f'(x)(f(x)-1)(f(x)-2)$$

We have to find those intervals for which $g'(x)$ is positive. For this purpose, we analyze the function $f(x)$ and draw its graph:



We make the following observations for different segments of the curve:

$$XA:\ \left.\begin{array}{l} f'(x) > 0 \\ f(x) - 1 > 0 \\ f(x) - 2 < 0 \end{array}\right\} g'(x) < 0$$

$$AM:\ \left.\begin{array}{l} f'(x) > 0 \\ f(x) - 1 > 0 \\ f(x) - 2 > 0 \end{array}\right\} g'(x) > 0$$

The X-coordinates of A and B, *i.e.*, x_1 and x_2 are given by solving $\frac{x^2-x+1}{x^2+x+1} = 2$:

$$x_1 = \frac{-3-\sqrt{5}}{2}, \quad x_2 = \frac{-3+\sqrt{5}}{2}$$

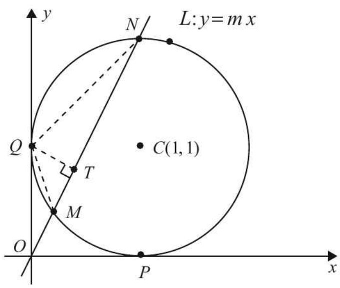
$$MB:\ \left.\begin{array}{l} f'(x) < 0 \\ f(x) - 1 > 0 \\ f(x) - 2 > 0 \end{array}\right\} g'(x) < 0 \qquad BC:\ \left.\begin{array}{l} f'(x) < 0 \\ f(x) - 1 > 0 \\ f(x) - 2 < 0 \end{array}\right\} g'(x) > 0$$

$$CN:\ \left.\begin{array}{l} f'(x) < 0 \\ f(x) - 1 < 0 \\ f(x) - 2 < 0 \end{array}\right\} g'(x) < 0 \qquad NY:\ \left.\begin{array}{l} f'(x) > 0 \\ f(x) - 1 < 0 \\ f(x) - 2 < 0 \end{array}\right\} g'(x) > 0$$

Therefore, $g(x)$ is increasing for $x \in (\frac{-3-\sqrt{5}}{2}, -1) \cup (\frac{-3+\sqrt{5}}{2}, 0) \cup (1, \infty)$. The options (A), (C) and (D) are correct.

S8. Differentiating $f(x)$ with respect to x (under the integral sign), it can easily be seen that $f(x)$ has 5 extreme points: 3 minima (at $x = 0, \pm 3$) and 2 maxima (at $x = \pm 1$). The correct option is (C). The completion of the solution is left to the reader as an exercise.

S9.



The equation of the circle is

$$(x-1)^2+(y-1)^2=1 \qquad (1)$$

We assume the equation of the line to be

$$L: y = mx \qquad (2)$$

Substituting (2) in (1), we have a quadratic in x:

$$(1+m^2)x^2-2(1+m)x+1=0 \qquad (3)$$

Assuming $M \equiv (x_1, mx_1)$, $N \equiv (x_2, mx_2)$, where x_1 and x_2 are the roots of (3), we have

$$MN = \sqrt{(x_1-x_2)^2+(mx_1-mx_2)^2} = \sqrt{1+m^2}\,|x_1-x_2|$$
$$= \sqrt{1+m^2}\sqrt{(x_1+x_2)^2-4x_1x_2}$$

Using the values of x_1+x_2 and x_1x_2 from (3), we have

$$MN = 2\sqrt{2}\sqrt{\frac{m}{1+m^2}}$$

Now, $Q \equiv (0, 1)$, so $QT = \frac{1}{\sqrt{1+m^2}}$. Thus,

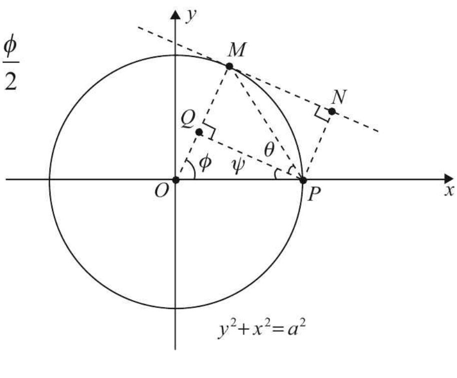
$$\text{area}(\Delta MNQ) = \Delta(m) = \sqrt{2}\left(\frac{\sqrt{m}}{1+m^2}\right) = \frac{\sqrt{2}}{m^{-1/2}+m^{3/2}}$$

To find $\Delta_{\max}$, we find the minimum value of $f(x) = m^{\frac{-1}{2}} + m^{\frac{3}{2}}$. This can easily be shown to be obtained when $m = \frac{1}{3}$. Thus, the equation of L for Δ to be maximum is

$$x-3y=0$$

The correct option is (C).

S10. There is no loss of generality in assuming P to be the point $(a, 0)$. We assume $M \equiv (a\cos\phi, a\sin\phi)$. We make the following observations:



(1) $\psi = \frac{\pi}{2}-\phi$, $\angle OMP = \angle OPM = \frac{\pi}{2}-\frac{\phi}{2}$, $PM = 2a\sin\frac{\phi}{2}$

$$\Rightarrow \quad \theta = \frac{\pi}{2}-\frac{\phi}{2}-\psi = \frac{\phi}{2}$$

(2) $MN = PQ = OP\cos\psi = a\sin\phi$

(3) $PN = QM = PM\sin\theta = 2a\sin\frac{\phi}{2}\cdot\sin\frac{\phi}{2} = 2a\sin^2\frac{\phi}{2}$

Thus,

$$\text{area}(\Delta PMN) = A(\phi) = \frac{1}{2}\times MN\times PN = a^2\sin\phi\sin^2\frac{\phi}{2}$$

We now find $A_{\max}$:

$$A'(\phi) = a^2\left(\cos\phi\sin^2\frac{\phi}{2} + \sin\phi\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\frac{\phi}{2}\cos\phi + \cos\frac{\phi}{2}\sin\phi = \sin\frac{3\phi}{2} = 0$$

$$\Rightarrow \quad \frac{3\phi}{2} = \pi \quad \Rightarrow \quad \phi = \frac{2\pi}{3}$$

Thus, $A_{\max} = \frac{3\sqrt{3}}{8}a^2$. The correct option is (C).

S11. We have

$$f(x) = \left|\sin x + \cos x + \frac{1}{\sin x\cos x} + \frac{\sin x + \cos x}{\sin x\cos x}\right|$$

Using $\sin x + \cos x = t$, and therefore $\sin x\cos x = \frac{t^2-1}{2}$, we have

$$f(t) = \left|t + \frac{2(1+t)}{t^2-1}\right| = \left|t + \frac{2}{t-1}\right| = \left|(t-1) + \frac{2}{t-1} + 1\right| \le 2\sqrt{2} + 1$$

The maximum value is $2\sqrt{2}+1$. The correct option is (D). Note that no differentiation was used.

S12. Substitute $(x+\frac{1}{x})^3 = a$ and $x^3 + \frac{1}{x^3} = b$. Thus,

$$f(x) = \left|\frac{a^2-b^2}{a+b}\right| = |a-b| = \left|\left(x+\frac{1}{x}\right)^3 - x^3 - \frac{1}{x^3}\right| = 3\left|x + \frac{1}{x}\right| \ge 6$$

The minimum value of $|f(x)|$ is therefore 6. The correct option is (B). No differentiation was used in this example. In general, if the answer is obtainable by simple algebraic techniques, then differentiation can be avoided.

S13. We should first note that by "most economical speed" s, what is implied is *that* speed with respect to the ground for which the boat can cover the maximum possible distance *with respect to the ground* for a given amount of fuel. Let us denote the distance travelled with respect to the ground by x, the velocity *with respect to the stream* by v, and the amount of petrol being burnt by f. Then,

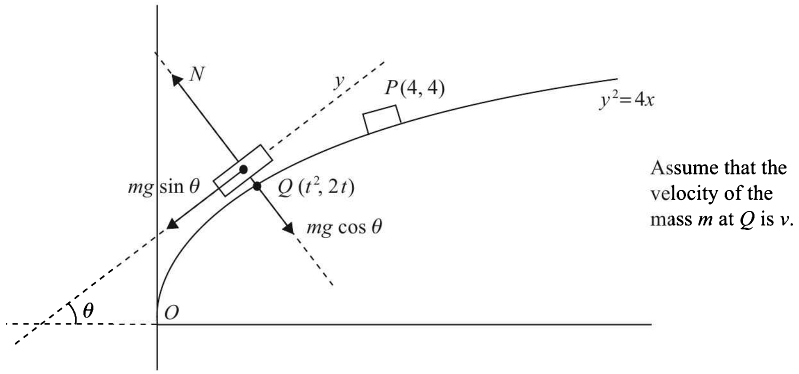
$$\frac{df}{dt} = \lambda v^3 \text{ for some } \lambda, \quad \frac{dx}{dt} = v - c \quad \Rightarrow \quad \frac{df}{dx} = \frac{\lambda v^3}{v-c} = F(v) \quad \text{(say)}$$

Note that $F(v)$ represents the amount of petrol used per unit distance travelled (with respect to the ground). We therefore need to minimize $F(v)$:

$$F'(v) = \lambda\frac{3v^2(v-c) - v^3}{(v-c)^2} = 0 \quad \Rightarrow \quad v = \frac{3c}{2}$$

For this value of v, the speed of the boat s with respect to the ground is $\frac{3c}{2} - c$ *i.e.*, $\frac{c}{2}$ miles/hour, and so c/s has the value 2.

S14. Assume that Q is the point $(t^2, 2t)$:



Step 1: Energy conservation between P and Q

$$\frac{1}{2}mv^2 = mg(4-2t) \quad \Rightarrow \quad v^2 = 4g(2-t) \tag{1}$$

Step 2: $N = 0$ at Q; only $mg\cos\theta$ provides the centripetal force

$$mg\cos\theta = \frac{mv^2}{r_Q} \quad \Rightarrow v^2 = gr_Q\cos\theta \tag{2}$$

Step 3: Find r_Q, $\cos\theta$ for the point Q.

$$y^2 = 4x \quad \Rightarrow \quad y' = \frac{2}{y} = \frac{1}{t} \quad \left(\text{thus,} \quad \tan\theta = \frac{1}{t}\right)$$

$$\Rightarrow \quad y'' = -\frac{2}{y^2}y' = -\frac{1}{2t^3}$$

$$\Rightarrow \quad \cos\theta = \frac{t}{\sqrt{1+t^2}},\ r_Q = \frac{(1+(\frac{1}{t})^2)^{\frac{3}{2}}}{\frac{1}{2t^3}} = 2(t^2+1)^{\frac{3}{2}} \tag{3}$$

Step 4: Use the values of (1) and (3) in (2).

$$4g(2-t) = g\cdot 2(t^2+1)^{\frac{3}{2}}\cdot\frac{t}{(t^2+1)^{\frac{1}{2}}} = 2gt(t^2+1)$$

$$\Rightarrow \quad 2(2-t) = t(t^2+1)$$

$$\Rightarrow \quad t^3 + 3t = 4$$

It should be evident that $t = 1$ is what we were looking for.

(a) $Q \equiv (1,2)$. The correct option is (D).
(b) $r_Q \equiv 4\sqrt{2}$. The correct option is (C).
(c) $v^2 = 4g \quad \Rightarrow \quad v = 2\sqrt{g}$ m/s. The correct option is (B).

S15. Note that $g(x) = (f(x)f'(x))'$. Now, $f(a) = f(e) = 0$, and $f(x)$ also has at least one zero each in (b, c) and (c, d). Thus, in $[a, e]$, $f(x)$ has at least 4 zeroes, which implies that $f'(x)$ has at least 3 zeroes. This further implies that $f(x)f'(x)$ has at least 7 zeroes in $[a, e]$. $g(x)$, which is the derivative of $f(x)f'(x)$, thus has at least 6 zeroes in $[a, e]$. The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

S16. We have

$$g'(x) = f''(x) - f'(x) = (2f'(x) - 2f(x)) - f'(x) = f'(x) - 2f(x)$$

For the sake of convenience, we will write $f(x), g(x), F(x)$ as f, g, F respectively, and $f(-x), g(-x)$ as f_-, g_- respectively. Now,

$$F = (f^2 + g^2)(f_-^2 + g_-^2)$$

Differentiating with respect to x, we have

$$F' = (2ff' + 2gg')(f_-^2 + g_-^2) + (f^2 + g^2)(-2f_-f_-' - 2g_-g_-') \tag{1}$$

Using $g' = f' - 2f$ and $g = f' - f$, the first term in (1) becomes

$$\begin{aligned}(2ff' + 2gg')(f_-^2 + g_-^2) &= (2ff'' + 2(f' - f)(f' - 2f))(f_-^2 + (f_-' - f_-)^2) \\ &= 2(f'^2 + 2f^2 - 2ff')(f_-'^2 + 2f_-^2 - 2f_-f_-')\end{aligned}$$

The second term in (1) will be the exact negative of this term, and thus $F' = 0$. This means that $F(x)$ is a constant function, *i.e.*, $F(1) = F(0) = 1$.

S17. We note that both x^3 and $e^{\frac{x}{2}}$ are strictly increasing on $\mathbb{R}$, and thus $f(x)$ is a strictly increasing function on $\mathbb{R}$. This means that the inverse of $f(x)$ is uniquely defined. In particular,

$$f(0) = 1 \quad \Rightarrow \quad f^{-1}(1) = g(1) = 0$$

Now, since $f(f^{-1}(x)) = x$, we have by differentiating this:

$$f'(f^{-1}(x))(f^{-1}(x))' = 1$$

Since $f^{-1}(x) = g(x)$, we have

$$f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

Therefore,

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)}$$

Finally, since $f'(x) = 3x^2 + \frac{1}{2}e^{\frac{x}{2}}$, we have $f'(0) = \frac{1}{2}$, and thus $g'(1) = 2$.

S18. The left side is a GP with common ratio $\frac{x}{y}$. Summing it and rearranging the result slightly yields

$$x^{n+1} - y^{n+1} = x^2 - y^2 \tag{1}$$

Substituting $y = 2x$ gives $x^{n-1} = \dfrac{3}{2^{n+1} - 1}$ (2)

We need to find the slope of the curve in (1) at the point with the x-coordinate given by (2). Now, differentiate (1) to obtain y':

$$y' = \left.\frac{(n+1)x^n - 2x}{(n+1)y^n - 2y}\right|_{y=2x} = \frac{(n+1)x^n - 2x}{(n+1)2^n x^n - 4x} = \frac{(n+1)x^{n-1} - 2}{2^n(n+1)x^{n-1} - 4}$$

Using (2), we obtain y' as $\frac{3n+5-2^{n+2}}{3n\cdot 2^n + 3\cdot 2^n - 2^{n+3} + 4}$.

S19. Consider the continuous and differentiable function $g(x) = f(x) - \sin x$. We have $g\left(\frac{1}{n}\right) = 0 \ \forall \ n \geq 1, n \in \mathbb{Z}$. Since g is continuous in any infinitesmally small interval around $x = 0$, $g = 0$ for at least one point in this interval, *i.e.*, $\lim_{x\to 0} g(x) = 0$. Thus, $g(0) = 0$, which implies that $f(0) = 0$. Now,

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h)}{h}$$

Since h is small, h can be assumed to lie in the interval $[\frac{1}{N+1}, \frac{1}{N}]$ for some large N. But since $g(\frac{1}{N+1}) = g(\frac{1}{N}) = 0$, by the continuity of $g(x)$, we have $g(h) = 0$, which implies that $g'(0) = 0$. Thus,

$$f'(0) - \cos 0 = 0 \quad \Rightarrow \quad f'(0) = 1$$

S20. We have $f''(x) = g'(x)$ and $f'(x) = g''(x) = g(x)$. Thus,

$$f''(x) = f(x) \tag{1}$$

$$g''(x) = g(x) \tag{2}$$

$$f'(x) = g(x) \tag{3}$$

$$g'(x) = f(x) \tag{4}$$

From (1) and (2), we can deduce the forms of f and g:

$$f(x) = Ae^x + Be^{-x} \quad A, B \in \mathbb{R}$$
$$g(x) = Ce^x + De^{-x} \quad C, D \in \mathbb{R}$$

Since $f(0) = 1$, and $g(0) = 0$, we have

$$A + B = 1, C + D = 0 \tag{5}$$

Also, from (3) and (4), we can obtain the following additional conditions:

$$A = C, B = -D \tag{6}$$

From (5) and (6),

$$A = B = \frac{1}{2}, C = \frac{1}{2}, D = -\frac{1}{2}$$

Thus,

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

It can be now be easily verified that $f(x)$ has a minimum at $x = 0$ with value $f(0) = 1$. Thus, $f(x) = 0$ has no real roots.

S21. We evaluate the derivative of the given function:

$$f'(x) = \frac{a + (2a-3)x^2 + (a-1)x^4}{(1+x^2)^2}$$

$$\geq \frac{\frac{9}{8} + (\frac{9}{4} - 3)x^2 + (\frac{9}{8} - 1)x^4}{(1+x^2)^2} = \frac{(x^2-3)^2}{8(1+x^2)^2} \geq 0 \ \forall x \in \mathbb{R}$$

Thus, $f(x)$ is increasing $\forall\ x \in \mathbb{R}$.

S22. We have $f'(x) = (x^2 + (b+2)x + b + c)e^x$. Thus,

$$f > 0 \quad \Leftrightarrow \quad b^2 - 4c < 0$$

and

$$f' > 0 \quad \Leftrightarrow \quad b^2 - 4c + 4 < 0$$

(a) Let $f > 0\ \forall x \in \mathbb{R}$. Thus, $b^2 - 4c < 0$, but nothing can be said about the sign of $b^2 - 4c + 4$, *i.e.*, it may be positive. Thus, $f > 0\ \forall x \in \mathbb{R}$ does not necessarily imply $f' > 0\ \ \forall\ x \in \mathbb{R}$.

(b) Let $f' > 0\ \ \forall x \in \mathbb{R}$. Thus, $b^2 - 4c + 4 < 0$, which necessarily implies that $b^2 - 4c < 0$ *i.e.*, $f > 0\ \forall x \in \mathbb{R}$

S23. **(a)** We have

$$f'(x) = 2e^x + ae^{-x} + (2a+1)$$

$$= e^{-x}(2e^{2x} + (2a+1)e^x + a) \tag{1}$$

For $f(x)$ to increase for all x, we must have $f'(x) > 0\ \ \forall\ x \in \mathbb{R}$. In the product in (1), the term e^{-x} is positive for all x. Therefore, what we require is that

$$2e^{2x} + (2a+1)e^x + a > 0 \ \ \forall x \in \mathbb{R} \tag{2}$$

If we represent e^x by y, then (2) can be stated equivalently as follows:

$$2y^2 + (2a+1)y + a > 0 \quad \forall y \in \mathbb{R}^+$$

Therefore, the function $g(y) = 2y^2 + (2a+1)y + a$ must have either both zeroes as negative or non-real zeroes:

Case I

$g(y)=0$ has both roots as negative

$\Rightarrow \quad D>0, S<0, P>0$

$$\Rightarrow \begin{cases} (2a+1)^2-8a>0 \\ -\dfrac{(2a+1)}{2}<0 \\ \dfrac{a}{2}>0 \end{cases}$$

$\Rightarrow \quad a>0$

Case II

$g(y)=0$ has non-real roots

$\Rightarrow \quad D<0$

$\Rightarrow$ This is not possible (verify)

Thus, $f(x)$ increases for all x if $a>0$.

(b) Note that x^2+x+2 is positive for all $x \in \mathbb{R}$. On the other hand, $x^2+5x+6<0$ if $x \in (-3,-2)$. Thus,

$$\frac{x^2+x+2}{x^2+5x+6}<0 \quad \Rightarrow \quad x \in (-3,-2)$$

The point of minimum of $f(x)$ should lie in this interval. We have

$$f'(x)=a^2-3x^2=0 \quad \Rightarrow \quad x=\pm\frac{a}{\sqrt{3}}$$

Also, $$f''(x)=-6x=\begin{cases} -\dfrac{6a}{\sqrt{3}} & \text{for} \quad x=\dfrac{a}{\sqrt{3}} \\ +\dfrac{6a}{\sqrt{3}} & \text{for} \quad x=-\dfrac{a}{\sqrt{3}} \end{cases}$$

It is evident that if a is positive, then $x=-\frac{a}{\sqrt{3}}$ is the point of minimum, since $f''(-\frac{a}{\sqrt{3}})>0$. Thus,

$$-3<-\frac{a}{\sqrt{3}}<-2 \quad \Rightarrow \quad 2\sqrt{3}<a<3\sqrt{3}$$

S24. We have $g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x)$. Since $f''(x)<0 \ \forall \ x \in (0,2)$, $f'(x)$ decreases on (0, 2). We observe that

(i) when $\dfrac{x}{2}<2-x \ \left(i.e., \ x<\dfrac{4}{3}\right)$, $f'\left(\dfrac{x}{2}\right)>f'(2-x)$,

(ii) while when $\dfrac{x}{2}>2-x\left(i.e., \ x>\dfrac{4}{3}\right)$, $f'\left(\dfrac{x}{2}\right)<f'(2-x)$.

Thus, $g'(x)\begin{cases} >0 & \forall \ x \in \left(0,\frac{4}{3}\right) \\ <0 & \forall \ x \in \left(\frac{4}{3},2\right) \end{cases}$. We see that $g(x)$ increases on $(0,\frac{4}{3})$.

S25.

$$f'(x)=\begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$=\{(x+b^2)(x+c^2)-b^2c^2\}+\{(x+a^2)(x+c^2)-a^2c^2\}+\{(x+a^2)(x+b^2)-a^2b^2\}$$

$$=3x^2+2(a^2+b^2+c^2)x$$

$f(x)$ is decreasing when $f'(x)<0$:

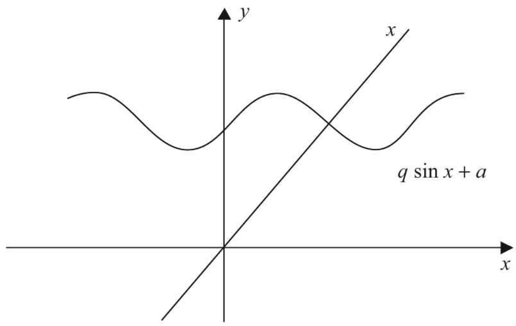
$$\Rightarrow \quad 3x^2+2(a^2+b^2+c^2)x<0 \quad \Rightarrow \quad x\in\left(-\frac{2}{3}(a^2+b^2+c^2),0\right)$$

Thus,

$$\lambda_{\min}=-\frac{2}{3}(a^2+b^2+c^2)$$

S26. To understand why there will be only one unique solution, we plot the graph of both

$$f(x)=x \text{ and } g(x)=q\sin x+a,$$



and consider the points of intersection: Visually, it clearly seems that there will be just one point of intersection, and hence a unique solution. To prove this rigorously, we assume that there are two distinct solutions, $x=x_1$ and $x=x_2$:

$$x_1=q\sin x_1+a$$
$$x_2=q\sin x_2+a$$

Subtracting, we have

$$x_1-x_2=q(\sin x_1-\sin x_2)$$

$$=2q\cos\left(\frac{x_1+x_2}{2}\right)\sin\left(\frac{x_1-x_2}{2}\right)$$

$$\Rightarrow \quad \frac{x_1-x_2}{2}=q\cos\left(\frac{x_1+x_2}{2}\right)\sin\left(\frac{x_1-x_2}{2}\right) \qquad (1)$$

However,

$$\left|q\cos\left(\frac{x_1+x_2}{2}\right)\sin\left(\frac{x_1-x_2}{2}\right)\right|=|q|\left|\cos\left(\frac{x_1+x_2}{2}\right)\right|\left|\sin\left(\frac{x_1-x_2}{2}\right)\right|$$

$$<1\cdot 1\cdot \sin\left(\frac{x_1-x_2}{2}\right)=\sin\left(\frac{x_1-x_2}{2}\right)$$

Thus, the only way (1) can hold is if $x_1 = x_2$, *i.e.*, a unique solution will exist. If $x = x_0$ is the unique solution, then

$$x_0 = q\sin x_0 + a$$

$$\Rightarrow \quad \frac{x_0 - a}{q} = \sin x_0 \quad \in [-1, 1] \quad \Rightarrow \quad x_0 \in \; [a-q, a+q]$$

S27. Observe that $f'(x) > 0 \;\forall\; x > 0 \;\Rightarrow\; f(x) > f(0) = 0 \;\forall\; x > 0$

(a) $$\frac{f'(x)}{f(x)} = \frac{a}{x} \;\Rightarrow\; f(x) = Cx^a \;\Rightarrow\; C > 0$$

$$\Rightarrow \; f'(x) = Cax^{a-1} \;\Rightarrow\; Ca > 0 \;\Rightarrow\; a > 0$$

$$\Rightarrow \; f''(x) = Ca(a-1)x^{\frac{a}{-2}} \;\Rightarrow\; a(a-1) < 0 \;\Rightarrow\; a < 1$$

$$\Rightarrow \; a \in (0, 1)$$

(b) $$\frac{f'(x)}{f(x)} = \frac{b}{x^2} \;\Rightarrow\; f(x) = Ce^{\frac{-b}{x}} \;\Rightarrow\; C > 0$$

$$\Rightarrow \; f'(x) = \frac{Cb}{x^2}e^{\frac{-b}{x}} \;\Rightarrow\; b > 0$$

$$\Rightarrow \; f''(x) = \frac{Cb}{x^4}e^{\frac{-b}{x}}(b - 2x)$$

We observe that for $x < \frac{b}{2}$, $f''(x) > 0$, which is contradictory.

$\Rightarrow$ No such b exists.

S28. Note that the domain of $f(x)$ is $x > 0$. Now,

$$f'(x) = \frac{1}{8x} - b + 2x = \frac{16x^2 - 8bx + 1}{8x}$$

The zeroes of the numerator are $r_1, r_2 = \frac{b}{4} \pm \frac{\sqrt{b^2-1}}{4}$. If $b < 1$, the zeroes are non-real, which means that $f'(x) > 0 \;\forall\; x \in \mathbb{R}$, *i.e.*, $f(x)$ is monotonically increasing on $(0, \infty)$, so that there will be no maxima and minima points. Now,

$$f''(x) = \frac{16x^2 - 1}{8x^2} \quad \text{(verify)}$$

If $b = 1$, then the zeroes of $f'(x)$ are $r_1, r_2 = \frac{1}{4}, \frac{1}{4}$; at $x = \frac{1}{4}$, $f''(x)$ is also zero, which means that for $b = 1$, $x = \frac{1}{4}$ is a point of inflexion ($f''(x)$ changes sign at $x = \frac{1}{4}$), but there still exists no maxima and minima points. We now consider the case where $b > 1$. In this case, we evaluate the signs of $f''(r_1)$ and $f''(r_2)$:

$$f''(r_1) = \frac{2}{r_1^2}\left(r_1^2 - \frac{1}{16}\right) = \frac{2}{r_1^2}\left(r_1 + \frac{1}{4}\right)\left(r_1 - \frac{1}{4}\right)$$

$$= \frac{2}{r_1^2}\left(\frac{b+1}{4} + \frac{\sqrt{b^2-1}}{4}\right)\left(\frac{b-1}{4} + \frac{\sqrt{b^2-1}}{4}\right)$$

We observe that since $b > 1$, both the brackets in this expression will have a net positive sign, which means that $f''(r_1) > 0$. On the other hand,

$$f''(r_2) = \frac{2}{r_2^2}\left(\frac{b+1}{4} - \frac{\sqrt{b^2-1}}{4}\right)\left(\frac{b-1}{4} - \frac{\sqrt{b^2-1}}{4}\right)$$

The left bracket in this expression will have a net positive sign. However, the right bracket will have a net negative sign because $\sqrt{b^2-1} > b-1$. Thus, $f''(r_2) < 0$. We conclude the following:

$b \in (0,1)$: $f(x)$ increases monotonically on $(0,\infty)$.

$b = 1$: $f(x)$increases monotonically on $(0,\infty)$.

$x = \frac{1}{4}$ is a point of inflexion.

$b \in (1,\infty)$: $x = \frac{b}{4} + \frac{\sqrt{b^2-1}}{4}$ is a point of minimum.

$x = \frac{b}{4} - \frac{\sqrt{b^2-1}}{4}$ is a point of maximum.

S29. Multiply both side of the given equality by $2f'(x)$:

$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)(f'(x))^2$$

$$\Rightarrow \quad (f^2(x) + (f'(x))^2)' = -2xg(x)(f'(x))^2 \qquad (1)$$

For $x > 0$, the right side (RHS) is negative, and thus $f^2(x) + (f'(x))^2$ is decreasing for $x > 0$. Similarly, for $x < 0$, it is increasing.

$\Rightarrow$ $f^2(x) + (f'(x))^2$ has a maximum at $x = 0$, because its derivative at $x = 0$ is 0 by (1).

$\Rightarrow$ $f^2(x) + (f'(x))^2 \le f(0) + (f'(0))^2$ for every x.

From this, it follows that $f(x)$ is bounded.

S30. From the geometry of the figure,

$$P \equiv (2at, at^2),\ Q \equiv (2at + 2l\cos\theta,\ at^2 + 2l\sin\theta)$$

From the fact that Q must satisfy $x^2 = 4ay$, we obtain

$$t = \tan\theta - \frac{l}{2a}\cos\theta \qquad \text{(verify)}$$

Therefore, the height is

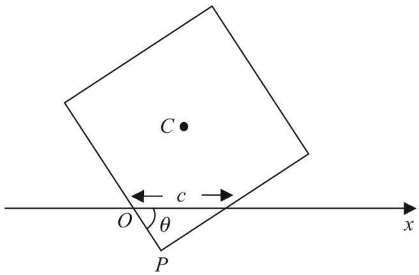
$$h = at^2 + l\sin\theta = a\tan^2\theta + \frac{l^2}{4a}\cos^2\theta$$

$h(\theta)$ has extreme (minima) points for $\cos^2\theta = \frac{2a}{l}$ or $\sin\theta = 0$, with the correspondening values of h as $l - a$ and $\frac{l^2}{4a}$ respectively. The greater of these two values will be determined by the relative magnitudes of l and a.

S31. Assume O to be the origin and the coordinate axes as indicated. Using this axes,

$$P \equiv (c\cos\theta\cos\theta, -c\cos\theta\sin\theta) \equiv (c\cos^2\theta, -c\cos\theta\sin\theta)$$

Also, $CP = \sqrt{2}\,a$ and the angle made by CP with the x-axis is $90° - \theta + 45°$. Thus, the y-coordinate of C is



$$y_C = -c\sin\theta\cos\theta + \sqrt{2}\,a\cos(\theta - 45°)$$

$$= a(\sin\theta + \cos\theta) - \frac{c\sin 2\theta}{2}$$

We wish to minimize y_C:

$$\frac{dy_C}{d\theta} = a(\cos\theta - \sin\theta) - c\cos 2\theta = 0$$

$$\Rightarrow \quad (\cos\theta - \sin\theta)(a - c\sin\theta - c\cos\theta) = 0$$

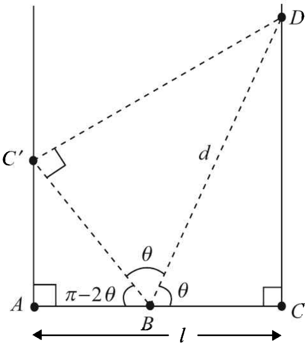
$$\Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2}$$

To ascertain which of the these points might be a minimum, we evaluate the double derivatives (verify these values):

$$\left.\frac{d^2y_C}{d\theta^2}\right|_{\pi=\frac{\pi}{4}} = 2c - \sqrt{2}a; \quad \left.\frac{d^2y_C}{d\theta^2}\right|_{\sin 2\theta = \frac{a^2 - c^2}{c^2}} = \frac{-a^3}{c^2} + 2c\left(\frac{a^2}{c^2} - 1\right)$$

Depending on the values of a and c, we can now ascertain the minima points. When $a = c$, it can be easily seen that $\theta = \frac{\pi}{4}$ will be the minima point, and the required distance willl be $(\sqrt{2} - \frac{1}{2})a$.

S32.



Note that when the paper is folded, C 'goes to' C'. Assume that f is the fraction of AC that is folded over. We make the following observations

(1) $\angle CBD = \angle DBC' = \theta$ (say)

$\angle BC'D = \dfrac{\pi}{2}$, $\angle ABC' = \pi - 2\theta$

(2) $AB = (1-f)l$, $BC = fl$

(3) $d = BC\sec\theta = fl\sec\theta$; also

$d = BC'\sec\theta = AB\sec(\pi - 2\theta)\sec\theta$

$= -(1-f)l\sec\theta\sec 2\theta$

Thus,

$$fl\sec\theta = -(1-f)l\sec\theta\sec 2\theta \Rightarrow f + (1-f)\sec 2\theta = 0 \Rightarrow f = \frac{\sec 2\theta}{\sec 2\theta - 1}$$

$$\Rightarrow d(\theta) = \frac{l\sec\theta\sec 2\theta}{\sec 2\theta - 1} = \frac{l}{\cos\theta(1-\cos 2\theta)} = \frac{l/2}{\sin^2\theta\cos\theta}$$

To minimize d, we maximize $h(\theta) = \sin^2\theta\cos\theta$:

$$h'(\theta) = 2\sin\theta\cos^2\theta - \sin^3\theta = 0$$

$$\Rightarrow \tan\theta = \sqrt{2} \Rightarrow \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}, \cos\theta = \frac{1}{\sqrt{3}}$$

Thus, $h_{\max} = \frac{2}{3\sqrt{3}}$, so that $d_{\min} = \frac{3\sqrt{3}}{4}l$.

S33. Yes. The solution involves a simple application of the Rolle's theorem. Consider $f(x)$ given by

$$f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1} + \cdots + a_nx$$

We note that $f(0) = f(1) = 0$, and thus $f'(x) = 0$ must have at least one root in (0, 1).

S34. Assume $x_1 < x_2$, which implies that $x_1 < \frac{x_1+x_2}{2} < x_2$. By LMVT, there exist

$$\zeta_1 \in \left(x_1, \frac{x_1+x_2}{2}\right) \text{ and } \zeta_2 \in \left(\frac{x_1+x_2}{2}, x_2\right)$$

such that

$$f'(\zeta_1) = \frac{f(\frac{x_1+x_2}{2}) - f(x_1)}{\frac{x_2-x_1}{2}} \text{ and } f'(\zeta_2) = \frac{f(x_2) - f(\frac{x_1+x_2}{2})}{\frac{x_2-x_1}{2}}$$

Subtracting these, we have

$$f'(\zeta_2)-f'(\zeta_1)=\frac{f(x_1)+f(x_2)-2f(\frac{x_1+x_2}{2})}{\frac{x_2-x_1}{2}}$$

Now,

$$f'(\zeta_2)-f'(\zeta_1)=(\zeta_2-\zeta_1)f''(\zeta)(\text{for some } \zeta \in (\zeta_1,\zeta_2))$$

$$>0$$

$$\Rightarrow \quad \frac{f(x_1)+f(x_2)}{2}>f\left(\frac{x_1+x_2}{2}\right)$$

S35. **(a)** Assume $h(x)=f(x)g(x)-f(a)g(x)-f(x)g(b)$. Thus,

$$h(a)=h(b)=-f(a)g(b)$$

By the Rolle's theorem, there exists $\xi \in (a,b)$ such that $h'(\xi)=0$, *i.e.*,

$$f'(\xi)g(\xi)-f'(\xi)g(b)+f(\xi)g'(\xi)-f(a)g'(\xi)=0$$

$$\Rightarrow \quad \frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$$

(b) Consider the function

$$F(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}-\frac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

Note that $F(a)=F(b)=F(c)$. By applying the Rolle's theorem twice, we reach the conclusion that $F''(\xi)=0$ for some $\xi \in (a,c)$:

$$\Rightarrow \quad \begin{vmatrix} f(a) & f(b) & f''(\xi) \\ g(a) & g(b) & g''(\xi) \\ h(a) & h(b) & h''(\xi) \end{vmatrix}=\frac{2}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} \tag{1}$$

Now, we construct another function $G(x)$ given by

$$G(x)=\begin{vmatrix} f(a) & f(x) & f''(\xi) \\ g(a) & g(x) & g''(\xi) \\ h(a) & h(x) & h''(\xi) \end{vmatrix}-\frac{(x-a)}{b-a}\begin{vmatrix} f(a) & f(b) & f''(\xi) \\ g(a) & g(b) & g''(\xi) \\ h(a) & h(b) & h''(\xi) \end{vmatrix}$$

Since $G(a)=G(b)=0$, we apply the Rolle's theorem and conclude that there exists a $\eta \in (a,b)$ such that $G'(\eta)=0$:

$$\Rightarrow \begin{vmatrix} f(a) & f'(\eta) & f''(\xi) \\ g(a) & g'(\eta) & g''(\xi) \\ h(a) & h'(\eta) & h''(\xi) \end{vmatrix} = \frac{1}{b-a}\begin{vmatrix} f(a) & f(b) & f''(\xi) \\ g(a) & g(b) & g''(\xi) \\ h(a) & h(b) & h''(\xi) \end{vmatrix} \qquad (2)$$

From (1) and (2), the conclusion follows.

S36. (a) Assume $\quad f(x) = (1+x^2)e^x - k$

$$\Rightarrow \quad f'(x) = (1+x)^2 e^x \geq 0$$

Therefore, $x = -1$ is a point of inflection. Now, as $x \to -\infty$, $f \to -k$ and as $x \to \infty$, $f \to \infty$. Also $f(0) = 1-k$. Along with the fact that $f(x)$ is increasing, we now have

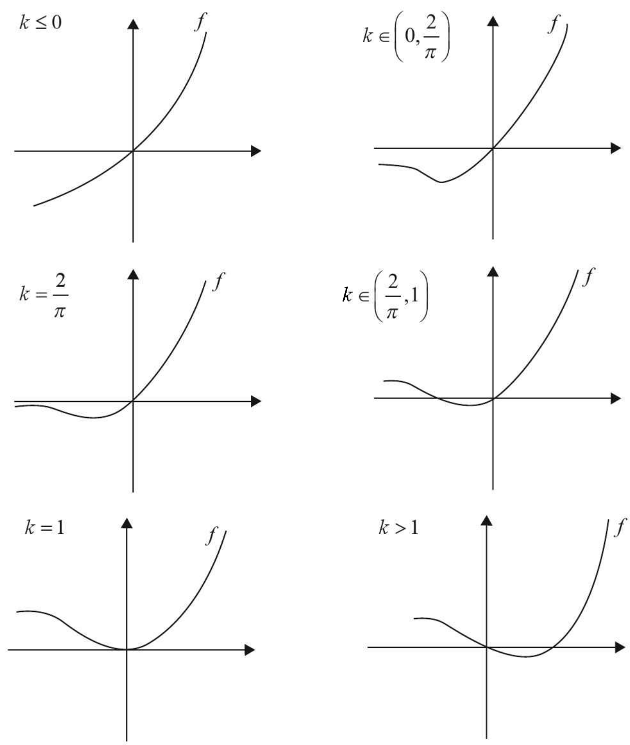
$$k < 0 \quad \Rightarrow \quad f(x) > 0 \text{ for every } x$$

$$k \in (0,1) \quad \Rightarrow \quad f(x) = 0 \text{ for some } x < 0$$

$$k > 1 \quad \Rightarrow \quad f(x) = 0 \text{ for some } x > 0$$

Therefore, for $k > 0$, $f(x) = 0$ has only one real root

(b) Assume $f(x) = e^x - 1 - k\tan^{-1}x$. As $x \to -\infty$, $f \to \frac{k\pi}{2} - 1$ and as $x \to \infty$, $f \to \infty$. Also, $f(0) = 0$, f $f'(x) = e^x - \frac{k}{1+x^2}$ and $f'(0) = 1-k$. Note that if $k < \frac{2}{\pi}$, then $f(-\infty) < 0$, while for $k > \frac{2}{\pi}$, $f(-\infty) > 0$. Also, for $k = 1$, $f'(0) = 0$, while for $k < 1$, $f'(0) > 0$ and for $k > 1$, $f'(0) < 0$. Based on these facts, we divide the situation into six cases:



Therefore, $k \in (-\infty, \frac{2}{\pi}] \cup \{1\}$.

S37. Comparing the expansion given in the question with the (standard) infinite Taylor's expansion, we have

$$\frac{h^n}{n!} f^n(x+h\theta) = \frac{h^n}{n!} f^n(x) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(x) + \frac{h^{n+2}}{(n+2)!} f^{n+2}(x) + \cdots \infty$$

$$\Rightarrow \quad \frac{f^n(x+h\theta) - f^n(x)}{h} = \frac{1}{n+1} f^{n+1}(x) + \frac{h \cdot n!}{(n+2)!} f^{n+2}(x) + \cdots \infty$$

Applying $\lim\limits_{h\to 0}$ on both sides, we have

$$\lim_{h\to 0} \frac{f^n(x+h\theta) - f^n(x)}{h} = \frac{f^{n+1}(x)}{n+1}$$

$$\Rightarrow \quad \theta \lim_{h\to 0} \frac{f^n(x+h\theta) - f^n(x)}{h\theta} = \frac{f^{n+1}(x)}{n+1}$$

$$\Rightarrow \quad \theta . f^{n+1}(x) = \frac{f^{n+1}(x)}{n+1}$$

$$\Rightarrow \quad \theta = \frac{1}{n+1}$$

Integration

PART-A: Summary of Important Concepts

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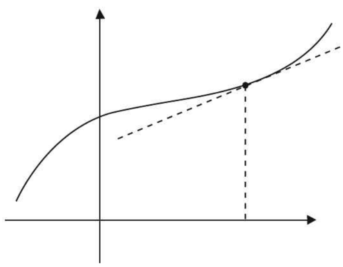
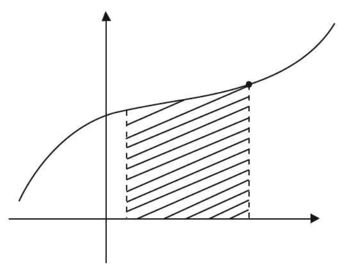
1. What is Integration?

Integral Calculus is one of the two fundamental branches of Calculus, the other being Differential Calculus, and enables us to calculate the areas under arbitrary curves, as Differential Calculus helps us to find the slopes of arbitrary curves. This association should always be perfectly clear in your mind:

DIFFERENTIATION
relates to
SLOPES of TANGENTS to CURVES



INTEGRATION
relates to
AREAS under CURVES



1.1 The Newton–Leibnitz Theorem

This theorem is one of the foundation stones of Calculus, and says the following. If $g'(x) = f(x)$, then $g(x)$ is a function known as the anti-derivative of $f(x)$, such that the area under the curve $f(x)$ from $x = a$ to $x = b$ will be given by

$$\boxed{\int_{x=a}^{x=b} f(x)dx = g(b) - g(a)}$$

That is, to evaluate the area under $f(x)$ between two points $x = a$ and $x = b$, we find the anti-derivative of $f(x)$, which is another function $g(x)$ such that $g'(x) = f(x)$, and we then calculate the area as $g(b) - g(a)$. For example:

$$\underbrace{\int_0^1 x^2 dx}_{f(x)} = \underbrace{\left.\frac{x^3}{3}\right|_0^1}_{\text{The anti-derivative of } f(x)} = \underbrace{\frac{1}{3} - \frac{0}{3} = \frac{1}{3}}_{\text{The area under } f(x) \text{ between } x=0 \text{ and } x=1}$$

1.2 Indefinite and Definite Integration

The process of finding only the anti-derivative of a function is referred to as Indefinite Integration, whereas the process of completely determining the area under the curve is referred to as Definite Integration. The names are self-explanatory; in Indefinite Integration, we are evaluating the general anti-derivative, without specifying any particular values or points between which we are calculating area; in Definite Integration, we are calculating the area between two specific points. To summarize:

$$\underbrace{\overbrace{f(x)\xrightarrow[\text{anti-derivative}]{\text{Find the}} g(x)}^{\text{INDEFINITE INTEGRATION}} \xrightarrow[\substack{\text{under } f(x) \text{ between}\\ x=a \text{ and } x=b}]{\text{Calculate area}} g(b)-g(a)}_{\text{DEFINITE INTEGRATION}}$$

2. Basic Indefinite Integrals

Listed below are some simple and widely encountered indefinite integrals, and it is advisable to have these memorized.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C; \quad n \neq -1$*

2. $\int \dfrac{1}{x} dx = \ln x + C$

3. $\int e^x dx = e^x + C$

4. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

5. $\int \sin x dx = -\cos x + C$

6. $\int \cos x dx = \sin x + C$

7. $\int \sec^2 x dx = \tan x + C$

8. $\int \operatorname{cosec}^2 x dx = -\cot x + C$

9. $\int \sec x \tan x dx = \sec x + C$

10. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$

11. $\int \cot x dx = \ln|\sin x| + C$

12. $\int \tan x dx = -\ln|\cos x| + C$

13. $\int \sec x dx = \ln|\sec x + \tan x| + C$

14. $\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + C$

15. $\int \dfrac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\dfrac{x}{a} + C$

16. $\int \dfrac{-1}{\sqrt{a^2 - x^2}} dx = -\sin^{-1}\dfrac{x}{a} + C_1 = \cos^{-1}\dfrac{x}{a} + C_2$

17. $\int \dfrac{1}{a^2 + x^2} dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

18. $\int \dfrac{-1}{a^2 + x^2} dx = \dfrac{-1}{a}\tan^{-1}\dfrac{x}{a} + C_1 = \dfrac{1}{a}\cot^{-1}\dfrac{x}{a} + C_2$

19. $\int \dfrac{1}{x\sqrt{x^2 - a^2}} dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{-1}{x\sqrt{x^2 - a^2}} dx = \dfrac{-1}{a}\sec^{-1}\dfrac{x}{a} + C_1$

$$= \frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + C_2$$

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*In particular, $\int dx = x + C$

3. Integration by Rearrangements

One of the simplest integration techniques is to rearrange the given function into simpler forms by some rearrangement or algebraic manipultion, where the simple forms are easily integrable. For example:

$$\int \frac{\cos x - \cos 2x}{1-\cos x} dx = \int \frac{\cos x - (2\cos^2 x - 1)}{1-\cos x} dx = \int \frac{(2\cos x + 1)(1-\cos x)}{(1-\cos x)} dx$$

$$= \int (2\cos x + 1) dx = 2\sin x + x + C.$$

Observe, how the rearrangement led to a simpler expression that was easily integrable. In trigonometric integrals, combination formulae and half-angle and multiple-angle formulae come handy:

(1) $\sin 2x \cos 4x \cos 5x = \frac{1}{2}(2\sin 2x \cos 4x)\cos 5x = \frac{1}{2}(\sin 6x - \sin 2x)\cos 5x$

$$= \frac{1}{4}\{(2\sin 6x \cos 5x) - (2\sin 2x \cos 5x)\}$$

$$= \frac{1}{4}\{\sin 11x + \sin x - \sin 7x + \sin 3x\}$$

$$= \frac{1}{4}\{\sin x + \sin 3x - \sin 7x + \sin 11x\}$$

$$\Rightarrow \quad \int \sin 2x \cos 4x \cos 5x \; dx = \frac{-\cos x}{4} - \frac{\cos 3x}{12} + \frac{\cos 7x}{28} - \frac{\cos 11x}{44} + C$$

(2) $\sin^3 x \cos^3 x = (\sin x \cos x)^3 = \frac{(2\sin x \cos x)^3}{8} = \frac{\sin^3 2x}{8}$

$$= \frac{1}{8}\left\{\frac{3\sin 2x - \sin 6x}{4}\right\} \text{(Triple angle formula)}$$

$$= \frac{3}{32}\sin 2x - \frac{1}{32}\sin 6x$$

$$\Rightarrow \quad \int \sin^3 x \cos^3 x \; dx = \frac{-3}{64}\cos 2x + \frac{1}{192}\cos 6x + C$$

4. Integration by substitutions

4.1 The technique of substitution

Many times, we encounter functions whose integrals cannot be obtained from their original expressions; however, an appropriate substitution might reduce the given function to another function whose integral is obtainable. This method of integration by substitution is used extensively to evaluate integrals. As we progress along this section, we will develop certain rules of thumb that will tell us what substitutions to use where. Also, multiple substitutions might be possible for the same function. Therefore, integration by substitution is more of an art and you can develop the knack of it only by extensive practice (and of course, some thinking!).

Illustration 1: Evaluate $\int \frac{x^7}{(1-x^2)^5}\,dx$.

Working: This example will serve to show that multiple substitutions are possible for the same function.

(a) Notice that the numerator, x^7dx, can be written as $x^6 \cdot xdx$. If we substitute $x^2 = y$, you will obtain $xdx = \frac{dy}{2}$, so that the entire integral can be expressed in terms of y. However, the integral will become

$$\frac{1}{2}\int \frac{y^3}{(1-y)^5}\,dy$$

which still cannot be integrated directly because of the denominator $(1-y)^5$. What we therefore do is substitute $(1 - x^2) = y$ instead of $x^2 = y$, because then the denominator will be simplified further directly:

$$1-x^2 = y \quad \Rightarrow \quad xdx = \frac{-dy}{2}$$

$$I = \int \frac{x^7}{(1-x^2)^5}\,dx = \int \frac{x^6 \cdot x}{(1-x^2)^5}\,dx = -\frac{1}{2}\int \frac{(1-y)^3}{y^5}\,dy$$

$$= \frac{1}{2}\int \left\{ \frac{y^3 - 1 - 3y^2 + 3y}{y^5} \right\} dy = \frac{1}{2}\int \left\{ y^{-2} - 3y^{-3} + 3y^{-4} - y^{-5} \right\} dy$$

$$= \frac{1}{2}\left\{ -\frac{1}{y} + \frac{3}{2y^2} - \frac{1}{y^3} + \frac{1}{4y^4} \right\} + C = \frac{1 - 4y + 6y^2 - 4y^3}{8y^4} + C$$

$$\Rightarrow \quad I = \frac{1 - 4(1-x^2) + 6(1-x^2)^2 - 4(1-x^2)^3}{8(1-x^2)^2} + C$$

(b) The denominator contains the term $(1 - x^2)$. Think of a substitution that could cause the denominator to reduce to a single term; this substitution should be trignometric:

$$x = \sin\theta \text{ so that } dx = \cos\theta\, d\theta$$

$$I = \int \frac{x^7}{(1-x^2)^5}\,dx = \int \frac{\sin^7\theta\cos\theta}{(1-\sin^2\theta)^5}\,d\theta = \int \frac{\sin^7\theta\cos\theta}{\cos^{10}\theta}\,d\theta = \int \tan^7\theta\sec^2\theta\,d\theta$$

Notice now that the substitution $\tan\theta = y$ will reduce I to a simple integrable form:

$$\tan\theta = y \quad \Rightarrow \quad \sec^2\theta d\theta = dy$$

$$I = \int \tan^7\theta\sec^2\theta d\theta = \int y^7 dy = \frac{y^8}{8} + C = \frac{\tan^8\theta}{8} + C$$

$$= \frac{x^8}{8(1-x^2)^4} + C \qquad (\because x = \sin\theta)$$

Are the answers we have obtained in the two approaches different? No, they are not. Try to understand how.

Illustration 2: Evaluate $\int \frac{\tan x}{a + b\tan^2}\,dx$.

Working: We first reduce this expression to another form involving sin and cos terms:

$$I = \int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx$$

If you observe the expression for I carefully, you will realise that a simple substitution is now possible:

$$a\cos^2 x + b\sin^2 x = y$$

$$\Rightarrow \quad (-2a\sin x\cos x + 2b\sin x\cos x)dx = dy \quad \Rightarrow \quad \sin x\cos x dx = \frac{dy}{2b-2a}$$

Thus we have obtained the numerator in terms of the derivative of the denominator:

$$I = \frac{1}{2(b-a)}\int\frac{dy}{y} = \frac{\ln|y|}{2(b-a)} + C = \frac{\ln|a\cos^2 x + b\sin^2 x|}{2(b-a)} + C$$

4.2 Common Substitutions

Listed below are some frequently encountered expression forms wherein some specific substitution works best in each case. It is strongly suggested that you always keep these substitutions in mind:

Expression	**Can be Reduced by Substitution**
$a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
$a^2 + x^2$	$x = a\tan\theta$ or $x = a\cot\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $x = a$ cosec θ
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-a}{b-x}}$ or $\sqrt{(x-a)(x-b)}$	$x = a\cos^2\theta + b\sin^2\theta$
$\sin^m x\cos^n x$ m is an odd integer n is an even integer	$\cos x = t$
$\sin^m x\cos^n x$ n is an odd integer, m is an even integer	$\sin x = t$
$\sin^m x\cos^n x$ Both m, n are odd integers	$\sin x = t$ or $\cos x = t$
$\sin^m x\cos^n x$ Both m, n are even integers	Use multiple-angle formulae to convert the given expression into one involving only linear trigonometric terms.

Based on the technique of substitution, here are some standard integrals which come up frequently in problems of Calculus:

Integrals	**Substitution Used**	**Result**
1. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$x = a\sin\theta$	$\sin^{-1}\frac{x}{a}+C$
2. $\int \frac{1}{a^2+x^2}\,dx$	$x = a\tan\theta$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
3. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx$	$x = a\sec\theta$	$\frac{1}{a}\sec^{-1}\frac{x}{a}+C$
4. $\int \frac{1}{x^2-a^2}\,dx$	$x = a\sec\theta$ (Alternatively, the expression can be split into separate fractions)	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|+C$
5. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$x = a\tan\theta$	$\ln\left\|x+\sqrt{x^2+a^2}\right\|+C$
6. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$x = a\sec\theta$	$\ln\left\|x+\sqrt{x^2-a^2}\right\|+C$

In many integrals, the following interesting technique can be used. Suppose that $L(x) = px+q$ is linear in x, while $Q(x) = ax^2+bx+c$ is a quadratic in x, and we have an integral of the form $\int L(x)/Q(x)dx$ or $\int L(x)/\sqrt{Q(x)}\,dx$. In such a scenario, we can express $L(x)$ in terms of $Q'(x)$:

$$L(x) = \alpha Q'(x)+\beta \quad \text{for some } \alpha,\ \beta \in \mathbb{R}$$

$$\Rightarrow \quad px+q = \alpha(2ax+b)+\beta$$

α and β can be easily evaluated. Lets see this technique in action to see why it is useful:

Illustration 3: Evaluate the following integrals:

$$\text{(a)} \quad \int \frac{x+2}{x^2+2x+2}\,dx \qquad \text{(b)} \quad \int \frac{3x-1}{\sqrt{x^2+2x+2}}\,dx$$

Working:

(a) We find constants α and β such that

$$x+2 = \alpha(x^2+2x+2)'+\beta = \alpha(2x+2)+\beta = 2\alpha x+2\alpha+\beta$$

Thus, $\alpha = \frac{1}{2}$ and $\beta = 1$

$$\Rightarrow \quad I = \int \frac{x+2}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \underbrace{\frac{1}{2}\int \frac{dt}{t}}_{(\text{where } t = x^2+2x+2)} + \int \frac{1}{(x+1)^2+1}\,dx = \frac{1}{2}\ln|t| + \tan^{-1}(x+1)+C$$

$$= \frac{1}{2}\ln(x^2+2x+2)+\tan^{-1}(x+1)+C$$

(b) Again, assume α and β such that

$$3x-1=\alpha(x^2+2x+2)'+\beta=2\alpha x+2\alpha+\beta$$

Thus, $\qquad \alpha=\frac{3}{2}$ and $\beta=-4$

$$\Rightarrow \quad I=\int\frac{3x-1}{\sqrt{x^2+2x+2}}\,dx=\frac{3}{2}\int\frac{2x+1}{\sqrt{x^2+2x+2}}\,dx-4\int\frac{1}{\sqrt{x^2+2x+2}}\,dx$$

$$=\underbrace{\frac{3}{2}\int\frac{dt}{\sqrt{t}}}_{(\text{where } t\,=\,x^2+2x+2)}-4\int\frac{1}{\sqrt{(x+1)^2+1}}\,dx$$

$$=3t^{1/2}-4\ln\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

$$=3\sqrt{x^2+2x+2}-4\ln\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

Assume now that we have to integrate an expression of the form $\frac{P(x)}{Q(x)}$ where $Q(x)$ is a quadratic polynomial while $P(x)$ is a polynomial with degree $n\geq 2$. In such a case, we can divide $P(x)$ by $Q(x)$ to obtain a quotient $Z(x)$ and a remainder $R(x)$ whose degree is less than 2:

$$\frac{P(x)}{Q(x)}=Z(x)+\frac{R(x)}{Q(x)}$$

The right hand side can now be integrated easily. Another frequently encountered type of integrals consists of rational expressions of linear trigonometric terms:

Illustration 4: Evaluate the following integrals:

$$\text{(a)}\quad \int\frac{1}{1+\sin x}\,dx \qquad \text{(b)}\quad \int\frac{1}{1+\sin x+\cos x}\,dx$$

Working:

For such functions, we use the following half-angle formulae:

$$\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}, \qquad \cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

Making these substitutions converts the given integrals into a form where the substitution $t=\tan\frac{x}{2}$ is possible.

(a) $$I=\int\frac{1}{1+\sin x}\,dx=\int\frac{1}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx$$

$$=\int\frac{1+\tan^2 x/2}{1+2\tan x/2+\tan^2 x/2}\,dx=\int\frac{\sec^2 x/2}{(1+\tan x/2)^2}\,dx$$

We now substitute $\tan\frac{x}{2}=t$. Thus, $\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$:

$$\Rightarrow \quad I=2\int\frac{dt}{(1+t)^2}=\frac{-2}{1+t}+C=\frac{-2}{1+\tan x/2}+C$$

(b) $I = \int \frac{1}{1+\sin x+\cos x}\,dx = \int \frac{1}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx$

$$= \int \frac{\sec^2\frac{x}{2}}{2+2\tan\frac{x}{2}}\,dx$$

Substitute $\tan\frac{x}{2} = t$. Thus, $\sec^2\frac{x}{2}dx = 2dt$

$$\Rightarrow \quad I = \int \frac{dt}{1+t} = \ln|1+t| + C = \ln\left|1+\tan\frac{x}{2}\right| + C$$

These two examples should make it clear that a general integral of the form $I = \int \frac{1}{a\sin x + b\cos x + c}\,dx$ can always be integrated using the mentioned substitutions. Let us look at the general case itself:

$$I = \int \frac{1}{a\sin x+b\cos x+c}\,dx = \int \frac{1}{\frac{2a\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{b-b\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+c}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{(c-b)\tan^2\frac{x}{2}+2a\tan\frac{x}{2}+(c+b)}\,dx$$

The substitution $\tan\frac{x}{2} = t$ reduces this integral to

$$I = 2\int \frac{dt}{(c-b)t^2+2at+(c+b)} = 2\int \frac{dt}{At^2+Bt+C}$$

We already know how to evaluate integrals of this form.

5. Integration by expression into partial fractions

This technique helps in evaluating the anti-derivatives of algebraic rational functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials, and $g(x)$ can be expressed as a product of factors of smaller degrees. At our level, we will encounter problems where $g(x)$ is a product of linear, or at the most quadratic, factors. Therefore, $g(x)$ could be of the following general forms:

- $g(x) = L_1(x)L_2(x)\cdots L_n(x)$ (n linear factors)
- $g(x) = L_1(x)\cdots L_r^k(x)\cdots L_n(x)$ $\left(\begin{array}{l}n \text{ linear factors; the } r\text{th} \\ \text{factor is repeated } k \text{ times}\end{array}\right)$
- $g(x) = L_1^{k_1}(x)L_2^{k_2}(x)\cdots L_n^{k_n}(x)$ $\left(\begin{array}{l}n \text{ linear factors, the } i\text{th factor} \\ \text{is repeated } k_i \text{ times}\end{array}\right)$
- $g(x) = L_1(x)L_2(x)\cdots L_n(x)Q_1(x)Q_2(x)\cdots Q_m(x)$ $\left(\begin{array}{l}n \text{ linear factors and} \\ m \text{ quadratic factors}\end{array}\right)$
- $g(x) = \cdots Q_r^k(x)\cdots$ $\left(\begin{array}{l}\text{a particular quadratic factor} \\ \text{repeats more than once}\end{array}\right)$
- A combination of any of the above.

Suppose that the degree of $g(x)$ is n and that of $f(x)$ is m. If $m \geq n$, we can always divide $f(x)$ by $g(x)$ to obtain a quotient $q(x)$ and a remainder $r(x)$ whose degree would be less than n:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

If $m < n$, $\frac{f(x)}{g(x)}$ is termed a *proper rational function*. The partial fraction expansion technique says that a proper rational function can be expressed as a sum of simpler rational functions, each possessing one of the factors of $g(x)$. The simpler rational functions are called *partial fractions*. Once we have split the original rational function using this technique, we can easily integrate the partial fractions separately.

Illustration 5: Evaluate $\int \frac{x^2-8x+7}{(x-5)^2(x+2)^2}\, dx$.

Working: We first express this expression as a sum of partial fractions:

$$\frac{(x-1)(x-7)}{(x-5)^2(x+2)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

Cross-multiplying, we obtain:

$$(x-1)(x-7) = A(x-5)(x+2)^2 + B(x+2)^2 + C(x-5)^2(x+2) + D(x-5)^2$$

$$\text{Put } x = 5 \quad \Rightarrow \quad B = \frac{4 \times -2}{7^2} = \frac{-8}{49}$$

$$\text{Put } x = 5 \quad \Rightarrow \quad D = \frac{-3 \times -9}{7^2} = \frac{27}{49}$$

To obtain A and C, we can now compare coefficients on both sides. Comparing the coefficients of x^3 on both sides, we obtain

$$0 = A + C \quad \Rightarrow \quad A = -C \tag{1}$$

Comparing the constant terms on both sides, we obtain

$$7 = -20A + 4B + 50C + 25D \tag{2}$$

Using (1) in (2), we obtain

$$7 = 20C + 4B + 50C + 25D = 70C - \frac{32}{49} + \frac{675}{49}$$

$$\Rightarrow \quad 70C = \frac{-400}{49} \quad \Rightarrow \quad C = \frac{-40}{343} \quad \Rightarrow \quad A = \frac{40}{343}$$

Thus, the required integral is

$$I = \frac{40}{343} \ln|x-5| + \frac{8}{49(x-5)} - \frac{40}{343} \ln|x+2| - \frac{27}{49(x+2)} + C$$

We observe that if $g(x)$ has a linear factor $L(x)$ repeated n times, there will be n partial fractions corresponding to $L(x)$, of the form

$$\frac{A_1}{L(x)}, \frac{A_2}{L^2(x)}, \ldots, \frac{A_n}{L^n(x)}$$

Similarly, if $g(x)$ has a quadratic factor $Q(x)$ repeated m times, there will be m partial fractions corresponding to $Q(x)$, of the form

$$\frac{A_1x+B_1}{Q(x)}, \frac{A_2x+B_2}{Q^2(x)}, \ldots, \left(\frac{A_mx+B_m}{Q^m(x)}\right)$$

6. Integration by parts

Using integration by parts, we can in principle calculate the integral of the product of any two arbitrary functions. You should be very thorough with the use of this technique, since it will be extensively required in solving integration problems. Let $f(x)$ and $g(x)$ be two arbitrary functions. We need to evaluate $\int f(x)g(x)\,dx$. The rule for integration by parts says that:

$$\boxed{\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left\{f'(x)\int g(x)dx\right\}dx}$$

Translated into words (which makes it easier to remember!), this rule says that: *The integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Derivative of the first function) × (integral of the second function)}.*

Theoretically, we can choose any of the two functions in the product as the first function and the other as the second function. However, a little observation of the expression above will show you that since we need to deal with the integral of the second function ($\int g(x)dx$, above)), we should choose the second function in such a way so that it is easier to integrate; consequently, the first function should be the one that is more difficult to integrate out of the two functions. We can thus define a priority list pertaining to the choice of the first function, corresponding to the degree of difficulty in integration:

$\boxed{I}$ –inverse trigonometric functions $\boxed{L}$ –logarithmic function $\boxed{A}$ –algebraic functions $\boxed{T}$ –trigonometric functions $\boxed{E}$ –exponential function	Decreasing order of difficulty in carrying out integration. For example, inverse trigonometric functions are the most difficult to integrate while exponential functions are the easiest. Thus, we should choose the first function in this order.

The boxed letters should make it clear to you why this rule of thumb for the selection of the first function is referred to as the ILATE rule. It is important to realise that the ILATE rule is just a guide that serves to facilitate the process of integration by parts; it is not a rule that always has to be followed; you can choose your first function contrary to the ILATE rule also if you wish to (and if you are able to integrate successfully with your choice). However, the ILATE rule works in most of the cases and is therefore widely used.

Illustration 6: Evaluate $\int(\ln x)^2\,dx$.

Working: We choose unity as the second function and apply integration by parts:

$$I = \int \underset{\text{Ist}}{(\ln x)^2}\cdot\underset{\text{IInd}}{1}\,dx = x(\ln x)^2 - \int\left(\frac{2\ln x}{x}\right)\cdot x\,dx = x(\ln x)^2 - 2\int \ln \underset{\uparrow}{x}\,dx$$

Again, apply integration by parts, taking unity as the second function.

$$= x(\ln x)^2 - 2\left\{x\ln x - \int\frac{1}{x}\cdot x\,dx\right\} = x(\ln x)^2 - 2x\ln x + 2x + C$$

Illustration 7: Evaluate $\int \sin^{-1} x\, dx$.

Working: Here again, we choose unity as the second function:

$$I = \int \underset{\text{Ist}}{\sin^{-1} x} \cdot \underset{\text{IInd}}{1}\, dx = x\sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} \cdot x\, dx = x\sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} \text{ (how?)}$$

$$= x\sin^{-1} x + \sqrt{t} + C = x\sin^{-1} x + \sqrt{1-x^2} + C$$

Illustration 8: Evaluate $\int x \tan^{-1} x\, dx$.

Working: Using the ILATE rule, we choose $\tan^{-1} x$ as the first function:

$$I = \int \underset{\text{Ist}}{\tan^{-1} x} \cdot \underset{\text{IInd}}{x}\, dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1}{1+x^2} \cdot x^2\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\, dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left\{\int dx - \int \frac{1}{1+x^2}\, dx\right\}$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

Using the technique of integration by parts, the following frequently encountered forms of integrals can be easily evaluated:

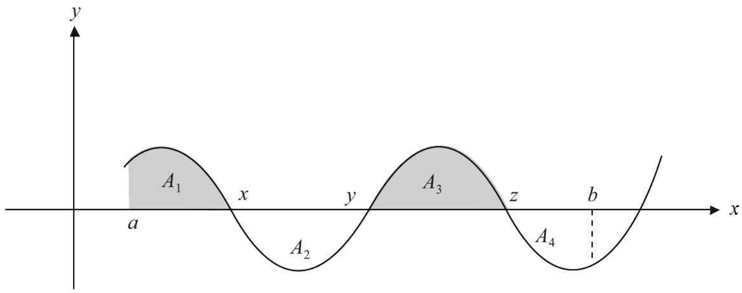
Integral	Result
1. $\int \sqrt{a^2+x^2}\, dx$	$\frac{1}{2}\left\{x\sqrt{a^2+x^2} + a^2 \ln\left\lvert x + \sqrt{x^2+a^2}\right\rvert\right\} + C$
2. $\int \sqrt{a^2-x^2}\, dx$	$\frac{1}{2}\left\{x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}\right\} + C$
3. $\int \sqrt{x^2-a^2}\, dx$	$\frac{1}{2}\left\{x\sqrt{x^2-a^2} - a^2 \ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert\right\} + C$
4. $\int e^{ax} \sin bx\, dx$	$\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$
5. $\int e^{ax} \cos bx\, dx$	$\frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

7. Fundamental Properties of Definite Integrals

7.1 Suppose that $f(x) < 0$ on some interval $[a, b]$. Then, the area under the curve $y = f(x)$ from $x = a$ to $x = b$ will be negative in sign, *i.e.*,

$$\int_a^b f(x)dx < 0$$

This property means that, for example, if $f(x)$ has the following form:



then $\int_a^b f(x)dx$ will equal $A_1 - A_2 + A_3 - A_4$ and not $A_1 + A_2 + A_3 + A_4$.

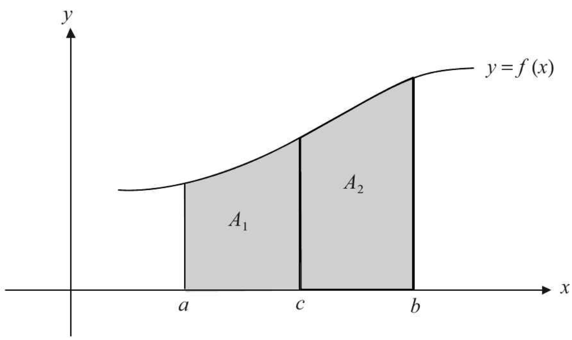
7.2 The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal in magnitude but opposite in sign to the area under the same curve from $x = b$ to $x = a$, *i.e.*,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

7.3 The area under the curve $y = f(x)$ from $x = a$ to $x = b$ can be written as the sum of the area under the curve from $x = a$ to $x = c$ and from $x = c$ to $x = b$, that is

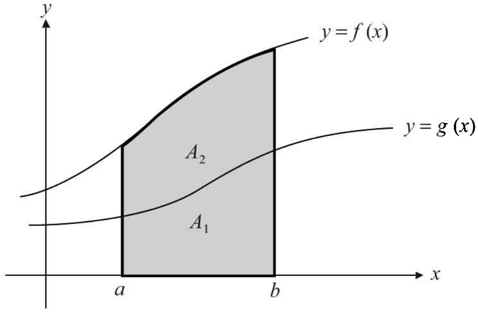
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let us consider an example of this. Let $c \in (a, b)$.



It is clear that the area under the curve from $x = a$ to $x = b, A$, is $A_1 + A_2$.

7.4 Let $f(x) > g(x)$ on the interval $[a, b]$. Then, $\int_a^b f(x) > \int_a^b g(x)dx$. This is because the curve of $f(x)$ lies above the curve of $g(x)$, or equivalently, the curve of $f(x) - g(x)$ lies above the x-axis for $[a, b]$.



This is an example where $f(x) > g(x) > 0$.

$$\int_a^b f(x)dx = A_1 + A_2$$

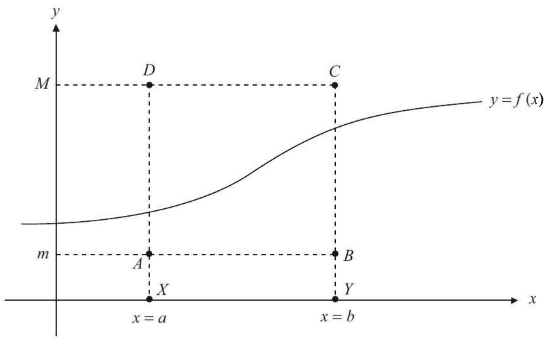
while $\int_a^b g(x)dx = A_1$

Similarly, if $f(x) < g(x)$ on the interval $[a, b]$, then $\int_a^b f(x)dx < \int_a^b g(x)dx$.

7.5 For the interval $[a, b]$, suppose $m < f(x) < M$. That is, m is a lower-bound for $f(x)$ while M is an upper bound. Then,

$$m(b-a) < \int_a^b f(x)dx < M(b-a)$$

This is obvious once we consider the following figure:

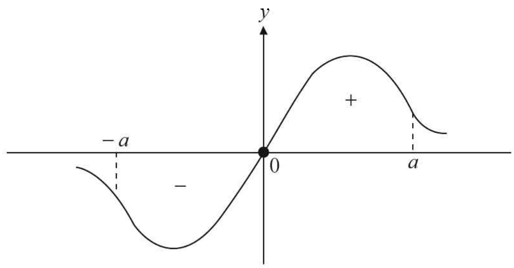


Observe that area(rect $AXYB$) $< \int_a^b f(x)dx <$ area(rect $DXYC$).

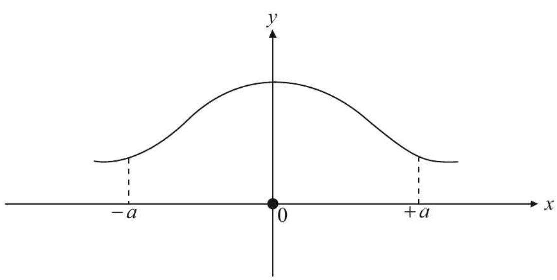
7.6 $\displaystyle\int_a^b \left(f_1(x)+f_2(x)\right)dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

7.7 For an odd function $\left(f(-x) = -f(x)\right)$, we have $\int_{-a}^{+a} f(x)dx = 0$, as is evident from the graph below:



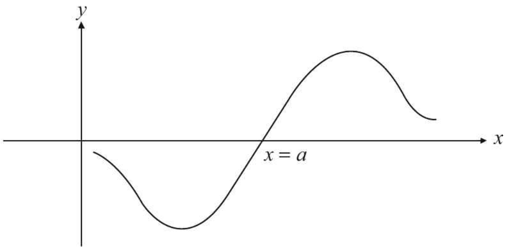
For an even function $\left(f(-x) = f(x)\right)$, we have $\int_{-a}^{+a} f(x)dx = 2\int_0^a f(x)dx$:

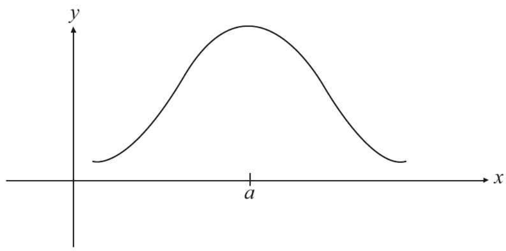


If $f(x)$ is odd (or even) about $x = a$ instead of $x = 0$, we have the following situations:

$$f(x) = -f(2a-x) \qquad\qquad f(x) = f(2a-x)$$

$$\Rightarrow \int_0^{2a} f(x)dx = 0 \qquad \Rightarrow \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$$





7.8 The *average* value of a function $f(x)$ over the interval $[a, b]$ is given by

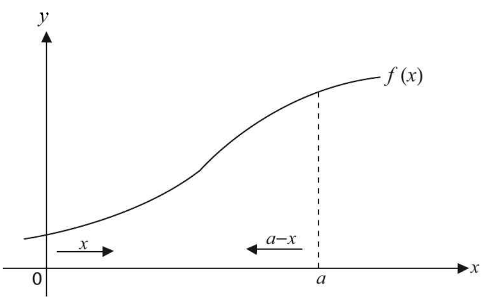
$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

This value is attained for at least one $c \in (a, b)$, under the constraint that f is continuous.

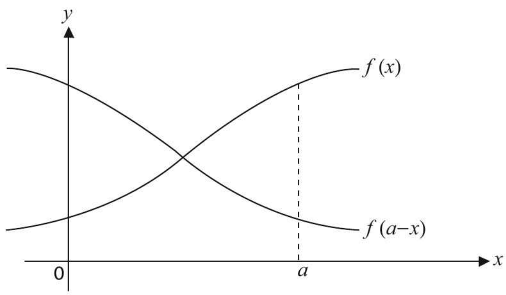
8. Advanced Properties of Definite Integrals

8.1 $\int_0^a f(x) = \int_0^a f(a-x)dx$

The following figure provides justification for this property:



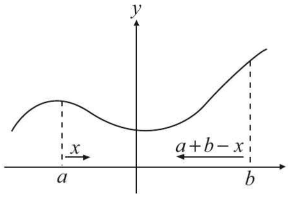
As x progressess from 0 to a, the variable $a - x$ progressess from a to 0. Thus, whether we use x or $a - x$, the entire interval $[0, a]$ is still covered.



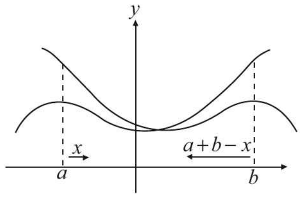
The function $f(a-x)$ can be obtained from the function $f(x)$ by first flipping $f(x)$ along the y-axis and then shifting it right by a units. Notice that in the interval $[0, a]$, $f(x)$ and $f(a-x)$ describe precisely the same area.

8.2 $\int_0^b f(x)dx = \int_0^b f(a+b-x)dx$

This property is a generalization of the previous property and its justification is analogous to the one above.



As the variable x varies from a to b, the variable $a+b-x$ varies from b to a. Thus, whether we use x or $a+b-x$, the entire interval $[a, b]$ is covered in both the cases and the areas wil be the same.



The graph of $f(a+b-x)$ can be obtained from the graph of $f(x)$ by first flipping the graph of $f(x)$ along the y-axis and then shifting it $(a+b)$ units towards the right; the areas described by $f(x)$ and $f(a+b-x)$ in the interval $[a,b]$ are precisely the same.

8.3 $\int_0^{2a} f(x)dx = \int_0^{a} \{f(x) + f(2a-x)\}\, dx$

The justification for this property is described below:

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_a^{2a} f(x)dx.$$

To evaluate $\int_a^{2a} f(x)dx$, we can equivalently use the variable $(2a-x)$ instead of x, but the limits of integration will change from (a to $2a$) to (0 to a). This is because as x varies from 0 to a, $2a-x$ will vary from ($2a$ to a) covering the same interval $[a, 2a]$. Thus,

$$\int_a^{2a} f(x)dx = \int_0^{a} f(2a-x)\, dx$$

Hence, the stated assertion is valid.

8.4 $\int_0^{b} f(x)dx = (b-a)\int_0^{1} f\left(a+(b-a)t\right)dt$

This property enables us to change the limits of the integrals from (a, b) to $(0, 1)$. In fact, the limits can be changed to any other values (a', b'), as follows:

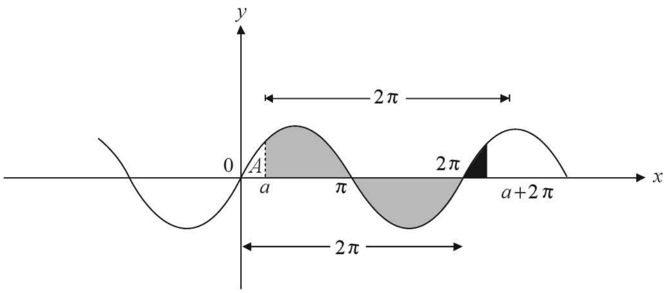
$$\int_a^{b} f(x)dx = \int_{a'}^{b'} f\left(a + \left(\frac{b-a}{b'-a'}\right)(t-a')\right)\left(\frac{b-a}{b'-a'}\right)dt$$

It will be instructive for the reader to prove this property.

8.5 If $f(x)$ is a periodic function with period T, then the area under $f(x)$ for n periods would be n times the area under $f(x)$ for one period, *i.e.*,

$$\int_0^{nT} f(x)\, dx = n\int_0^{T} f(x)\, dx$$

Now, consider the periodic function $f(x) = \sin x$ as an example. The period of $\sin x$ is 2π.



Suppose we intend to calculate $\int_a^{a+2\pi} \sin x\, dx$ as depicted above. Notice that the darkly shaded area in the interval $[2\pi, a+2\pi]$ can precisely cover the area marked as A. Thus,

$$\int_a^{a+2\pi} \sin x\, dx = \int_0^{2\pi} \sin x\, dx$$

This will hold true for every periodic function, *i.e.*,

$$\int_a^{a+T} f(x)\, dx = \int_0^{T} f(x)\, dx \text{ (where } T \text{ is the period of } f(x))$$

This also implies that

$$\int_a^{a+nT} f(x)\, dx = \int_0^{nT} f(x)\, dx = n\int_0^{T} f(x)\, dx$$

$$\text{and} \int_{a+nT}^{b+nT} f(x)\, dx = \int_a^{b} f(x)\, dx \quad \text{and} \quad \int_a^{b+nT} f(x)\, dx = \int_a^{b} f(x)\, dx + n\int_0^{T} f(x)\, dx$$

9. Differentiating an integral with variable limits

The anti-derivative $g(x)$ of a function $f(x)$ is defined as

$$g(x) = \int_0^{x} f(t)dt, \text{ so that } g'(x) = f(x)$$

Now consider an integral of the following form:

$$h(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt$$

That is, the limits of integration are themselves functions of x. The anti-derivative $g(x)$ is a special case of $h(x)$ with $\psi(x) = x$ and $\phi(x) = a$. Now, how do we evaluate $h'(x)$? Since $g(x)$ is the anti-derivative of $f(x)$, we have:

$$h(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt = g(t)\Big|_{\phi(x)}^{\psi(x)} = g(\psi(x)) - g(\phi(x))$$

$$\Rightarrow \quad h'(x) = g'(\psi(x))\psi'(x) - g'(\phi(x))\phi'(x)$$

$$= f(\psi(x))\psi'(x) - f(\phi(x))\phi'(x)$$

Illustration 9: Evaluate $f'(x)$ if $f(x) = \int_x^{x^2} (t^2+1)dt$.

Working: Let us first find out $f(x)$ using straight forward integration:

$$f(x) = \left(\frac{t^3}{3} + t\right)\Bigg|_x^{x^2} = \frac{x^6}{3} + x^2 - \frac{x^3}{3} - x$$

$$\Rightarrow \quad f'(x) = 2x^5 - x^2 + 2x - 1$$

Now we redo this using the technique we have just encountered:

$$\begin{aligned} f'(x) &= ((x^2)^2+1)(x^2)' - ((x)^2+1)(x)' \\ &= (x^4+1)(2x) - (x^2+1)(1) = 2x^5 - x^2 + 2x - 1 \end{aligned}$$

10. Leibniz Integral Rule: Differentiation under the integral sign

Consider a function in two variables x and y, *i.e.*,

$$z = f(x, y)$$

Let us consider the integral of z with respect to x, from a to b, *i.e.*,

$$I = \int_a^b f(x, y)dx$$

For this integration, the variable is only x and not y. y is essentially a constant for the integration process. Therefore, after we have evaluated the definite integral and put in the integration limits, y will still remain in the expression of I. This means that I is a function of y.

$$\Rightarrow \quad I(y) = \int_a^b f(x, y)dx$$

The above relation can be differentiated with respect to y as follows:

$$I'(y) = \frac{d}{dy}\left(\int_a^b f(x, y)\,dx\right) = \int_a^b \frac{\partial f(x, y)}{\partial y}dx$$

Here, $\frac{\partial f(x,y)}{\partial y}$ stands for the partial derivative of $f(x, y)$ with respect to y, that is , the derivative of $f(x, y)$ with respect to y, treating x as a constant.

Illustration 10: Evaluate $I = \int_0^1 \frac{x^k - 1}{\ln x}dx$.

Solution: Observe that I will be a function of k. Instead of carrying out direct integration, we proceed as follows:

$$\frac{dI(k)}{dk} = \int_0^1 \frac{\partial}{\partial k}\left(\frac{x^k - 1}{\ln x}\right)dx = \int_0^1 \frac{x^k \ln x}{\ln x}dx = \int_0^1 x^k dx = \frac{1}{k+1}$$

Thus,

$$dI(k) = \frac{dk}{k+1}$$

Integrating both sides, we obtain

$$I(k) = \ln(k+1) + C \qquad (1)$$

To obtain C, note from the original definition of I that $I(0) = 0$. Using this in (1), we obtain

$$0 = \ln 1 + C \quad \Rightarrow \quad C = 0$$

Thus,

$$I(k) = \ln(k+1)$$

11. Definite integral as a limit of a sum

The area under the curve is the sum of an infinitely large number of rectangles with infinitesimally small widths. These rectangles precisely give the area under the curve in the limit that their number tends to infinity, *i.e.*,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh) \quad \text{where} \quad h = \frac{b-a}{n}$$

In particular, notice that if the lower limit a is 0, then

$$\int_0^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{b}{n} f\left(\frac{br}{n}\right) \qquad (1)$$

We sometimes encounter series of the form as in the right hand side of (1). We observe that such series can be summed using definite integrals.

Illustration 11: Find the sum of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n} \text{ as } n \to \infty$$

Solution: The given series can be written concisely as

$$S = \sum_{r=0}^{2n} \frac{1}{n+r} = \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+(r/n)}$$

Comparing this with the right hand side of (1), we see that S can be expressed as the integral of a function $f(x) = \frac{1}{1+x}$ from 0 to 2, because since r varies till $2n$, r/n varies till 2. Thus,

$$S = \int_0^2 \frac{1}{1+x} dx = \ln 3$$

Illustration 12: Find the sum of the series

$$\frac{n}{n^2+1^2} + \frac{1}{n^2+2^2} + \ldots + \frac{1}{n^2+n^2} \text{ as } n \to \infty$$

Working: We have

$$S = \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+(r/n)^2}$$

We observe that S can be written in integral form as:

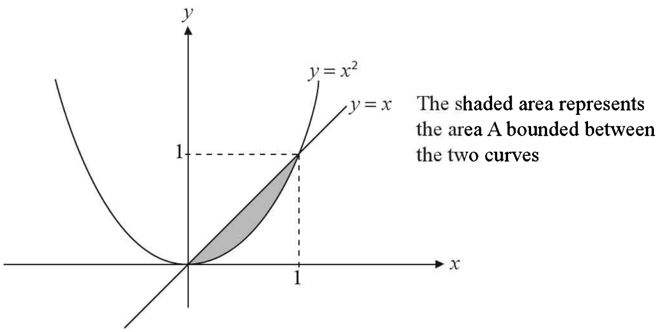
$$S = \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}.$$

12. Calculating Areas

The calculation of areas using definite integrals involves no new concepts other than those we have already studied. However, the important thing is to draw an accurate pictorial reprensentation of whatever area you are required to calculate.

Illustration13: Find the area bounded between the curves $y = x^2$ and $y = x$.

Working: The bounded area is depicted in the figure below:

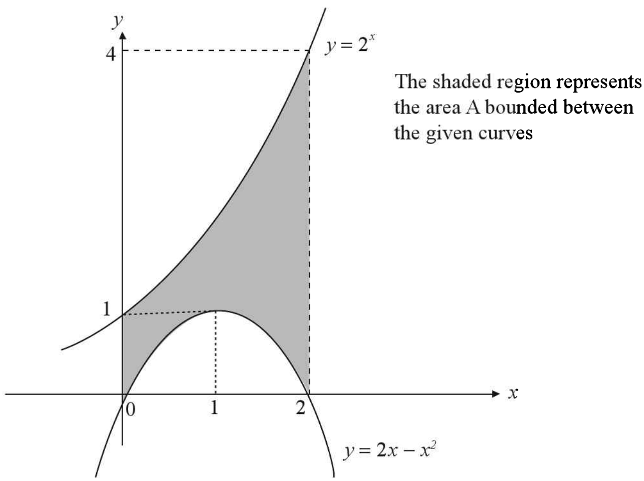


Thus,

$$A = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{6} \text{ sq units}$$

Illustration 14: Find the area of the region bounded between $x = 0$, $x = 2$, $y = 2^x$ and $y = 2x - x^2$.

Working: The bounded region has been sketched below. Verify for the validity of the figure drawn:



We have,

$$A = \int_0^2 \left(2^x - (2x - x^2)\right) dx = \left(\frac{2^x}{\ln 2} - x^2 + \frac{x^3}{3}\right)\Bigg|_0^2$$

$$= \frac{4}{\ln 2} - 4 + \frac{8}{3} - \frac{1}{\ln 2} = \left(\frac{3}{\ln 2} - \frac{4}{3}\right) \text{sq units}$$

IMPORTANT IDEAS AND TIPS

1. *Techniques of Integration.* Deciding which technique to use (and how to use it) to integrate a given function is the most important skill to be learnt for this chapter. Broadly speaking, as soon as we look at a function to be integrated, we will have to decide between using one of the following techniques:

(a) *Rearrangement and/or Substitution*: First, we try to see whether a rearrangement of the given function is possible to reduce or simplify it to an alternate form which consists of integrable terms. These integrable terms might be integrable directly (they may be some standard forms) or they may be integrable using some substitution(s) (or even another technique). With a lot of practice, it will start becoming obvious to you when a substitution will be needed. For many expressions, you will know what substitution will work best, but there will be cases when you might have to think of a new, as yet unseen, substitution. In many scenarios, a combination of rearrangements and substitutions will be required.

(b) *A Split Into Partial Fractions.* Deciding when to do this is straightforward. Whenever we have a rational algebraic function of the form $\frac{f(x)}{g(x)}$, where $g(x)$ can be factorized into linear or quadratic factors (at our level, we won't deal with higher order factors), we can split the function into partial fractions and integrate easily.

(c) *Integration by Parts.* In most cases, it will be easy to decide that integration by parts is required. For example, consider the function $f(x) = x \tan x$. This consists of a product of a linear algebraic function and a trigonometric term, two entirely "different" kinds of terms. This means that rearrangement or substitution will not work, and we will have to integrate by parts. In some cases, it might not be evident that integration by parts is possible - such examples and problems have been included in the next sections. In these cases, the only guide can be intuition, which comes from a lot of problem solving.

(d) *Recursion.* Like integration by parts, it is generally easy to decide that setting up a recursion is possible for a given problem. The integral should be expressible in terms of an *order* or *index*, say *n*. For example, as soon as we encounter the integral $\int_0^\infty e^{-x} x^n dx$, we can tell that a recursion can be set up, using integration by parts, which will help us express this *n*th order integral in terms of a lower order integral.

Mastering integration is all about *matching structures*: deciding which technique will work by trying to match the structure of the given integral to structures of integrals which you already know of.

2. *Simplifying Definite Integrals.* There are a lot of properties of definite integrals which help in their simplification, and deciding which property to apply might seem to be a challenge. Once again, *matching structures* is the key skill. The following critical observations help:

(a) Is the function even or odd about a given point?

(b) Does the function have any periodic character?

(c) Does the substitution $x \to a+b-x$ (where a, b are the integration limits) help in any way?

You must always ask yourself these questions when trying to simplify a definite integral. If the answer to any of these questions is yes, then you must use the corresponding fact in your simplification.

Integration

PART-B: Illustrative Examples

Example 1

What is the value of $\int \frac{x^3}{(x+1)^2}\,dx$?

(A) $\frac{x^2}{2} - x + 3\ln(x+1) + \frac{2}{x+1} + C$ (C) $\frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + C$

(B) $\frac{x^2}{2} - 2x + 4\ln(x+1) + \frac{3}{x+1} + C$ (D) $\frac{3x^2}{2} - x + 2\ln(x+1) + \frac{1}{x+1} + C$

Solution: The numerator has a degree higher then the denominator which hints that some reduction of this rational expression is possible. This reduction can be accomplished if we somehow rearrange the numerator in such a way that it leads to a cancellation of common factors with the denominator; since the denominator is $(x + 1)^2$, we try to rearrange the numerator in terms of $(x + 1)$:

$$\int \frac{x^3}{(x+1)^2}\,dx = \int \frac{(x^3+1)-1}{(x+1)^2}\,dx$$

$$= \int \left\{ \frac{(x+1)(x^2-x+1)}{(x+1)^2} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ \frac{x^2-x+1}{(x+1)} - \frac{1}{(x+1)^2} \right\} dx = \int \left\{ \frac{x^2-x-2+3}{x+1} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ \frac{(x+1)(x-2)+3}{(x+1)} - \frac{1}{(x+1)^2} \right\} dx = \int \left\{ (x-2) + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + C$$

The correct option is (C). ■

Example 2

What is the value of $\int \frac{x^8}{x^{12}-1}\,dx$?

(A) $\frac{1}{3}\tan^{-1}x^3 + \frac{1}{6}\ln\left|\frac{x^3-1}{x^3+1}\right| + C$ (B) $\frac{1}{6}\tan^{-1}x^3 + \frac{1}{12}\ln\left|\frac{x^3-1}{x^3+1}\right| + C$

(C) $\frac{1}{3}\tan^{-1}\left(\frac{x^3}{2}\right) + \frac{1}{6}\ln\left|\frac{x^3-1}{x^3+1}\right| + C$ (D) $\frac{1}{6}\tan^{-1}\left(\frac{x^3}{2}\right) + \frac{1}{12}\ln\left|\frac{x^3-1}{x^3+1}\right| + C$

Solution: To simplify this integral, we need to make an appropriate substitution, such as $x^n = t$ where $n < 12$; n should of course be a factor of 12 and the numerator $(x^8 dx)$ should be expressible in terms of t and the differential of t. A little thought will show that $x^3 = t$ is the appropriate substitution possible here:

$$x^3 = t \quad \Rightarrow \quad 3x^2\, dx = dt$$

$$\Rightarrow \quad I = \int \frac{x^8}{x^{12}-1} dx = \int \frac{(x^3)^2 . x^2}{(x^3)^4 - 1} dx$$

$$= \frac{1}{3} \int \frac{t^2}{t^4 - 1} \quad (t = x^3)$$

$$= \frac{1}{3} \int \frac{t^2}{(t^2-1)(t^2+1)} dt = \frac{1}{6} \int \frac{(t^2-1)+(t^2+1)}{(t^2-1)(t^2+1)} dt$$

$$= \frac{1}{6} \left\{ \int \frac{1}{t^2+1} dt + \int \frac{1}{t^2-1} dt \right\} = \frac{1}{6} \left(\tan^{-1} t + \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \right) + C$$

$$= \frac{1}{6} \tan^{-1} x^3 + \frac{1}{12} \ln \left| \frac{x^3-1}{x^3+1} \right| + C$$

The correct option is (B). ■

Example 3

What is the value of $\int \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$?

(A) $\frac{1}{20} \ln \left| \frac{5 - 4\cos x + 4\sin x}{5 + 4\cos x - 4\sin x} \right| + C$ (B) $\frac{1}{20} \ln \left| \frac{5 + 4\cos x - 4\sin x}{5 - 4\cos x + 4\sin x} \right| + C$

(C) $\frac{1}{40} \ln \left| \frac{5 - 4\cos x + 4\sin x}{5 + 4\cos x - 4\sin x} \right| + C$ (D) $\frac{1}{40} \ln \left| \frac{5 + 4\cos x - 4\sin x}{5 - 4\cos x + 4\sin x} \right| + C$

Solution: In our search for an appropriate substitution, we must look into how we can express the numerator $(\sin x + \cos x)dx$ in terms of the differential of the denominator (or some part of the denominator). Notice that $(\sin x + \cos x)$ is the derivative of $(-\cos x + \sin x)$ What if we could express the deno minator in terms of $(-\cos x + \sin x)$? We could then use the substitution $(-\cos x + \sin x) = y$. This is what we proceed to do:

$$9 + 16 \sin 2x = 9 - 16(-1 + 1 - \sin 2x)$$

$$= 9 - 16\{-1 + (\sin x - \cos x)^2\} = 25 - 16(-\cos x + \sin x)^2$$

Thus, we have succeded in expressing the denominator in terms of $(-\cos x + \sin x)$. Now, we substitute

$$(-\cos x + \sin x) = y \quad \Rightarrow \quad (\sin x + \cos x)dx = dy$$

$$\Rightarrow \quad I = \int \frac{dy}{25 - 16y^2} = \frac{1}{16} \int \frac{dy}{(\frac{25}{16}) - y^2}$$

This integral is straightforward to evaluate, and the rest of the solution is left to the reader as an exercise. The answer is:

$$I = \frac{1}{40}\ln\left|\frac{5-4\cos x+4\sin x}{5+4\cos x-4\sin x}\right| + C$$

Therefore, the correct option is (C). ■

Example 4

What is the value of $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$?

(A) $\frac{3}{2}x^{2/3}+6\tan^{-1}\sqrt[6]{x}+C$ (B) $\frac{1}{2}x^{2/3}+3\tan^{-1}\sqrt[6]{x}+C$

(C) $\frac{3}{2}x^{4/3}+4\tan^{-1}\sqrt[6]{x}+C$ (D) $\frac{1}{2}x^{4/3}+3\tan^{-1}\sqrt[6]{x}+C$

Solution: We have either cube roots or sixth roots in this expression. The LCM of (3, 6) is 6, so that the substitution $x = t^6$ can reduce this expression to an entirely rational form:

$$x = t^6$$

$$\Rightarrow \quad dx = 6t^5 dt$$

$$\Rightarrow \quad I = 6\int \frac{t^6+t^4+t}{t^6(1+t^2)}.t^5 dt = 6\int \frac{t^5+t^3+1}{1+t^2}dt = 6\left\{\int t^3 dt + \int \frac{1}{1+t^2}dt\right\}$$

$$= \frac{3}{2}t^4 + 6\tan^{-1} t + C = \frac{3}{2}x^{2/3} + 6\tan^{-1}\sqrt[6]{x} + C$$

The correct option is (A). ■

Example 5

What is the value of $\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$?

(A) $\frac{2}{3}\left[\frac{x-1}{x+2}\right]^{1/4}+C$ (B) $\frac{4}{3}\left[\frac{x-1}{x+2}\right]^{1/4}+C$

(C) $\frac{2}{3}\left[\frac{x+2}{x-1}\right]^{1/4}+C$ (D) $\frac{4}{3}\left[\frac{x+2}{x-1}\right]^{1/4}+C$

Solution: Observe the exponents carefully; $\frac{3}{4}$ is 1/4 less than 1 while $\frac{5}{4}$ is 1/4 more than 1. Therefore, we can write the denominator as

$$(x-1)^{3/4}(x+2)^{5/4} = (x-1)(x+2)\left(\frac{x+2}{x-1}\right)^{1/4}$$

We thus use the substitution

$$\frac{x+2}{x-1} = t^4 \quad \Rightarrow \quad x = \frac{t^4+2}{t^4-1} \text{ and } dx = \frac{-12t^3}{(t^4-1)^2}dt$$

$$\text{Thus, } I = \int \frac{dx}{(x-1)(x-2)(\frac{x+2}{x-1})^{\frac{1}{4}}} = \int \frac{-12t^3 dt}{(\frac{3}{t^4-1})(\frac{3t^4}{t^4-1})\cdot t\cdot(t^4-1)^2}$$

$$= -\frac{4}{3}\int\frac{dt}{t^2} = \frac{4}{3t} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

The correct option is (B). ■

Example 6

What is the value of $\int \frac{1}{\sin^6 x + \cos^6 x} dx$?

(A) $\tan^{-1}(\sin x - \cos x) + C$ (B) $\tan^{-1}(\tan x - \cot x) + C$

(C) $\tan^{-1}(\sec x - \cos x) + C$ (D) $\sec^{-1}(\sin x - \cos x) + C$

Solution: We convert this expression into one involving tan x and sec x terms:

$$I = \int \frac{\sec^6 x}{1+\tan^6 x} dx$$

We could now use the substitution $\tan x = t$ so that $\sec^2 x dx = dt$ Thus,

$$I = \int \frac{\sec^4 x}{1+t^6} dt = \int \frac{(1+t^2)^2}{1+t^6} dt$$

$$= \int \frac{(1+t^2)^2}{(1+t^2)(1-t^2+t^4)} dt \text{ (By factorizing } t^6 + 1 \text{ in the denominator)}$$

$$= \int \frac{1+t^2}{1-t^2+t^4} dt \qquad (1)$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-1} dt \quad \begin{Bmatrix} \text{Division of the numerator} \\ \text{and denominator by } t^2 \end{Bmatrix}$$

$$= \int \frac{1+\frac{1}{t^2}}{(t-\frac{1}{t})^2+1} dt$$

$$= \int \frac{1}{y^2+1} dy \quad \left\{\text{By the substitution } t - \frac{1}{t} = y\right\}$$

$$= \tan^{-1} y + C = \tan^{-1}\left(t - \frac{1}{t}\right) + C = \tan^{-1}(\tan x - \cot x) + C$$

The correct option is (B). ■

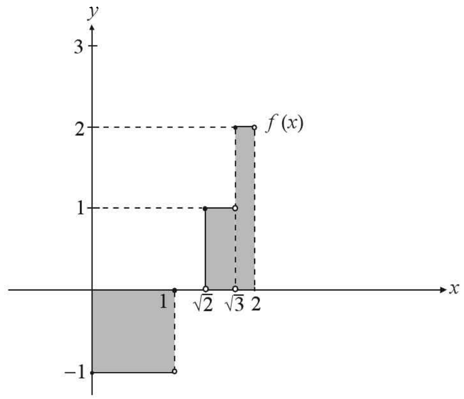
Example 7

What is the value of $\int_0^2 [x^2 - 1] dx$?

(A) $3 - \sqrt{2} + \sqrt{3}$ (B) $3 + \sqrt{2} - \sqrt{3}$

(C) $3 - (\sqrt{2} + \sqrt{3})$ (D) $3 + \sqrt{2} + \sqrt{3}$

Solution: The function to be integrated as been sketched below in the region of interest:



This function is discontinuous; so we will have to split the interval of integration in such a way so that in each of the sub-intervals that we obtain, the function is continuous and can be integrated. Thus, if $[x^2-1]=f(x)$, then:

$$\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^{\sqrt{2}} f(x)dx+\int_{\sqrt{2}}^{\sqrt{3}} f(x)dx+\int_{\sqrt{3}}^{2} f(x)dx$$

$$=\int_0^1(-1)dx+\int_1^{\sqrt{2}}(0)dx+\int_{\sqrt{2}}^{\sqrt{3}}(1)dx+\int_{\sqrt{3}}^{2}(2)dx$$

$$=-1+0+(\sqrt{3}-\sqrt{2})+2(2-\sqrt{3})=3-(\sqrt{2}+\sqrt{3})$$

The correct option is (C). In general, for any discontinuous function *f*(*x*) whose integral we need to evaluate, the approach described above is followed. *f*(*x*) is separately integrated in sub–intervals where it is continuous and the results so obtained are added. ■

Example 8

What is the value of $\int_0^{n\pi+V}|\sin x|dx$, where n is a positive integer and $V=[0,\pi)$?

(A) $(2n-1)+\cos V$ (B) $(2n+1)+\cos V$ (C) $(2n-1)-\cos V$ (D) $(2n+1)-\cos V$

Solution: $f(x)=|\sin x|$ is periodic with period π. Therefore,

$$\int_0^{n\pi+V}|\sin x|\,dx=\int_0^V|\sin x|\,dx+n\int_0^\pi|\sin x|\,dx$$

$$=\int_0^V\sin x dx+n\int_0^\pi\sin x dx \quad \left[\because \begin{array}{l}\sin x\ge 0\\ \text{for } x\in[0,\pi)\end{array}\right]$$

$$=-\cos x\Big|_0^V+n(-\cos x)\Big|_0^\pi=1-\cos V+n(2)=(2n+1)-\cos V$$

The correct option is (D). ■

Example 9

If $m, n \in \mathbb{N}$, what is the value of $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$?

(A) $\dfrac{m!n!}{(m+n)!}$ (B) $\dfrac{(m+1)!(n+1)!}{(m+n)!}$ (C) $\dfrac{m!n!}{(m+n+1)!}$ (D) $\dfrac{(m-1)!(n-1)!}{(m+n+1)!}$

Solution: It should be obvious that we require a recursive relation involving $I_{m,n}$ and a lower order integral (with respect to one of the two variables m or n).

$$I_{m,n} = \int_0^1 \underset{\substack{\uparrow \\ \text{Ist} \\ \text{Function}}}{(1-x)^n} \underset{\substack{\text{IInd} \\ \text{Function}}}{x^m} \, dx = \left. \frac{(1-x)^n x^{m+1}}{m+1} \right|_0^1 + \frac{n}{m+1} \int_0^1 (1-x)^{n-1} x^{m+1} dx = 0 + \frac{n}{m+1} I_{m+1,n-1}$$

Thus, the required recursive relation is

$$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$$

We use this relation repeatedly now till n reduces to 0:

$$\begin{aligned} I_{m,n} = \frac{n}{m+1} I_{m+1,n-1} &= \frac{n(n-1)}{(m+1)(m+2)} I_{m+2,n-2} \\ &\vdots \\ &= \frac{n(n-1)...1}{(m+1)(m+2)...(m+n)} I_{m+n,0} \\ &= \frac{m!n!}{(m+n)!} I_{m+n,0} \end{aligned}$$

$I_{m+n,0}$ is easy to evaluate. Verify that it is $\frac{1}{m+n+1}$. Thus,

$$I_{m,n} = \frac{m!n!}{(m+n+1)!}$$

The correct option is (C). ■

Example 10

What is the value of the following limit?

$$\lim_{n \to \infty} \frac{[(n+1)(n+2)...(n+n)]^{1/n}}{n}$$

(A) $\dfrac{1}{e}$ (B) $\dfrac{2}{e}$ (C) $\dfrac{4}{e}$ (D) $\dfrac{8}{e}$

Solution: Let L represent the given limit. We have,

$$\ln L = \frac{1}{n}\ln\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\} = \frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)$$

Thus, $\ln L = \int_0^1 \ln(1+x)dx$

$$= x\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}dx = \ln 2 - \int_0^1\left(\frac{x+1-1}{x+1}\right)dx$$

$$= \ln 2 - 1 + \ln 2 = 2\ln 2 - 1 = \ln\left(\tfrac{4}{e}\right)$$

This implies that

$$L = \frac{4}{e}$$

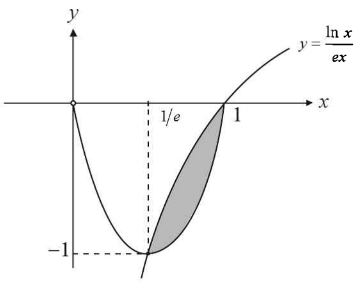
The correct option is (C). ■

Example 11

What is the area of the region bounded between the curves $y = ex\ln x$ and $y = \frac{\ln x}{ex}$?

(A) $\dfrac{e^2-1}{2e}$ (B) $\dfrac{e^2-3}{2e}$ (C) $\dfrac{e^2-5}{4e}$ (D) $\dfrac{e^2-6}{5e}$

Solution: The two curves are plotted below:



The point of intersection is determined by equating the equation of the two curves, *i.e.*,

$$ex\ln x = \frac{\ln x}{ex}$$

$$\Rightarrow \quad \ln x = 0 \ \text{ or } e^2x^2 = 1$$

$$\Rightarrow \quad x = 1 \ \text{ or } x = \frac{1}{e}$$

The shaded area represents the area A between two curves

From the figure, it is clear that

$$A = \int_{1/e}^{1}\left(\frac{\ln x}{ex} - ex\ln x\right)dx = \int_{1/e}^{1}\frac{\ln x}{ex}dx - e\int_{1/e}^{1}x\ln x\,dx$$

$$= \left\{\frac{1}{e}\int_{-1}^{0}t\,dt\right\} - e\left\{\frac{x^2}{2}\ln x\Big|_{1/e}^{1} - \frac{1}{2}\int_{1/e}^{1}x\,dx\right\} = \frac{-1}{2e} - e\left\{\frac{1}{2e^2} - \frac{1}{4}\left(1-\frac{1}{e^2}\right)\right\}$$

$$= -\frac{1}{2e} - \frac{1}{2e} + \frac{e}{4}\left(\frac{e^2-1}{e^2}\right) = -\frac{1}{e} + \frac{e^2-1}{4e}$$

$$= \frac{e^2-5}{4e} \text{ sq units}$$

The correct option is (C). ■

Example 12

What is the area of the region containing the points whose (x, y) coordinates satisfy the following relation?

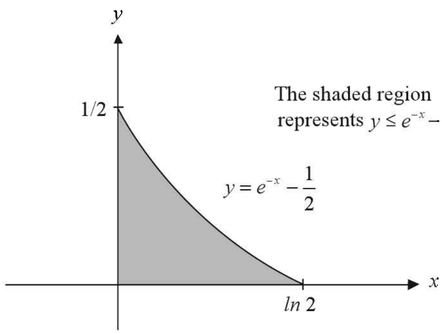
$$|y| + \frac{1}{2} \le e^{-|x|}$$

(A) $1 - \ln 2$ (B) $1 + \ln 2$ (C) $2(1 - \ln 2)$ (D) $2(1 + \ln 2)$

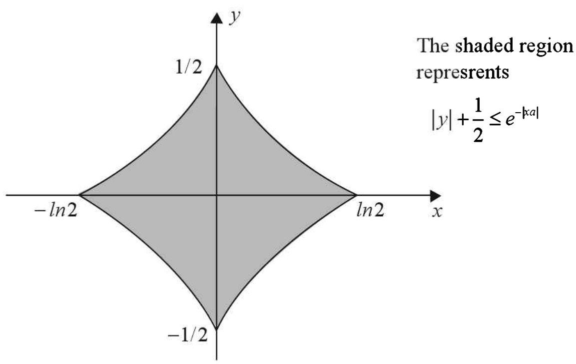
Solution: If you observe the given relation carefully, you will realise that whatever region we obtain will be symmetric about both the x-axis and y-axis. This means that we only need to plot the region in the first quadrant. The regions in the other quadrants can then automatically be obtained by reflecting symmetrically the region in the first quadrant into all the other quadrants. So let us now consider just the first quadrant. In this quadrant, both $x, y > 0$, so that the given relation can be simply written as

$$y + \frac{1}{2} \le e^{-x} \quad \Rightarrow \quad y \le e^{-x} - \frac{1}{2}$$

We now plot this region for $x > 0, y > 0$:



Our required region is therefore,



The required area is:

$$A = 4\int_0^{\ln 2}\left(e^{-x} - \frac{1}{2}\right)dx = 4\left(e^{-x} - \frac{x}{2}\right)\Bigg|_0^{\ln 2}$$

$$= 4\left(\frac{1}{2} - \frac{1}{2}\ln 2\right) = 2(1 - \ln 2) \text{ squnits}$$

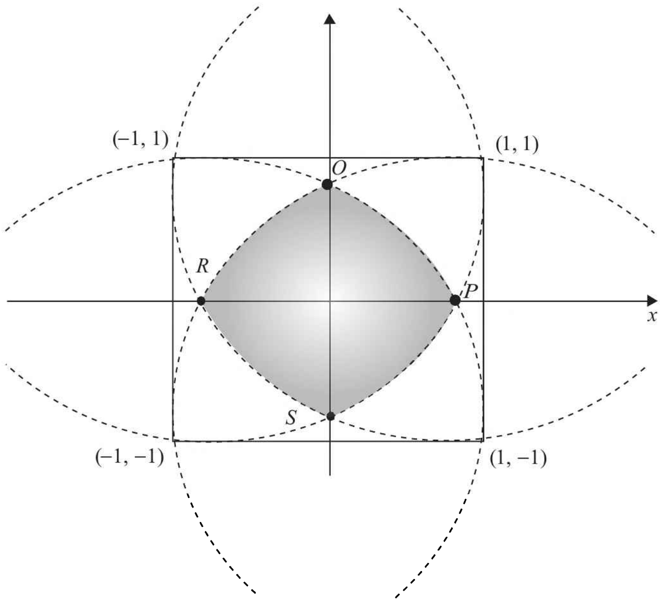
The correct option is (C). ■

Example 13

A square has its vertices at (1, 1), (1, –1), (–1, –1) and (–1, 1). Four circles of radius 2 are drawn, one centred at each vertex of the square. What is the area common to these four circles?

(A) $2\left(\frac{\pi}{3}-\sqrt{3}\right)$ (B) $4\left(\frac{\pi}{3}-\sqrt{3}\right)$ (C) $2\left(\frac{\pi}{3}+\frac{1}{\sqrt{3}}\right)$ (D) $4\left(\frac{\pi}{3}+\frac{1}{\sqrt{3}}\right)$

Solution



Each circle has been partially drawn.
The area common to these four circles has been shaded.

Let us evaluate the area in the first quadrant. For that, we need to evaluate the x-coordinate of P. Notice that the circle centred at $(-1, -1)$ is the one which intersects the x-axis at P. The equation of this circle is given by:

$$(x+1)^2+(y+1)^2=4 \tag{1}$$

When $y=0$, $x=\sqrt{3}-1$. Thus, $P \equiv (\sqrt{3}-1,\ 0)$. The area in the first quadrant is now:

$$A_{1^{st}} = \int_0^{\sqrt{3}-1} \left(\sqrt{4-(x+1)^2}-1\right) dx \quad \left\{\begin{matrix}\text{we used (1) to write down the equation} \\ \text{of the circle in an explicit form.}\end{matrix}\right\}$$

$$= \left\{-x+\frac{1}{2}(x+1)\sqrt{4-(x+1)^2}+\frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right)\right\}\Bigg|_0^{\sqrt{3}-1} \quad \text{(verify this step)}$$

$$= \left(-\sqrt{3}+1+\frac{\sqrt{3}}{2}+2\sin^{-1}\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{3}}{2}+2\sin^{-1}\frac{1}{2}\right)=\frac{\pi}{3}-\sqrt{3}$$

Therefore, the total bounded area is

$$A = 4\times A_{1^{st}} = 4\left(\frac{\pi}{3}-\sqrt{3}\right) \text{sq units}$$

The correct option is (B). ■

Example 14

What is the area of the region bounded by $x + 1 = 0$, $y = 0$, $y = x^2 + x + 1$ and the tangent to $y = x^2 + x + 1$ at $x = 1$?

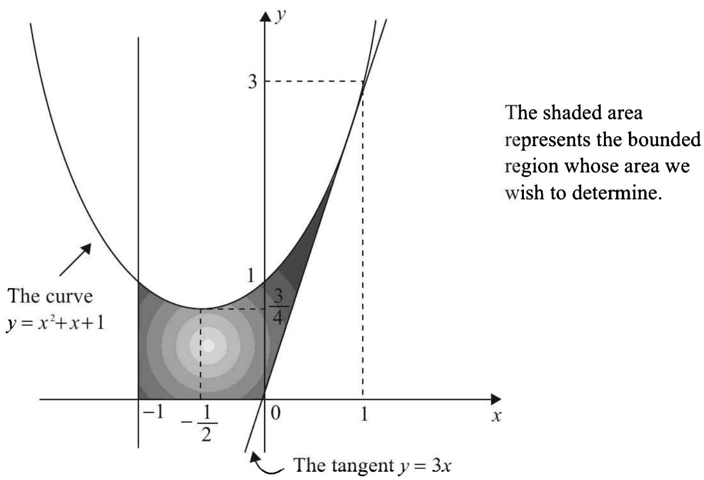
(A) $\frac{5}{6}$ (B) $\frac{7}{6}$ (C) $\frac{11}{6}$ (D) $\frac{13}{6}$

Solution: Let us first determine the equation of the said tangent:

$$\left.\frac{dy}{dx}\right|_{x=1} = (2x+1)\big|_{x=1}$$
$$= 3$$

Also, when $x = 1$, $y = 3$. The required equation is

$$y - 3 = 3(x-1) \Rightarrow y = 3x$$



From the figure above, it should be clear that the area can be calculated as described below:

$$A = \int_{-1}^{0}(x^2+x+1)\,dx + \int_{0}^{1}\{(x^2+x+1)-3x\}\,dx$$

$$= \left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)\Bigg|_{-1}^{0} + \left(\frac{x^3}{3}-x^2+x\right)\Bigg|_{0}^{1}$$

$$= -\left(-\frac{1}{3}+\frac{1}{2}-1\right)+\left(\frac{1}{3}-1+1\right)\Bigg|_{0}^{1} = \frac{7}{6}\text{ sq units}$$

The correct option is (B). ■

Example 15

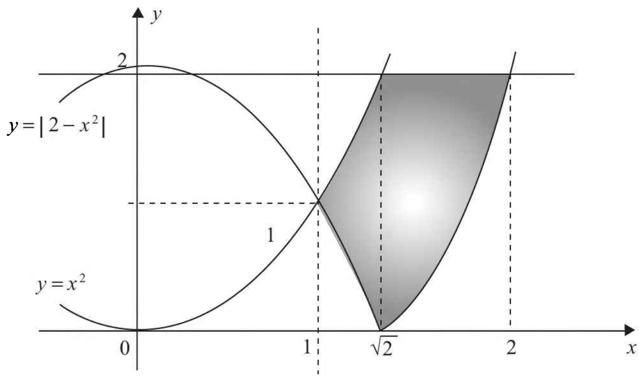
What is the area of the region bounded by the curves

$$y = x^2,\ y = |2 - x^2| \text{ and } y = 2,$$

which lies to the right of the line $x = 1$?

(A) $\dfrac{10-4\sqrt{2}}{3}$ (B) $\dfrac{12-5\sqrt{2}}{3}$ (C) $\dfrac{16-8\sqrt{2}}{3}$ (D) $\dfrac{20-12\sqrt{2}}{3}$

Solution: The specified region is sketched in the figure, below:



The required area can be evaluated as follows:

$$A = \int_1^{\sqrt{2}} (x^2 - (2 - x^2))\, dx + \int_{\sqrt{2}}^{2} (2 - (x^2 - 2))\, dx$$

$$= \left(\frac{2x^3}{3} - 2x\right)\Bigg|_1^{\sqrt{2}} + \left(4x - \frac{x^3}{3}\right)\Bigg|_{\sqrt{2}}^{2}$$

$$= \frac{20 - 12\sqrt{2}}{3} \text{ sq units} \quad \text{(verify the calculations)}$$

The correct option is (D). ■

SUBJECTIVE TYPE EXAMPLES

Example 16

Evaluate the following integrals:

(a) $\int \frac{1}{\sqrt{4x-x^2-3}}dx$ (b) $\int \frac{1}{x^2+2x+2}dx$ (c) $\int \frac{1}{(x+2)\sqrt{x^2+4x+3}}dx$

(d) $\int \frac{1}{x^2+6x+8}dx$ (e) $\int \frac{1}{\sqrt{4x^2+12x+10}}dx$ (f) $\int \frac{1}{\sqrt{25x^2+20x+3}}dx$

Solution: (a) $I = \int \frac{1}{\sqrt{4x-x^2-3}}dx = \int \frac{1}{\sqrt{1-(x-2)^2}}dx = \sin^{-1}(x-2)+C$

(b) $I = \int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx = \tan^{-1}(x+1)+C$

(c) $I = \int \frac{1}{(x+2)\sqrt{x^2+4x+3}}dx = \int \frac{1}{(x+2)\sqrt{(x+2)^2-1}}dx = \sec^{-1}(x+2)+C$

(d) $I = \int \frac{1}{x^2+6x+8}dx = \int \frac{1}{(x+3)^2-1}dx = \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right|+C$

(e) $I = \int \frac{1}{\sqrt{4x^2+12x+10}}dx = \int \frac{1}{\sqrt{(2x+3)^2+1}}dx = \frac{1}{2}\ln\left|(2x+3)+\sqrt{4x^2+12x+10}\right|+C$

(f) $I = \int \frac{1}{\sqrt{25x^2+20x+3}}dx = \int \frac{1}{\sqrt{(5x+2)^2-1}}dx = \frac{1}{5}\ln\left|(5x+2)+\sqrt{25x^2+20x+3}\right|+C$

■

Example 17

Evaluate the following integrals:

(a) $\int \frac{x^2+1}{x^4+1}dx$ (b) $\int \frac{x^2-1}{x^4+1}dx$ (c) $\int \frac{x^2+1}{x^4+x^2+1}dx$ (d) $\int \frac{1}{x^4+1}dx$

Solution: (a) We divide the numerator and denominator by x^2:

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

We now need to express the numerator as a differential of some term occurring in the denominator. The denominator $x^2+\frac{1}{x^2}$ can be written as $(x-\frac{1}{x})^2+2$. The derivative of $x-\frac{1}{x}$ is $1+\frac{1}{x^2}$. Thus, we substitute $x-\frac{1}{x}=t$. The integral reduces to $I = \int \frac{dt}{t^2+2}$. The value of the integral is

$$I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$$

(b) We follow the same approach as we did in the previous part:

$$I = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

However, this time we write the denominator $(x^2+\frac{1}{x^2})$ as $(x+\frac{1}{x})^2-2$ and substitute $(x+\frac{1}{x})=t$; this is because the numerator $(1-\frac{1}{x^2})$ is a derivative of $(x+\frac{1}{x})$, while in the previous example, the numerator was $(1+\frac{1}{x^2})$, which was a derivative of $(x-\frac{1}{x})$. Thus, we have:

$$x+\frac{1}{x}=t \quad \Rightarrow \quad I=\int\frac{dt}{t^2-2}$$

$$\Rightarrow \quad I=\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C \quad =\frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+C$$

(c) $$I=\int\frac{x^2+1}{x^4+x^2+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx=\int\frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+3}dx$$

The substitution $x-\frac{1}{x}=t$ reduces I to an 'integrable' form:

$$I=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+C$$

(d) This integral can easily be obtained from the integrals we've already evaluated in parts (a) and (b):

$$I=\int\frac{1}{x^4+1}dx=\frac{1}{2}\left\{\int\frac{x^2+1}{x^4+1}dx-\int\frac{x^2-1}{x^4+1}dx\right\}$$ ■

Example 18

Evaluate the following integrals:

(a) $\int\sin^3 x\cos^4 x dx$ (c) $\int\sin^3 x\cos^5 x\,dx$ (e) $\int\frac{1}{\sqrt[4]{\sin^3 x\cos^5 x}}dx$ (g) $\int\cos^8 x\,dx$

(b) $\int\sin^2 x\cos^5 x\,dx$ (d) $\int\cos^5 x dx$ (f) $\int\sin^4 x\cos^2 x\,dx$ (h) $\int\sec^4 x\,\mathrm{cosec}^2 x\,dx$

Solution: (a) Notice that the power of $\sin x$ is odd while that of $\cos x$ is even. What substitution can we possibly use? If we use $\sin x = t$, we will get $\cos x\,dx = dt$. Thus, we'll be left with $\int t^3\cos^3 x\,dt$, which contains $\cos^3 x$; this will be tedious to express in terms of t or $\sin x$. We therefore use the substitution $\cos x = t$:

$$\cos x=t \quad \Rightarrow \quad \sin x\,dx=-dt$$

$$\Rightarrow \quad I=\int\sin^3 x\cos^4 x\,dx=-\int\sin^2 x\cdot t^4 dt=-\int(1-t^2)t^4 dt$$

$$=\int(t^6-t^4)dt=\frac{t^7}{7}-\frac{t^5}{5}+C=\frac{\cos^7 x}{7}-\frac{\cos^5 x}{5}+C$$

(b) If you compare this integral with the one in part-(a), you'll see that the roles have been reversed. $\sin x$ has an even power while the power of $\cos x$ is odd. We therefore use the substitution $\sin x = t$:

$$\Rightarrow \quad \cos x\,dx=dt$$

$$\Rightarrow \quad I=\int t^2\cos^4 x\,dt=\int t^2(1-t^2)^2 dt$$

$$= \int t^2(t^4 - 2t^2 + 1)dt = \int (t^6 - 2t^4 + t^2)dt$$

$$= \frac{t^7}{7} - \frac{2t^5}{5} + \frac{t^3}{3} + C = \frac{\sin^7 x}{7} - \frac{2\sin^5 x}{5} + \frac{\sin^3 x}{3} + C$$

(c) Here, the powers of $\sin x$ and $\cos x$ are both odd. This suggests that we can substitute either $\sin x$ or $\cos x$ as a new variable. We use $\sin x = t$.

$$\Rightarrow \quad \cos x\, dx = dt$$

$$\Rightarrow \quad I = \int t^3 \cos^4 x\, dt = \int t^3 (1 - t^2)^2 dt = \int t^3(t^4 - 2t^2 + 1)dt$$

$$= \int (t^7 - 2t^5 + t^3)dt = \frac{t^8}{8} - \frac{t^6}{3} + \frac{t^4}{4} + C$$

$$= \frac{\sin^8 x}{8} - \frac{\sin^6 x}{3} + \frac{\sin^4 x}{4} + C$$

(d) The power of $\cos x$ is odd. As in part-(b), we use the substitution $\sin x = t$:

$$\Rightarrow \quad \cos x\, dx = dt$$

$$\Rightarrow \quad I = \int \cos^4 x\, dt = \int (1 - t^2)^2 dt = \int (t^4 - 2t^2 + 1)dt$$

$$= \frac{t^5}{5} - \frac{2t^3}{3} + t + C = \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3} + \sin x + C$$

(e) Observe that neither the substitution $\sin x = t$ or $\cos x = t$ would help. Some rearrangement needs to be done before we can substitute for something:

$$I = \int \frac{1}{\sqrt[4]{\sin^3 x \cos^5 x}} dx = \int \frac{1}{\sqrt[4]{\cos^8 x \cdot \frac{\sin^3 x}{\cos^3 x}}} dx$$

$$= \int \frac{1}{\cos^2 x \sqrt[4]{\tan^3 x}} dx = \int \frac{\sec^2 x}{\sqrt[4]{\tan^3 x}} dx$$

Now we use the substitution $\tan x = t$:

$$\Rightarrow \quad \sec^2 x\, dx = dt$$

$$\Rightarrow \quad I = \int \frac{dt}{t^{3/4}} = 4t^{\frac{1}{4}} + C = 4(\tan x)^{\frac{1}{4}} + C$$

Note that this example is one where the powers of $\sin x$ and $\cos x$ add up to a negative integer:

$$\left(-\frac{3}{4}\right) + \left(-\frac{5}{4}\right) = -2$$

(f) Both the powers are even. Observe that neither of the substitutions $\sin x = t$ or $\cos x = t$ would help. Therefore, we again try to convert this expression into another form which contains $\tan x$ and $\sec x$ terms, as in the previous example.

$$I = \int \sin^4 x \cos^2 x\, dx = \int \cos^6 x \cdot \frac{\sin^4 x}{\cos^4 x} dx = \int \frac{\tan^4 x}{\sec^6 x} dx$$

Not much headway! What do we do now? Neither the substitution $\tan x = t$ or $\sec x = t$ would help. Instead, we follow an alternative approach. We use multiple angle formulae to reduce the non-linear trigonometric terms in the integral expression to linear trigonometric terms:

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Similarly, $\qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$

$$\Rightarrow \quad I = \frac{1}{16}\int (1+\cos 2x)(3 - 4\cos 2x + \cos 4x)dx$$

$$= \frac{1}{16}\int \{3 - \cos 2x + \cos 4x - 4\cos^2 2x + \cos 2x \cos 4x\}\,dx$$

$$= \frac{1}{16}\int \left\{3 - \cos 2x + \cos 4x - \frac{4(1+\cos 4x)}{2} + \frac{\cos 6x + \cos 2x}{2}\right\} dx$$

$$= \frac{1}{16}\int \left\{1 - \frac{1}{2}\cos 2x - \cos 4x + \frac{1}{2}\cos 6x\right\} dx$$

$$= \frac{1}{16}\left(x - \frac{\sin 2x}{4} - \frac{\sin 4x}{4} + \frac{\sin 6x}{12}\right) + C$$

(g) The power of $\cos x$ is even . As in the previous example, we use multiple angle formulae to reduce the non-linear trignometric term $(\cos^8 x)$ to (a combination of) linear trignometric terms.

$$I = \int \cos^8 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^4 dx = \frac{1}{16}\int ((1+\cos 2x)^2)^2\, dx$$

$$= \frac{1}{16}\int (1 + 2\cos 2x + \cos^2 2x)^2\, dx = \frac{1}{16}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{64}\int (3 + 4\cos 2x + \cos 4x)^2\, dx$$

$$= \frac{1}{64}\int \left\{9 + 16\cos^2 2x + \cos^2 4x + 24\cos 2x + 6\cos 4x + 8\cos 2x \cos 4x\right\}$$

$$= \frac{1}{64}\int \left\{9 + 16\left(\frac{1+\cos 4x}{2}\right) + \frac{1+\cos 8x}{2} + 24\cos 2x + 6\cos 4x + 4\cos 6x + 4\cos 2x\right\} dx$$

$$= \frac{1}{64}\int \left\{\frac{35}{2} + 28\cos 2x + 14\cos 4x + 4\cos 6x + \frac{1}{2}\cos 8x\right\} dx$$

This expression involves purely linear trigonometric terms and can easily be integrated. Thus,

$$I = \frac{35}{128}x + \frac{7}{32}\sin 2x + \frac{7}{128}\sin 4x + \frac{1}{96}\sin 6x + \frac{1}{1024}\sin 8x + C$$

(h) $I = \displaystyle\int \sec^4 x\ \mathrm{cosec}^2 x\, dx = \int \frac{1}{\sin^2 x \cos^4 x}\, dx$

Again, neither the substitution $\sin x = t$ or $\cos x = t$ would help. We also cannot use multiple angle formulae as we've done in the previous two examples. However, as in part (e), we can convert this expression into one involving $\tan x$ and $\sec x$ terms.

$$I = \int \frac{1}{\sin^2 x \cos^4 x}\,dx = \int \frac{1}{\tan^2 x\,\cos^6 x}\,dx = \int \frac{\sec^6 x}{\tan^2 x}\,dx$$

Now, we use the substitution $\tan x = t$. Thus, $\sec^2 x\ dx = dt$

$$\Rightarrow \quad I = \int \frac{\sec^4 x}{t^2}\,dt = \int \frac{(1+\tan^2 x)^2}{t^2}\,dt = \int \frac{(1+t^2)^2}{t^2}\,dt$$

$$= \int \left(\frac{t^4 + 2t^2 + 1}{t^2} \right) dt = \int (t^2 + 2 + t^{-2})\,dt = \frac{t^3}{3} + 2t - \frac{1}{t} + C$$

$$= \frac{\tan^3 x}{3} + 2\tan x - \frac{1}{\tan x} + C$$

Note that like part (e), this example is also one where the powers of $\sin x$ and $\cos x$ add up to a negative integer. ■

Example 19

Evaluate the following integrals:

(a) $\displaystyle\int \frac{1}{(x-1)\sqrt{x+2}}\,dx$ (c) $\displaystyle\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx$

(b) $\displaystyle\int \frac{x}{(x-3)\sqrt{x+1}}\,dx$ (d) $\displaystyle\int \frac{x}{(x^2+2x+2)\sqrt{x+1}}\,dx$

Solution: (a) Substitute $x + 2 = t^2$

$$\Rightarrow \quad dx = 2t\,dt$$

$$\Rightarrow \quad I = \int \frac{1}{(x-1)\sqrt{x+2}}\,dx = \int \frac{1}{(t^2-3)\cdot t}\cdot 2t\,dt$$

$$= 2\int \frac{1}{t^2-3}\,dt = \frac{1}{\sqrt{3}} \ln\left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| + C = \frac{1}{\sqrt{3}} \ln\left| \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + C$$

(b) Substitute $x + 1 = t^2$

$$\Rightarrow \quad dx = 2t\,dt$$

$$\Rightarrow \quad I = \int \frac{x}{(x-3)\sqrt{x+1}}\,dx = \int \frac{(t^2-1)}{(t^2-4)\cdot t}\,2t\,dt$$

$$= 2\int \frac{t^2-1}{t^2-4}\,dt = 2\int \left\{ 1 + \frac{3}{t^2-4} \right\} dt$$

$$= 2t + \frac{3}{2} \ln\left| \frac{t-2}{t+2} \right| + C = 2\sqrt{x+1} + \frac{3}{2} \ln\left| \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right| + C$$

(c) Substitute $x+1=\frac{1}{t} \quad \Rightarrow \quad dx=-\frac{1}{t^2}dt$

$$\Rightarrow \quad I=\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx=\int \frac{t}{\sqrt{(\frac{1}{t}-1)^2+(\frac{1}{t}-1)+1}}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$=-\int \frac{1}{\sqrt{t^2-t+1}}\,dt \;=-\int \frac{1}{\sqrt{(t-\frac{1}{2})^2+\frac{3}{4}}}\,dt$$

$$=-\ln\left|\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right|+C$$

$$=-\ln\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+C$$

(d) Substitute $x+1=t^2 \quad \Rightarrow \quad dx=2t\,dt$

$$\Rightarrow \quad I=\int \frac{x}{(x^2+2x+2)\sqrt{x+1}}\,dx=\int \frac{t^2-1}{(t^4+1)\cdot t}\cdot 2t\,dt=2\int \frac{t^2-1}{t^4+1}\,dt$$

We have already seen how to evaluate integrals of this type. ■

Example 20

Evaluate $\int \frac{x^3+2x^2+x-7}{\sqrt{x^2+2x-3}}\,dx$.

Solution: We can find a quadratic polynomial ax^2+bx+c and a constant α such that

$$\int \frac{x^3+2x^2+x-7}{\sqrt{x^2+2x-3}}=(ax^2+bx+c)\sqrt{x^2+2x-3}+\alpha\int \frac{1}{\sqrt{x^2+2x-3}}\,dx$$

Differentiating both sides, we obtain:

$$\frac{x^3+2x^2+x-7}{\sqrt{x^2+2x-3}}=\frac{(ax^2+bx+c)(x+1)}{\sqrt{x^2+2x-3}}+(2ax+b)\sqrt{x^2+2x-3}+\frac{\alpha}{\sqrt{x^2+2x-3}}$$

$$\Rightarrow \quad x^3+2x^2+x-7=(ax^2+bx+c)(x+1)+(2ax+b)(x^2+2x-3)+\alpha$$

Comparing coefficients on both sides, we obtain:

$$a=\frac{1}{3},\; b=\frac{1}{6},\; c=\frac{5}{2},\; \alpha=-9$$

Once we have these values, the integral is straightforward to obtain. ■

Example 21

Evaluate the following integrals:

(a) $\displaystyle\int \frac{1}{1+\sin^2 x+2\cos^2 x}\,dx$ (b) $\displaystyle\int \frac{1}{(a\sin x+b\cos x)^2}\,dx$

Solution: We can solve these integrals easily by simply dividing the numerator and denominator by $\cos^2 x$, which converts the given expression into a modified form containing $\tan x$ and $\sec x$ terms:

$$\text{(a)}\quad I = \int \frac{1}{1+\sin^2 x+2\cos^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x+\tan^2 x+2}\,dx \qquad \left\{\begin{array}{l}\text{Now write the } \sec^2 x\\ \text{in the denominator}\\ \text{as } 1+\tan^2 x\end{array}\right\}$$

$$= \int \frac{\sec^2 x}{3+2\tan^2 x}\,dx$$

Substituting $\tan x = t$ reduces this integral to

$$I = \int \frac{dt}{3+2t^2} = \frac{1}{2}\int \frac{dt}{\left(\sqrt{\frac{3}{2}}\right)^2+t^2}$$

This can now be integrated easily.

$$\text{(b)}\quad I = \int \frac{1}{(a\sin x+b\cos x)^2}\,dx = \int \frac{\sec^2 x}{(a\tan x+b)^2}\,dx$$

$$= \int \frac{dt}{(at+b)^2} \qquad \{\text{Substituting } \tan x = t\}$$

$$= \frac{-1}{a(at+b)}+C = \frac{-1}{a(a\tan x+b)}+C$$

All integrals of the general form $I = \int \frac{1}{a+b\sin^2 x+c\cos^2 x}\,dx$ or $I = \int \frac{1}{(a\sin x+b\cos x)^2}\,dx$ can therefore be solved using the approach described above. ■

Example 22

Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$.

Solution: We have

$$I = \int \frac{\sqrt{\cos^2 x-\sin^2 x}}{\sin x}\,dx = \int \sqrt{\cot^2 x-1}\,dx$$

The substitution $\cot x = \sec\theta$ can help us simplify this integral:

$$\cot x = \sec\theta$$

$$\Rightarrow \quad -\operatorname{cosec}^2 x dx = \sec\theta \tan\theta d\theta$$

$$\Rightarrow \quad I = \int\sqrt{\sec^2\theta - 1}\cdot \frac{\sec\theta\tan\theta}{-\operatorname{cosec}^2 x}\, d\theta$$

$$= -\int\frac{\sec\theta\tan^2\theta}{1+\sec^2\theta}\, d\theta \quad \left\{\begin{matrix}\textit{Now convert this expression to one} \\ \textit{involving } \sin\theta \textit{ and } \cos\theta \textit{ terms}\end{matrix}\right\}$$

$$= -\int\frac{\sin^2\theta}{\cos\theta + \cos^3\theta}\, d\theta$$

$$= -\int\frac{\sin^2\theta}{\cos\theta(1+\cos^2\theta)}\, d\theta \quad \left\{\begin{matrix}\text{Now write } \sin^2 \\ \text{as } 1-\cos^2\theta\end{matrix}\right\}$$

$$= \int\frac{\cos^2\theta - 1}{\cos\theta(1+\cos^2\theta)}\, d\theta \quad = -\int\frac{(\cos^2\theta+1) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\, d\theta$$

$$= -\int\sec\theta d\theta + 2\int\frac{\cos\theta}{1+\cos^2\theta}\, d\theta$$

$$= -\ln|\sec\theta + \tan\theta| + 2\int\frac{\cos\theta}{2-\sin^2\theta}\, d\theta$$

Now use the substitution $\sin\theta = t$.

$$= -\ln|\sec\theta + \tan\theta| + 2\int\frac{dt}{\left(\sqrt{2}\right)^2 - t^2} = -\ln|\sec\theta + \tan\theta| + \frac{2}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + C$$

$$= -\ln\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C.$$ ■

Example 23

Evaluate $\int\sqrt{\tan\theta}\, d\theta$.

Solution: To get rid of the square-root and convert this expression into a purely rational one, we use the substitution $\tan\theta = x^2$. Thus,

$$\sec^2\theta\, d\theta = 2x\, dx \quad \Rightarrow \quad d\theta = \frac{2x\, dx}{\sec^2\theta} \quad = \frac{2x}{1+x^4}\, dx$$

$$\Rightarrow \quad I = \int\sqrt{x^2}\cdot\frac{2x}{1+x^4}\, dx = 2\int\frac{x^2}{1+x^4}\, dx = \int\frac{x^2+1}{x^4+1}\, dx + \int\frac{x^2-1}{x^4+1}\, dx$$

We have already evaluated both these integrals in the previous examples.

Example 24

Evaluate the following integrals:

(a) $\int (x+3)\sqrt{x^2+2x+3}\;dx$ (b) $\int \frac{x}{x-\sqrt{x^2-1}}\;dx$

Solution: (a) The derivative of x^2+2x+3 is $2x+3$. Thus, we can find α and β such that

$$(x+3) = \alpha(2x+3)+\beta$$

$$\Rightarrow \quad \alpha = \frac{1}{2},\; \beta = \frac{3}{2}$$

$$\Rightarrow \quad I = \frac{1}{2}\int (2x+3)\sqrt{x^2+2x+3}\;dx + \frac{3}{2}\int\sqrt{x^2+2x+3}\;dx$$

$$= \underbrace{\frac{1}{2}\int\sqrt{t}\;dt}_{(\text{where } t\,=\,x^2+2x+3)} + \underbrace{\frac{3}{2}\int\sqrt{(x+1)^2+2}\;dx}_{(\text{this is easy to integrate})}$$

$$= \frac{t^{3/2}}{3} + \frac{3}{4}\left\{(x+1)\sqrt{x^2+2x+3} + 2\ln\left|(x+1)+\sqrt{x^2+2x+3}\right|\right\} + C$$

$$= \frac{(x^2+2x+3)^{3/2}}{3} + \frac{3}{4}\left\{(x+1)\sqrt{x^2+2x+3} + 2\ln\left|(x+1)+\sqrt{x^2+2x+3}\right|\right\} + C$$

(b) We first rationalize the denominator:

$$I = \int\frac{x}{x-\sqrt{x^2-1}}\;dx \quad = \int x(x+\sqrt{x^2-1})\;dx$$

$$= \int x^2dx + \underbrace{\int x\sqrt{x^2-1}\;dx}_{(\text{substitute } x^2-1\,=\,t)} \quad = \frac{x^3}{3}+\frac{1}{2}\int\sqrt{t}\;dt$$

$$= \frac{x^3}{3}+\frac{t^{3/2}}{3}+C \quad = \frac{x^3}{3}+\frac{(x^2-1)^{3/2}}{3}+C$$

■

Example 25

Expand the following using partial fractions:

(a) $\frac{x^2-3x-4}{x^3-6x^2+11x-6}$ (b) $\frac{x^2+x+1}{(x-1)^2(x-2)(x-3)}$

Solution: (a) The denominator can be factorized:

$$x^3-6x^2+11x-6 = (x-1)(x-2)(x-3)$$

The partial fraction expansion is:

$$\frac{(x+1)(x-4)}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

By cross-multiplying, we obtain:

$$(x+1)(x-4) = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$\text{Put } x=1 \quad \Rightarrow \quad A = \frac{(2)\times(-3)}{(-1)\times(-2)} = -3$$

$$\text{Put } x=2 \quad \Rightarrow \quad B = \frac{(3)\times(-2)}{(1)\times(-1)} = 6$$

$$\text{Put } x=3 \quad \Rightarrow \quad C = \frac{(4)\times(-1)}{(2)\times(1)} = -2$$

Thus, the partial fraction expansion is

$$\frac{-3}{x-1} + \frac{6}{x-2} + \frac{-2}{x-3}$$

(b) The partial fraction expansion of this expression would be of the form

$$\frac{x^2+x+1}{(x-1)^2(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{x-3}$$

By cross-multiplying, we obtain:

$$x^2+x+1 = A(x-1)(x-2)(x-3) + B(x-2)(x-3) + C(x-1)^2(x-3) + D(x-1)^2(x-2)$$

$$\text{Put } x=1 \quad \Rightarrow \quad B = \frac{1+1+1}{(-1)\times(-2)} = \frac{3}{2}$$

$$\text{Put } x=2 \quad \Rightarrow \quad C = \frac{4+2+1}{1^2\times -1} = -7$$

$$\text{Put } x=3 \quad \Rightarrow \quad D = \frac{9+3+1}{2^2\times 1} = \frac{13}{4}$$

To obtain A, we compare the coefficients of x^3 on both sides. Thus,

$$0 = A + C + D$$

$$\Rightarrow \quad A = -(C+D) = -\left(-7+\frac{13}{4}\right) = \frac{15}{4}$$

The required partial fraction expansion is

$$\frac{\frac{15}{4}}{x-1} + \frac{\frac{3}{2}}{(x-1)^2} + \frac{-7}{x-2} + \frac{\frac{13}{4}}{x-3}. \qquad \blacksquare$$

Example 26

Evaluate $\int \frac{x^2-8x+7}{(x-5)^2(x+2)^2}\,dx$.

Solution: We first express this expression as a sum of partial fractions:

$$\frac{(x-1)(x-7)}{(x-5)^2(x+2)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

By cross-multiplying, we obtain:

$$(x-1)(x-7) = A(x-5)(x+2)^2 + B(x+2)^2 + C(x-5)^2(x+2) + D(x-5)^2$$

$$\text{Put } x=5 \quad \Rightarrow \quad B = \frac{4\times -2}{7^2} = \frac{-8}{49}$$

$$\text{Put } x=-2 \quad \Rightarrow \quad D = \frac{-3\times -9}{7^2} = \frac{27}{49}$$

To obtain A and C, we can now compare coefficients on both sides. Comparing the coefficients of x^3 on both sides, we obtain:

$$0 = A + C$$

$$\Rightarrow \quad A = -C \qquad (1)$$

Comparing the constant terms on both sides, we obtain

$$7 = 20C + 4B + 50C + 25D \qquad (2)$$

Using (1) in (2), we obtain:

$$7 = 20C + 4B + 50C + 25D = 70C - \frac{32}{49} + \frac{675}{49}$$

$$\Rightarrow \quad 70C = \frac{-400}{49} \quad \Rightarrow \quad C = \frac{-40}{343} \quad \Rightarrow \quad A = \frac{40}{343}$$

Thus, the required integral is:

$$I = \frac{40}{343}\ln|x-5| + \frac{8}{49(x-5)} - \frac{40}{343}\ln|x+2| - \frac{27}{49(x+2)} + C$$ ■

Example 27

Evaluate the following integrals:

(a) $\int \frac{1}{x(1+x+x^2+x^3)}\,dx$ (b) $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}\,dx$

Solution: (a) We have

$$I = \int \frac{1}{x(1+x+x^2(1+x))}\,dx = \int \frac{1}{x(1+x)(1+x^2)}\,dx$$

We find out the partial fraction expansion of this expression:

$$\frac{1}{x(1+x)(1+x^2)} = \frac{A}{x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

By cross-multiplying, we obtain

$$1 = A(1+x)(1+x^2) + Bx(1+x^2) + (Cx+D)x(1+x)$$

Put $x = 0 \quad \Rightarrow \quad A = 1$

Put $x = -1 \quad \Rightarrow \quad B = -\frac{1}{2}$

Compare the coefficients of $x^3 \quad \Rightarrow \quad 0 = A + B + C \Rightarrow \quad C = -\frac{1}{2}$

Compare the coefficients of $x^2 \quad \Rightarrow \quad 0 = A + C + D \Rightarrow \quad D = -\frac{1}{2}$

The partial fraction expansion is

$$\frac{1}{x} + \frac{(-\frac{1}{2})}{1+x} + \frac{(-\frac{1}{2})x + (-\frac{1}{2})}{1+x^2}$$

The integral is therefore:

$$\begin{aligned} I &= \int \frac{1}{x} dx - \frac{1}{2}\int \frac{1}{1+x} dx - \frac{1}{2}\int \frac{x}{1+x^2} dx - \frac{1}{2}\int \frac{1}{1+x^2} dx \\ &= \ln|x| - \frac{1}{2}\ln|1+x| - \frac{1}{4}\ln(1+x^2) - \frac{1}{2}\tan^{-1} x + C \end{aligned}$$

(b) We again find out the partial fraction expansion of the given expression:

$$\frac{3x^2+x+3}{(x-1)^3(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1}$$

We cross multiply to obtain

$$3x^2 + x + 3 = A(x-1)^2(x^2+1) + B(x-1)(x^2+1) + C(x^2+1) + (Dx+E)(x-1)^3$$

Put $x = 1 \quad \Rightarrow \quad C = \frac{7}{2}$ (1)

Compare the coefficients of $x^4 \quad \Rightarrow \quad 0 = A + D$ (2)

Compare the coefficients of $x^3 \quad \Rightarrow \quad 0 = -2A + B - 3D + E$ (3)

Compare the coefficients of $x^2 \quad \Rightarrow \quad 3 = 2A - B + C + 3D - 3E$ (4)

Compare the coefficients of $x \quad \Rightarrow \quad 1 = -2A + B - D + 3E$ (5)

Adding (4) and (5), we obtain $4 = C + 2D$:

$$\Rightarrow \quad D = \frac{1}{4} \text{ (from (1))}$$

$$\Rightarrow \quad A = \frac{-1}{4} \text{ (from (2))}$$

Adding (3) and (4), we obtain:

$$3 = C - 2E$$

$$\Rightarrow \quad E = \frac{1}{4} \text{ (again, using (1))}$$

Finally, from (5), $B = 0$. The partial fraction expansion is therefore

$$\frac{(-\frac{1}{4})}{x-1} + \frac{(\frac{7}{2})}{(x-1)^3} + \frac{(\frac{1}{4})x + (\frac{1}{4})}{x^2+1}$$

The integral is

$$I = -\frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{7}{2}\int \frac{1}{(x-1)^3}\,dx + \frac{1}{4}\int \frac{x}{x^2+1}\,dx + \frac{1}{4}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{1}{4}\ln|x-1| - \frac{7}{4(x-1)^2} + \frac{1}{8}\ln(x^2+1) + \frac{1}{4}\tan^{-1} x + C$$

$$= \frac{1}{4}\left\{\tan^{-1} x - \frac{7}{(x-1)^2} + \ln\left(\frac{\sqrt{x^2+1}}{|x-1|}\right)\right\} + C$$

■

Example 28

Evaluate the following integrals:

(a) $\int x \sin x \, dx$ (b) $\int (\ln x)^2 \, dx$ (c) $\int \sin^{-1} x \, dx$ (d) $\int x \tan^{-1} x \, dx$

Solution: (a) Using the ILATE rule, we let $f(x) = x$ be our first function:

$$I = \int \underset{\text{Ist}}{x} \underset{\text{IInd}}{\sin x}\, dx = -x\cos x - \int 1\cdot(-\cos x)\, dx = -x\cos x + \sin x + C$$

See what happens if you choose $\sin x$ as the first function.

(b) We choose unity as the second function and apply integration by parts:

$$I = \int \underset{\text{Ist}}{(\ln x)^2} \cdot \underset{\text{IInd}}{1}\, dx = x(\ln x)^2 - \int \left(\frac{2\ln x}{x}\right)\cdot x\, dx$$

$$= x(\ln x)^2 - 2\int \underset{\substack{\uparrow \\ \text{Again apply integration by parts, taking unity as the second function.}}}{\ln x}\, dx \qquad = x(\ln x)^2 - 2\left\{x\ln x - \int \frac{1}{x}\cdot x\, dx\right\}$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

(c) Here again, we choose unity as the second function:

$$I = \int \underset{\text{Ist}}{\sin^{-1} x} \cdot \underset{\text{IInd}}{1}\, dx \quad = x\sin^{-1} x - \int \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\substack{\text{Substitute } 1-x^2=t \\ \Rightarrow -x\,dx = dt/2}} \cdot x\, dx$$

$$= x\sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} \quad = x\sin^{-1} x + \sqrt{t} + C \quad = x\sin^{-1} x + \sqrt{1-x^2} + C$$

(d) Using the ILATE rule, we choose $\tan^{-1} x$ as the first function:

$$I = \int \underset{\text{Ist}}{\tan^{-1} x} \cdot \underset{\text{IInd}}{x}\, dx \quad = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1}{1+x^2} \cdot x^2\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\, dx \quad = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left\{\int dx - \int \frac{1}{1+x^2}\, dx\right\}$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

■

Example 29

Evaluate $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$.

Solution: Let $I_1 = \int e^{ax} \sin bx\, dx$ and $I_2 = \int e^{ax} \cos bx\, dx$:

$$I_1 = \int \underset{\text{Ist}}{e^{ax}} \underset{\text{IInd}}{\sin bx}\, dx \text{ (Use integration by parts)}$$

$$= \frac{-e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx\, dx$$

$$\Rightarrow \quad I_1 = \frac{-e^{ax}\cos bx}{b} + \frac{a}{b} I_2 \tag{1}$$

Similarly, we now apply integration by parts on I_2:

$$I_2 = \int \underset{\text{Ist}}{e^{ax}} \underset{\text{IInd}}{\cos bx}\, dx = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax} \sin bx\, dx$$

$$\Rightarrow \quad I_2 = \frac{e^{ax}\sin bx}{b} - \frac{a}{b} I_1 \tag{2}$$

Solving (1) and (2), we have (verify):

$$I_1 = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

$$I_2 = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

■

Example 30

Evaluate the following integrals

(a) $\int \tan^{-1}\sqrt{x}\,dx$ (b) $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx$

Solution: (a) We apply integration by parts, taking unity as the second function:

$$I = \int \underset{\text{Ist}}{\tan^{-1}\sqrt{x}} \cdot \underset{\text{IInd}}{1}\,dx = x\tan^{-1}\sqrt{x} - \int \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\cdot x\,dx$$

$$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{1+x}\,dx$$

To evaluate the integral on the right side above (call it I_2), we let $x = t^2$. Thus,

$$dx = 2tdt$$

$$I_2 = \int\frac{t^2}{1+t^2}\,dt = \int\left(1-\frac{1}{1+t^2}\right)dt$$

$$= t - \tan^{-1}t + C = \sqrt{x} - \tan^{-1}\sqrt{x} + C$$

Thus, $I = x\tan^{-1}\sqrt{x} + \sqrt{x} - \tan^{-1}\sqrt{x} + C$

(b) We can divide the given expression into two parts, $\frac{x}{\sqrt{1+x^2}}$ and $\tan^{-1}x$. The integration of the first expression is easy to carry out:

$$\int\frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int\frac{dt}{\sqrt{t}} \quad (\text{where } t = 1+x^2)$$

$$= \sqrt{t} = \sqrt{1+x^2}$$

Now we integrate our original expression by parts:

$$I = \int \underset{\text{Ist}}{\tan^{-1}x} \cdot \underset{\text{IInd}}{\frac{x}{\sqrt{1+x^2}}}\,dx = \sqrt{1+x^2}\tan^{-1}x - \int\frac{1}{1+x^2}\cdot\sqrt{1+x^2}\,dx$$

$$= \sqrt{1+x^2}\tan^{-1}x - \int\frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \sqrt{1+x^2}\tan^{-1}x - \ln\left|x+\sqrt{1+x^2}\right| + C$$

■

Example 31

Evaluate the following integrals:

(a) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$ (b) $\int e^x\left\{\frac{x^3-x+2}{(x^2+1)^2}\right\}dx$

Solution: (a) We have

$$I = \int e^{2x}\left(\frac{1+2\sin x\cos x}{2\cos^2 x}\right) dx = \int e^{2x}\left(\frac{1}{2}\sec^2 x + \tan x\right) dx$$

$$= \frac{1}{2}\int e^{t}\left(\frac{1}{2}\sec^2\frac{t}{2} + \tan\frac{t}{2}\right) dt \text{ (using the substitution } 2x = t)$$

$$= \frac{1}{2}e^{t}\tan\frac{t}{2} + C \text{ (how?)}$$

$$= \frac{1}{2}e^{2x}\tan x + C$$

(b) We try to express $\frac{x^3-x+2}{(x^2+1)^2}$ in the form $f(x) + f'(x)$.

Let, $$f(x) = \frac{Ax+B}{x^2+1}$$

$$\Rightarrow \quad f'(x) = \frac{A(x^2+1)-2x(Ax+B)}{(x^2+1)^2} = \frac{-Ax^2-2Bx+A}{(x^2+1)^2}$$

$$\Rightarrow \quad f(x) + f'(x) = \frac{Ax+B}{x^2+1} + \frac{-Ax^2-2Bx+A}{(x^2+1)^2}$$

$$= \frac{(Ax+B)(x^2+1)+(-Ax^2-2Bx+A)}{(x^2+1)^2}$$

$$= \frac{Ax^3+(B-A)x^2+(A-2B)x+(A+B)}{(x^2+1)^2}$$

Comparing the numerator of the expression above with (x^3-x+2), we obtain

$$A = B = 1$$

$$\Rightarrow \quad I = \int e^{x}\left\{\frac{x^3-x+2}{(x^2+1)^2}\right\} dx = \int e^{x}\left\{\frac{x+1}{x^2+1} + \frac{-x^2-2x+1}{(x^2+1)^2}\right\} dx$$

$$= e^{x}\left(\frac{x+1}{x^2+1}\right) + C \text{ (why?)}$$

■

Example 32

Evaluate the following integrals:

(a) $\displaystyle\int \frac{x^2+x+1}{\sqrt{x^2+2x+3}}\, dx$ (b) $\displaystyle\int (x^2+x+1)\sqrt{x^2+2x+3}\, dx$

Solution: Let us first find constants α, β and γ (common to both the questions) such that

$$x^2+x+1 = \alpha(x^2+2x+3) + \beta(2x+2) + \gamma$$

$$= \alpha x^2 + (2\alpha+2\beta)x + 3\alpha + 2\beta + \gamma$$

$$\Rightarrow \quad \alpha = 1, \; \beta = -\frac{1}{2}, \quad \gamma = -1$$

(a) $I_1 = \int \frac{x^2+x+1}{\sqrt{x^2+2x+3}}\,dx$

$$= \int \frac{x^2+2x+3}{\sqrt{x^2+2x+3}}\,dx - \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx - \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \sqrt{(x+1)^2+2}\,dx - \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx - \int \frac{1}{\sqrt{(x+1)^2+2}}\,dx$$

$$\begin{pmatrix}\text{use the substituion}\\ x^2+2x+3=t\end{pmatrix}$$

$$= \frac{1}{2}\{(x+1)\sqrt{x^2+2x+3} + 2\ln|(x+1)+\sqrt{x^2+2x+3}|\}$$
$$-\sqrt{x^2+2x+3} - \ln\left\{(x+1)+\sqrt{x^2+2x+3}\right\} + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+3} - \sqrt{x^2+2x+3} + C$$

$$= \frac{1}{2}(x-1)\sqrt{x^2+2x+3} + C$$

(b) Using the same values for α, β and γ, this integral now becomes

$$I_2 = \int (x^2+2x+3)^{3/2}dx - \frac{1}{2}\int (2x+2)\sqrt{x^2+2x+3}\,dx - \int \sqrt{x^2+2x+3}\,dx$$

The last two integrals can be evaluated easily. To evaluate

$$I_3 = \int (x^2+2x+3)^{3/2}dx = \int ((x+1)^2+2)^{3/2}dx,$$

we let $x+1 = \sqrt{2}\tan\theta$, so that $dx = \sqrt{2}\sec^2\theta d\theta$:

$$\Rightarrow \quad I_3 = \int 2^{3/2}\sec^3\theta \cdot \sqrt{2}\sec^2\theta d\theta = 4\int \sec^5\theta d\theta$$

$$\Rightarrow \quad I = \frac{I_3}{4} = \int \underset{\text{Ist}}{\sec^3\theta} \cdot \underset{\text{IInd}}{\sec^2 d\theta}$$

$$= \sec^3\theta\tan\theta - 3\int \sec^3\theta\tan^2\theta d\theta$$

$$= \sec^3\theta\tan\theta - 3\int \sec^3\theta(\sec^2\theta - 1)d\theta$$

$$= \sec^3\theta\tan\theta - 3\int \sec^5\theta d\theta + 3\int \sec^3\theta d\theta$$

$$= \sec^3\theta\tan\theta - 3I + 3I_4$$

where $I_4 = \int \sec^3\theta d\theta$

$$\Rightarrow \quad 4I = \sec^3\theta\tan\theta + 3I_4$$

To evaluate I_4, we again use integration by parts:

$$I_4 = \int \underset{\text{Ist}}{\sec\theta}\,\underset{\text{IInd}}{\sec^2\theta}\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta$$

$$= \sec\theta\tan\theta - I_4 \quad + \int \sec\theta\,d\theta$$

$$\Rightarrow \quad 2I_4 = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$\Rightarrow \quad I = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| + C$$

■

Example 33

Write reduction formulae (formulae which express the integral containing n in terms of integral(s) containing lower order indices) for the following integrals:

(a) $\int \sin^n x\,dx$ (c) $\int \tan^n x\,dx$ (e) $\int \sec^n x\,dx$

(b) $\int \cos^n x\,dx$ (d) $\int \cot^n x\,dx$ (f) $\int \operatorname{cosec}^n x\,dx$

Solution: (a) $I_n = \int \sin^n x\,dx = \int \underset{\text{Ist}}{\sin^{n-1} x}\cdot\underset{\text{IInd}}{\sin x}\,dx$

$$= -\cos x\cdot\sin^{n-1}x - (n-1)\int \sin^{n-2}x\cdot\cos x\cdot(-\cos x)\,dx$$

$$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$$

$$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$= -\cos x\sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow \quad I_n = -\frac{\cos x\sin^{n-1}x}{n} + \frac{(n-1)}{n}I_{n-2}$$

Observe that this reduction formula relates I_n to I_{n-2}. Using exactly the same approach, we obtain a similar relation for our next part.

(b) $I_n = \int \cos^n x\,dx = \dfrac{\cos^{n-1}x\sin x}{n} + \dfrac{(n-1)}{n}I_{n-2}$

(c) $I_n = \int \tan^n x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)dx$

$$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

Thus, $\quad I_n = \dfrac{\tan^{n-1}x}{n-1} - I_{n-2}$

(d) Using an analogous approach as in the part above, we obtain the required relation for this part:

$$I_n = \int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$$

(e) $I_n = \int \sec^n x dx = \int \underset{\text{Ist}}{\sec^{n-2} x} \cdot \underset{\text{IInd}}{\sec^2 x}\, dx$

$$= \tan x \cdot \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan x\, dx$$

$$= \tan x \cdot \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \tan x \cdot \sec^{n-2} x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$\Rightarrow \quad I_n = \frac{\tan x \cdot \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Using exactly the same approach, we obtain a similar relation for the next part.

(f) $I_n = \int \operatorname{cosec}^n x dx = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$ ■

Example 34

Suppose that $g(x)$ is an even function, and $f(x) = \int_0^x g(t)dt$. Is $f(x)$ even or odd, or neither?

Solution: We have

$$f(-x) = \int_0^{-x} g(t)dt.$$

If we let $t = -y$, the limits of integration change from (0 to $-x$) to (0 to x). Thus,

$$f(-x) = \int_0^x g(-y)(-dy)$$

$$= -\int_0^x g(y)dy \quad (\because\ g \text{ is even})$$

$$= -f(x)$$

Thus, $f(x)$ is odd. ■

Example 35

Find a (reasonable) lower bound and an upper bound for the integral

$$I = \int_0^1 \frac{1}{\sqrt{1+x^6}} dx$$

Solution: By this question, we mean that we need to find two values m and M between which I lies, *i.e.*,

$$m < I < M.$$

Technically, any number less than I, say –(1 billion), is a valid lower bound for I, and any number greater than I, say (1 billion), would be a valid upper bound. But how useful would it be to state the (trivial) fact that I lies between (– 1 billion) and (1 billion)? Not much. We want ‘tight’ bounds, *i.e.*, narrow ranges in which I could lie. Thus, $M - m$ should be as small as possible so

that we have an accurate idea about the approximate value of I. Let us obtain an upper bound first. We have,

$$1+x^6 > 1 \quad \text{for} \quad x \in (0,1)$$

$$\Rightarrow \quad \sqrt{1+x^6} > 1 \quad \text{for} \quad x \in (0,1)$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^6}} < 1 \quad \text{for} \quad x \in (0,1)$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{1+x^6}}dx < \int_0^1 1\cdot dx$$

$$\Rightarrow \quad I < 1$$

Thus we now know that I is less than 1. Let us obtain a 'good' lower bound now. Since $x \in (0,1)$, $x^6 < x$. Thus,

$$1+x^6 < 1+x < (1+x)^2 \quad \text{for} \quad x \in (0,1)$$

$$\Rightarrow \quad \sqrt{1+x^6} < 1+x \quad \text{for} \quad x \in (0,1)$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{1+x^6}}dx > \int_0^1 \frac{1}{1+x}dx$$

$$\Rightarrow \quad I > \ln 2$$

Thus, we now have a fair idea about the approximate value of I:

$$\ln 2 < I < 1$$

Try to obtain tighter bounds for yourself. ■

Example 36

Determine a positive integer $n \le 5$ such that

$$\int_0^1 e^x (x-1)^n dx = 16 - 6e$$

Solution: Since n will not turn to be a very large integer, one might be tempted to try out various values of n in the given relation, starting from 1 onwards, and see which one fits. This trial-and-error approach might quickly give a result in this particular example, but what would we do it n was possibly larger? The generally followed approach in such examples, where the integral can be characterised by a positive integer n (called the order of the integral), is to express it in terms of a lower order integral. If we denote the nth order integral by I_n, we should try to express I_n in terms of I_k where $k < n$. Such a relation is called a recursive relation. We can then simply use this relation repeatedly on any order and obtain the integral of the next order (instead of every time repeating the calculation of integration again). We will use this approach for the current example. Let

$$I_n = \int_0^1 e^x (x-1)^n dx$$

To simplify I_n, we let $x-1=t \Rightarrow dx = dt$, and the limits become –1 to 0. Thus,

$$I_n = \int_{-1}^{0} e^{t+1} t^n dt = e\int_{-1}^{0} e^t \cdot t^n \; dt$$

We now use integration by parts to solve this integral, taking t^n as the first function:

$$\begin{aligned} I_n &= e\left\{ t^n \cdot e^t \Big|_{-1}^{0} - n\int_{-1}^{0} (t^{n-1} \cdot \int e^t dt)dt \right\} \\ &= e\left\{ \frac{(-1)^{n+1}}{e} - n\int_{-1}^{0} t^{n-1} \cdot e^t dt \right\} \\ &= (-1)^{n+1} - nI_{n-1} \end{aligned} \tag{1}$$

We have thus established the relation between I_n and I_{n-1} in (1). Now observe that I_0 can be evaluated easily:

$$I_0 = e\int_{-1}^{0} e^t dt \; = e-1$$

Using (1) repeatedly, we can now obtain all the higher order integrals:

$$\begin{aligned} I_1 &= (-1)^{1+1} - 1 \cdot I_0 = 2-e \\ I_2 &= (-1)^{2+1} - 2 \cdot I_1 = -5+2e \\ I_3 &= (-1)^{3+1} - 3 \cdot I_2 = 16-6e \end{aligned}$$

$n = 3$ is therefore the positive integer we had set out to determine. Notice the power of the recursive relation that we obtained in (1). Using that relation, it was just a matter of minor calculations to successively determine I_1, I_2 and I_3 from I_0. Without (1), we would have to apply integration by parts *everytime*, had we used the trial-and-error approach. ■

Example 37

Evaluate $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

Solution: Let $f_n(x) = \int \sin^n x \, dx$

$$\begin{aligned} \Rightarrow \quad f_n(x)dx &= \int \sin^n x \, dx \\ &= \int \underset{\substack{\downarrow \\ \text{Ist} \\ \text{function}}}{\sin^{n-1} x} \cdot \underset{\substack{\downarrow \\ \text{IInd} \\ \text{function}}}{\sin x} \, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cdot \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx \\ &= -\sin^{n-1} x \cos x + (n-1) f_{n-2}(x) - (n-1) f_n(x) \\ \Rightarrow \quad nf_n(x) &= -\sin^{n-1} x \cos x + (n-1) f_{n-2}(x) \end{aligned}$$

$$\Rightarrow \quad f_n(x) = \frac{-\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) f_{n-2}(x)$$

$$\Rightarrow \quad I_n = f_n(x)\Big|_0^{\pi/2} = 0 + \left(\frac{n-1}{n}\right) I_{n-2}$$

Thus, our recursive relation is

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

We now use this repeatedly:

$$\begin{aligned} I_n &= \left(\frac{n-1}{n}\right) I_{n-2} \\ &= \frac{(n-1)(n-3)}{n(n-2)} I_{n-4} \\ &\quad \vdots \\ &= \left\{ \begin{matrix} \dfrac{(n-1)(n-3)...(2)}{n(n-2)...(3)} I_1 \quad (\text{if } n \text{ is odd}) \\ \text{or} \\ \dfrac{(n-1)(n-3)...(1)}{n(n-2)...(2)} I_0 \quad (\text{if } n \text{ is even}) \end{matrix} \right\} \end{aligned} \qquad (1)$$

Now, $I_1 = \int_0^{\pi/2} \sin x \, dx = 1$ and $I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$. Thus, I_n is now obtainable from (1). Notice that $J_n = \int_0^{\pi/2} \cos^n x \, dx$ will be the same as I_n (why?). ■

Example 38

Prove that if $k \in \mathbb{Z}^+$

$$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos(2k-1)x]$$

Hence or otherwise, prove that

$$\int_0^{\pi/2} \sin 2kx \cot x dx = \pi/2$$

Solution: Let us first directly try to prove the second part using the technique of recursion. Let

$$I_k = \int_0^{\pi/2} \sin 2kx \cot x \, dx$$

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \sin 2x \cot x \, dx \quad = 2\int_0^{\pi/2} \cos^2 x \, dx$$

$$= 2\int_0^{\pi/2} \left[\frac{1+\cos 2x}{2}\right] dx \quad = \frac{\pi}{2}$$

Also,
$$\begin{aligned} I_{k+1}-I_k &= \int_0^{\pi/2}(\sin(2k+2)x-\sin 2kx)\cot x\,dx \\ &= 2\int_0^{\pi/2}\cos(2k+1)x\sin x\cdot\cot x\,dx \quad = \int_0^{\pi/2}\left(2\cos x\cdot\cos(2k+1)x\right)dx \\ &= \int_0^{\pi/2}(\cos(2k+2)x+\cos 2kx)dx \quad = \left.\frac{\sin(2k+2)x}{2k+2}\right|_0^{\pi/2}-\left.\frac{\sin 2kx}{2k}\right|_0^{\pi/2} \\ &= 0-0=0 \end{aligned}$$

Thus,

$$I_k=\frac{\pi}{2}\text{ for all }k\in\mathbb{Z}^+$$

Now we'll use the first result mentioned in the question to prove the second part. The proof of the first result is simple:

$$\begin{aligned} &2\sin x[\cos x+\cos 3x+\cdots+\cos(2k-1)x] \\ &\quad=\sin 2x+(\sin 4x-\sin 2x)+\cdots+(\sin 2kx-\sin(2k-2)x) \\ &\quad=\sin 2kx \end{aligned}$$

Thus, the stated assertion is valid. Now,

$$\begin{aligned} I &= \int_0^{\pi/2}\sin 2kx\cot x\,dx \\ &= 2\int_0^{\pi/2}\cos x[\cos x+\cos 3x+\ldots\cos(2k-1)x]dx \quad \left\{\begin{matrix}\text{Using the}\\ \text{first result}\end{matrix}\right\} \\ &= \int_0^{\pi/2}\{1+\cos 2x+(\cos 4x-\cos 2x)+\ldots+\cos 2kx-\cos(2k-2)x\}dx \\ &= \int_0^{\pi/2}(1+\cos 2kx)dx \\ &= \frac{\pi}{2} \quad \{\cos 2kx\text{ integrates to }0\} \end{aligned}$$

■

Example 39

Evaluate these limits:

(a) $L_1=\lim\limits_{x\to 0}\dfrac{\int_0^x(1-\cos 2x)\,dx}{x\int_0^x\tan x\,dx}$ (b) $L_2=\lim\limits_{x\to 0}\dfrac{\left(\int_0^x e^x dx\right)^2}{\int_0^{x^2}e^{x^2}dx}$

Solution: Observe that both these limits are of the form $\frac{0}{0}$, and therefore, we can use the LH rule. To differentiate the integrals, we use the technique to differentiate under the integral sign:

(a) $$L_1 = \lim_{x\to 0} \frac{1-\cos 2x}{x\tan x + \int_0^x \tan x\, dx} \quad \left(\text{still of the form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\sin 2x}{x\sec^2 x + \tan x + \tan x} \qquad = \lim_{x\to 0} \frac{4\sin x\cos x}{x\sec^2 x + 2\tan x}$$

$$= \lim_{x\to 0} \frac{4\sin x\cos^3 x}{x + 2\sin x\cos x} \qquad = \lim_{x\to 0} \frac{4\cos^3 x}{\left(\frac{x}{\sin x}\right) + 2\cos x}$$

$$= \frac{4}{1+2} \qquad = \frac{4}{3}.$$

(b) $$L_2 = \lim_{x\to 0} \frac{\cancel{2}\left(\int_0^x e^x dx\right) e^x}{\cancel{2}x \cdot e^{x^4}} \quad \left(\text{still of the form } \tfrac{0}{0}\right)$$

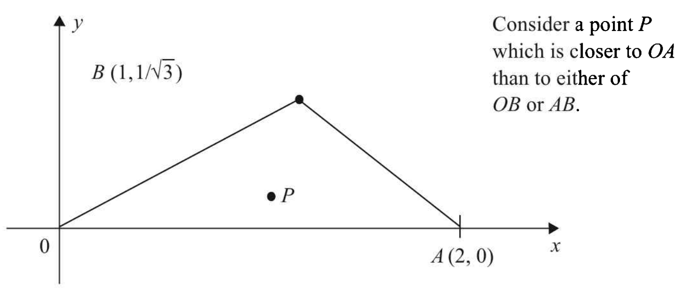
$$= \lim_{x\to 0}\left(\frac{e^x}{e^{x^4}}\right) \cdot \lim_{x\to 0}\left(\frac{\int_0^x e^x dx}{x}\right) \quad \left\{\begin{array}{l}\text{The second limit is of} \\ \text{the form } \tfrac{0}{0}\end{array}\right\}$$

$$= 1 \cdot \lim_{x\to 0} \frac{e^x}{1} = 1$$ ■

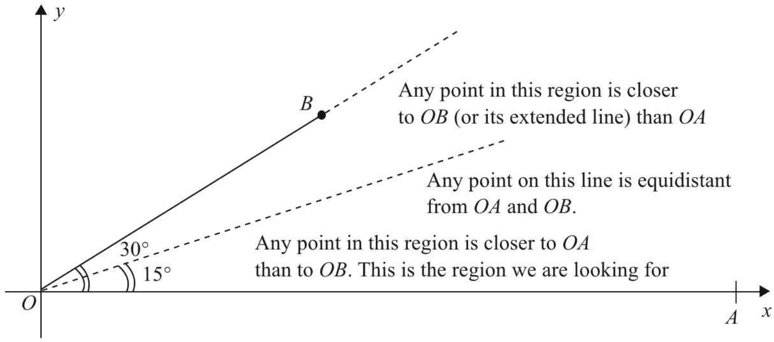
Example 40

Let O, (0, 0), $A(2, 0)$ and $B(1, 1/\sqrt{3})$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min\{d(P, OB), d(P, AB)\}$, where d denotes the distance from the point to the corresponding line. Sketch the area R and find its area.

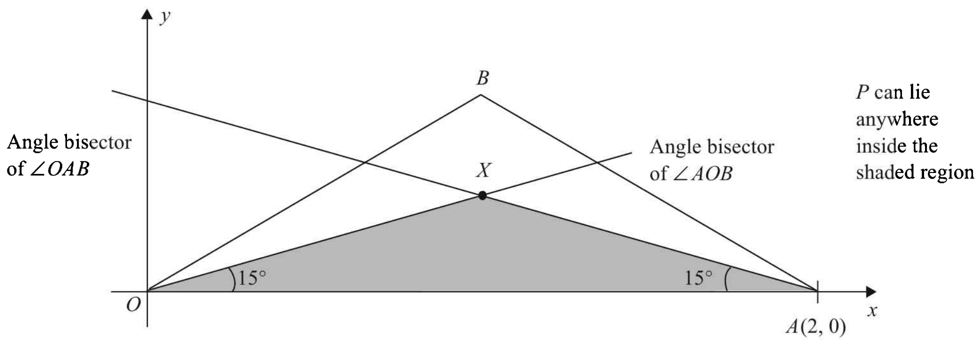
Solution: This question is more about correctly plotting the relevant region rather than calculating area. In fact, we need not even use definite integration here since we'll see that the region R is bounded by straight lines. Consider the triangle OAB as described in the question:



We need to find out the region in which P can possibly lie. Suppose that P was equidistant from OA and OB. Then P would lie on the angle bisector of $\angle AOB$. Since P is actually closer to OA, it lies 'below' the angle bisector, as shown in the next figure:



Since P also satisfies the constraint that it lies closer to OA than to AB, P will lie below the angle bisector of $\angle OAB$ too. Thus, P lies in the following region:



It is a matter of simple geometry now to evaluate the required area which is the area of ΔOAX. Verify that this is $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ sq units. ■

Example 41

A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at P is $a(y-1)+(x-1)=0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Hence obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

Solution: The slope of the given normal is obvious from the expression:

$$\frac{y-1}{x-1} = -\frac{1}{a} \Rightarrow m_N = -\frac{1}{a} \Rightarrow m_T = a \Rightarrow \left.\left(\frac{dy}{dx}\right)\right|_P = a$$

It is given that $\frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = ky$. Since,

$$\left.\frac{dy}{dx}\right|_{(1,1)} = a \Rightarrow k = a$$

Thus,

$$\frac{dy}{dx} = ay \Rightarrow \frac{dy}{y} = adx$$

$\Rightarrow \ln y = ax + \ln C$ (We took the constant of integration as $\ln C$ instead of C so that the final expression for y is simpler.)

$$\Rightarrow y = Ce^{ax}$$

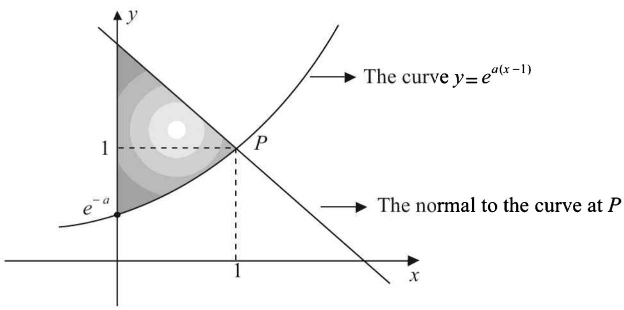
Since this curve passes through (1, 1),

$$1 = Ce^{a\cdot 1} \quad \Rightarrow C = e^{-a}$$

Thus, the equation of the curve is

$$y = e^{a(x-1)}$$

Let us now proceed to evaluate the bounded area, which is sketched below:



The equation of the normal has already been provided:

$$y = 1 + \frac{1}{a}(1-x)$$

Thus, the required area is:

$$A = \int_0^1 \left\{ \left(1 + \frac{1}{a}(1-x)\right) - e^{a(x-1)} \right\} dx = \left(x + \frac{x}{a} - \frac{x^2}{2a} - \frac{e^{a(x-1)}}{a} \right)\Bigg|_0^1$$

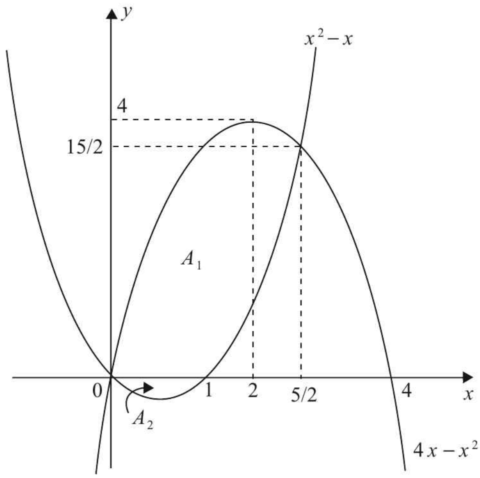
$$= \left(1 + \frac{1}{a} - \frac{1}{2a} - \frac{1}{a}\right) - \left(-\frac{e^{-a}}{a}\right) = \left(1 - \frac{1}{2a} + \frac{1}{ae^a}\right) \text{sq units}$$

■

Example 42

In what ratio does the x-axis divide the area of the region bounded by the parabola $y = 4x - x^2$ and $y = x^2 - x$?

Solution: The two parabolas and the region they bound have been sketched below:



We need to evaluate $\frac{A_1}{A_2}$

The intersection of the curves can be evaluated by equating the equation of the curves, *i.e,*.

$4x - x^2 = x^2 - x$

The total area $A_1 + A_2$ can be evaluated as the area of the region bounded between the two curves:

$$A_1 + A_2 = \int_0^{5/2} \left\{(4x - x^2) - (x^2 - x)\right\} dx$$

$$= \int_0^{5/2} (5x - 2x^2)\, dx = \left(\frac{5x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_0^{5/2}$$

$$= \frac{125}{8} - \frac{125}{12} = \frac{125}{24} \text{sq units}$$

The area A_2 is

$$A_2 = \left|\int_0^1 (x^2 - x)\, dx\right| = \left|\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1\right| = \frac{1}{6} \text{sq units}$$

The area A_1 is therefore

$$A_1 = \frac{125}{24} - \frac{1}{6} = \frac{121}{24} \text{sq units}$$

Thus, the required ratio is

$$\frac{A_1}{A_2} = \frac{121}{4}$$

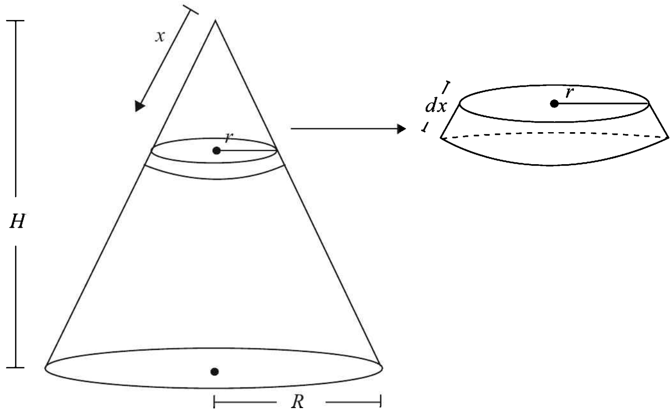
■

Example 43

Find the lateral surface area and the volume of a cone of height H and base radius R.

Solution: This question is an easy application question of definite integration. Let us evaluate the surface area first.

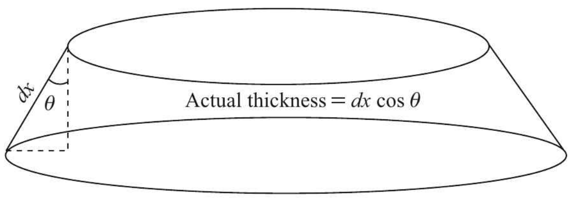
Take a variable x as depicted along side. We analyse an infinitesmally small strip of width dx now.



Note from the geometry of the cone that $\frac{r}{x} = \frac{R}{L}$, where L is the slant height of the cone, *i.e.*, $L = \sqrt{R^2 + H^2}$. The surface area of this elemental strip is $2\pi r\, dx$. We have:

$$\text{Lateral surface area } A = \int_0^L 2\pi r dx = \frac{2\pi R}{L}\int_0^L x dx = \pi RL.$$

To calculate the volume, we again use the elemental strip analysed in the figure below. The thickness of this strip is not dx but $dx \cos\theta$, where $\cos\theta = \frac{H}{L}$.



The volume of this strip is therefore $\pi r^2 dx \cos\theta$. Thus, the total volume is

$$V = \int_0^L \pi r^2 dx \cos\theta = \pi \cdot \frac{R^2}{L^2} \cdot \cos\theta \int_0^L x^2 dx = \frac{\pi R^2 H}{L^3} \cdot \frac{L^3}{3} = \frac{1}{3}\pi R^2 H$$

Integration

PART-C: Advanced Problems

P1. What is the value of $\int \frac{1}{\sin(x-a)\cos(x-b)}dx$?

(A) $\dfrac{1}{\sin(a-b)}\ln\left|\dfrac{\sin(x-a)}{\cos(x-b)}\right|+C$ (C) $\dfrac{1}{\sin(a-b)}\ln\left|\dfrac{\cos(x-a)}{\sin(x-b)}\right|+C$

(B) $\dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\sin(x-a)}{\cos(x-b)}\right|+C$ (D) $\dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\cos(x-a)}{\sin(x-b)}\right|+C$

P2. What is the value of $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$?

(A) $-2\ln\left(\dfrac{1+\sqrt{1-x}}{\sqrt{x}}\right)+2\cos^{-1}\sqrt{x}+C$ (C) $-2\ln\left(\dfrac{1-\sqrt{1-x}}{\sqrt{x}}\right)+\cos^{-1}\sqrt{x}+C$

(B) $-\ln\left(\dfrac{1+\sqrt{1-x}}{\sqrt{x}}\right)+\cos^{-1}\sqrt{x}+C$ (D) $-2\ln\left(\dfrac{1-\sqrt{1-x}}{\sqrt{x}}\right)-2\cos^{-1}\sqrt{x}+C$

P3. What is the value of $\int \frac{1}{\sqrt{\sin^3 x\sin(x+a)}}dx$?

(A) $\dfrac{-2\sqrt{\sin a+\tan x\cos a}}{\sin a}+C$ (C) $\dfrac{-2\sqrt{\sin a+\cot x\cos a}}{\sin a}+C$

(B) $\dfrac{-2\sqrt{\cos a+\tan x\sin a}}{\sin a}+C$ (D) $\dfrac{-2\sqrt{\cos a+\cot x\sin a}}{\sin a}+C$

P4. For $x>0$, let $f(x)=\int\limits_1^x \dfrac{\ln t}{1+t}\,dt.$

(a) Find the function $f(x)+f(\frac{1}{x})$.

(b) The value of $f(e)+f\left(\frac{1}{e}\right)$ is equal to

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{3}{4}$ (D) 1

P5. Let $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. What is the value of $J-I$?

(A) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+C$ (B) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}+e^x+1}{e^{2x}-e^x+1}\right|+C$

(C) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}+e^x-1}{e^{2x}-e^x+1}\right|+C$ (D) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}+e^x+1}{e^{2x}-e^x-1}\right|+C$

P6. The integral $I = \int \frac{x^n - 1}{x^{n+1}\sqrt{2x^{2n} - 2x^n + 1}}\, dx$ is equal to

(A) $\dfrac{\sqrt{2 - x^{-n} + 2x^{-2n}}}{n} + C$ (C) $\dfrac{\sqrt{2 - x^{-n} - 2x^{-2n}}}{n} + C$

(B) $\dfrac{\sqrt{2 - 2x^{-n} + x^{-2n}}}{n} + C$ (D) $\dfrac{\sqrt{2 + x^{-n} - 2x^{-2n}}}{2n} + C$

P7. Let $f(x) = \frac{\sin x}{(1 + \sin^n x)^{\frac{1}{n}}}$, and $g(x) = \underbrace{f \circ f \circ f \circ \ldots \circ f(x)}_{n \text{ times}}$. What is the value of $\int \sin^{n-2} x \cos x g(x) dx$?

(A) $\dfrac{(1 + n\sin^{n-1} x)^{1+\frac{1}{n}}}{n^2 - n} + C$ (C) $\dfrac{(1 + n\sin^n x)^{1-\frac{1}{n}}}{n^2 - n} + C$

(B) $\dfrac{(1 - n\sin^{n-1} x)^{1+\frac{1}{n}}}{n^2 + n} + C$ (D) $\dfrac{(1 - n\sin^n x)^{1-\frac{1}{n}}}{n^2 + n} + C$

P8. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ *and* $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$, for $n = 1, 2, 3, \ldots$. Which of the following is / are true?

(A) $S_n < \dfrac{\pi}{3\sqrt{3}}$ (B) $S_n > \dfrac{\pi}{3\sqrt{3}}$ (C) $T_n < \dfrac{\pi}{3\sqrt{3}}$ (D) $T_n > \dfrac{\pi}{3\sqrt{3}}$

P9. Let $f(x)$ be a periodic function with period a, and $I = \int_0^a f(x) dx$. If

$$I_1 = \sum_{k=1}^{n} \int_{1+a}^{1+ka+a} f(kx)\, dx = \lambda I,$$

the value of λ is

(A) $\dfrac{n(n-1)}{2}$ (B) $\dfrac{n(n+1)}{2}$ (C) n^2 (D) $\dfrac{n^2 + 1}{2}$

P10. What is the value of $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$?

(A) $\dfrac{\pi^2}{4}$ (B) $\dfrac{\pi^2}{2}$ (C) π^2 (D) $2\pi^2$

P11. What is the value of $\int_0^{\pi/2} x^2 \text{cosec}^2 x\, dx$?

(A) $\dfrac{\pi}{2} \ln 2$ (B) $\pi \ln 2$ (C) $2\pi \ln 2$ (D) $\pi \ln 8$

P12. The value of $\int_0^{\pi/2} \frac{\sin^3 x}{(1 + \cos^2 x)\sqrt{1 + \cos^2 x + \cos^4 x}} dx$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{2\pi}{3}$

P13. Let $\lambda = (\frac{\pi}{4})^{1/3}$. The value of $\int_{-\lambda}^{\lambda} \frac{x^2}{(1 + \sin^2 x^3)(1 + e^{x^7})} dx$ is

(A) $\dfrac{\sqrt{2}}{6} \tan^{-1} \sqrt{2}$ (B) $\dfrac{2\sqrt{2}}{3} \tan^{-1} \sqrt{2}$ (C) $\dfrac{4\sqrt{2}}{3} \tan^{-1} \sqrt{2}$ (D) $\dfrac{8\sqrt{2}}{3} \tan^{-1} \sqrt{2}$

P14. What is the value of $\int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x} dx$?

(A) $\dfrac{\pi}{\cos\alpha}(\pi - \alpha)$ (B) $\dfrac{\pi}{\sin\alpha}(\pi - \alpha)$ (C) $\dfrac{2\pi}{\cos\alpha}(\pi - \alpha)$ (D) $\dfrac{2\pi}{\sin\alpha}(\pi - \alpha)$

P15. What is the least positive value of a for which the equation

$$\int_0^x (t^2 - 8t + 13)dt = x \sin \frac{a}{x}$$

has a solution?

(A) π (B) 2π (C) 3π (D) 4

P16. What is the solution for x : $\int_{\ln 2}^{x} \frac{1}{\sqrt{e^t - 1}} dt = \frac{\pi}{6}$?

(A) ln 3 (B) ln 4 (C) ln 5 (D) ln 6

P17. What is the value of $\int_0^{\pi/2} \frac{x \sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx$ (where a, $b > 0$)?

(A) $\dfrac{\pi}{2a^2 b(a+b)}$ (B) $\dfrac{\pi}{2ab^2(a+b)}$ (C) $\dfrac{\pi}{4a^2 b(a+b)}$ (D) $\dfrac{\pi}{4ab^2(a+b)}$

P18. What is the value of $I = \int_0^1 \ln(\sqrt{1-x} + \sqrt{1+x})dx$?

(A) $\ln\sqrt{2} + \dfrac{\pi}{2} - \dfrac{1}{4}$ (B) $\ln\sqrt{2} + \dfrac{\pi}{4} - \dfrac{1}{2}$ (C) $-\ln\sqrt{2} + \dfrac{\pi}{4} + \dfrac{1}{2}$ (D) $-\ln\sqrt{2} - \dfrac{\pi}{2} + 1$

P19. What is the value of $I = \int_0^1 \cot^{-1}(1 + x^2 - x)dx$?

(A) $\dfrac{\pi}{2} - \ln 2$ (B) $\dfrac{\pi}{2} + \ln 2$ (C) $\pi - \ln 2$ (D) $\pi + \ln 2$

P20. What is the value of $I = \int_0^\pi \frac{x \sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi} dx$?

(A) $\dfrac{2}{\pi^2}$ (B) $\dfrac{4}{\pi^2}$ (C) $\dfrac{8}{\pi^2}$ (D) $\dfrac{16}{\pi^2}$

P21. What is the value of $I = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$?

(A) π^2 (B) $2\pi^2$ (C) $3\pi^2$ (D) $4\pi^2$

P22. What is the value of $\int_0^\pi \frac{x^2 \sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi}$?

(A) $\dfrac{2}{\pi}$ (B) $\dfrac{4}{\pi}$ (C) $\dfrac{8}{\pi}$ (D) $\dfrac{16}{\pi}$

P23. What is the value of the ratio $\frac{\int_0^{\pi/2} f(\sin 2x)\sin x \, dx}{\int_0^{\pi/4} f(\cos 2x)\cos x \, dx}$?

(A) 1 (B) $\sqrt{2}$ (C) $\dfrac{1}{\sqrt{2}}$ (D) 2

P24. What is the value of the ratio $\frac{\int_0^\pi x^3 \log \sin x \, dx}{\int_0^\pi x^2 \log(\sqrt{2}\sin x) \, dx}$?

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{2\pi}{3}$ (C) $\dfrac{3\pi}{4}$ (D) $\dfrac{3\pi}{2}$

P25. What is the value of $\int_0^\infty \frac{dx}{(x + \sqrt{1+x^2})^n}$?

(A) $\dfrac{n}{2(n^2-1)}$ (B) $\dfrac{n}{n^2-1}$ (C) $\dfrac{2n}{n^2-1}$ (D) $\dfrac{4n}{n^2-1}$

P26. What is the value of $I = \int_{-1}^{+1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2} dx$?

(A) $2\sqrt{2} - \dfrac{\pi}{2} + 2$ (B) $4\sqrt{2} - \dfrac{\pi}{2} + 1$ (C) $4\sqrt{2} - \pi - 2$ (D) $4\sqrt{2} - \pi + 2$

P27. What is the value of the following integral?

$$\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx.$$

(A) $\frac{12}{5}\left(e\cos\frac{1}{2}-\frac{e}{2}\sin\frac{1}{2}+1\right)$ (B) $\frac{24}{5}\left(e\cos\frac{1}{2}+\frac{e}{2}\sin\frac{1}{2}-1\right)$

(C) $\frac{12}{5}\left(e\cos\frac{1}{2}+\frac{e}{4}\sin\frac{1}{2}-1\right)$ (D) $\frac{24}{5}\left(e\cos\frac{1}{2}+\frac{2e}{3}\sin\frac{1}{2}-1\right)$

P28. What is the value of $I=\int_0^{\infty} e^{-ax}\,x^n dx$?

(A) $\frac{(n-1)!}{a^n}$ (B) $\frac{n!}{a^n}$ (C) $\frac{(n-1)!}{a^{n+1}}$ (D) $\frac{n!}{a^{n+1}}$

P29. What is the value of $I=\int_0^1 \frac{x^{\alpha}-x^{\beta}}{\ln x}dx$?

(A) $\ln\left(\frac{\alpha}{\beta}\right)$ (B) $2\ln\left(\frac{\alpha}{\beta}\right)$ (C) $\ln\left(\frac{\alpha+1}{\beta+1}\right)$ (D) $\ln\left(\frac{\alpha-1}{\beta-1}\right)$

P30. What is the value of $I=\int_0^{\infty} \frac{\tan^{-1}ax}{x(1+x^2)}dx\ (a\ge 0)$?

(A) $\frac{\pi}{4}\ln(1+a)$ (B) $\frac{\pi}{3}\ln(1+a)$ (C) $\frac{\pi}{2}\ln(1+a)$ (D) $\pi\ln(1+a)$

P31. (a) What is the value of $\lim_{n\to\infty}((1+(\frac{1}{n})^4)(1+(\frac{2}{n})^4)^{1/2}(1+(\frac{3}{n})^4)^{1/3}(1+(\frac{4}{n})^4)^{1/4}...2^{1/n})$? You can use the following series:

$$1-\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}-...\infty=\frac{\pi^2}{12}$$

(A) $e^{\frac{\pi^2}{12}}$ (B) $e^{\frac{\pi^2}{24}}$ (C) $e^{\frac{\pi^2}{36}}$ (D) $e^{\frac{\pi^2}{48}}$

(b) What is the value of $\lim_{n\to\infty}(\Pi_{r=1}^{n}(1+\frac{r^2}{n^2})^{2r/n^2})$?

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$ (C) $\frac{6}{e}$ (D) $\frac{8}{e}$

P32 Consider the region R defined as

$$R=\left\{(x,y)\,\middle|\, \begin{matrix} x^2+y^2\le 100 \\ \sin(x+y)\ge 0\end{matrix}\right\}$$

What is the area of R?

(A) 25π (B) 40π (C) 50π (D) 64π (E) None of these

P33. A curve $y=f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the co-ordinate axis. If the curve divides the area formed by these lines and the coordinate axes in the ratio $m{:}n$, what is the equation of the curve? In the following options, k represents an arbitrary constant.

(A) $y^2=kx^{m/n}$ (B) $y=kx^{m/n}$ (C) $y=kx^{2m/n}$ (D) $y=kx^{1+m/n}$

P34. What is the positive value of b for which the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) None of these

P35. Let $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$. What is the area of the region bounded by the curve $y = f(x)$, $x-$axis, $x = 0$ and $x = 1$?

(A) $\dfrac{58}{27}$ (B) $\dfrac{61}{27}$ (C) $\dfrac{67}{27}$ (D) $\dfrac{71}{27}$

SUBJECTIVE TYPE EXAMPLES

P36. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$.

P37. Evaluate the integral

$$I = \int \sin^{-1}(x+\sqrt{x^2+a^2})dx.$$

P38. Evaluate the integral

$$I = \int \left(3x + \frac{6x^2 \sin^2(\frac{x}{2})}{x - \sin x}\right) \frac{(x-\sin x)^{3/2}}{\sqrt{x}} dx$$

P39. Evaluate $\int \frac{(x+1)}{x(1+xe^x)^2} dx$.

P40. Evaluate the indefinite integral

$$\int \cos 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$$

P41. Evaluate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)} dx$.

P42. Evaluate $\int (x-2)\sqrt{\frac{1+x}{1-x}}\, dx$.

P43. Evaluate $\int \frac{1}{x^6-a^6+a^2x^2(x^2-a^2)} dx$.

P44. Evaluate $\int \frac{x^2}{(x\sin x+\cos x)^2} dx$.

P45. For any natural number m, evaluate

$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m} dx, x > 0$$

P46. (a) Evaluate $\int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x})dx$.

(b) Evaluate $\int_0^{\pi/2}(\sqrt{\sin x}+\sqrt{\cos x})^{-4} dx$.

P47. Let $I = \int_0^a \frac{\cos x}{\sin x+\cos x} dx$ and $J = \int_0^a \frac{\sin x}{\sin x+\cos x} dx$, $a \in [0, \frac{3\pi}{4})$.

(a) By considering $I + J$ and $I - J$, find the value of I.

(b) Evaluate $\int_0^{\pi/2} \frac{\cos x}{a\sin x+b\cos x} dx$.

(c) Evaluate $\int_0^{\pi/2} \frac{\cos x+4}{3\sin x+4\cos x+25} dx$.

P48. Find a simplified expression for the function $f(x)$ given by

$$f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt, \quad x \in \left(0, \frac{\pi}{2}\right)$$

P49. f, g, h are continuous functions on $[0, a]$ such that $f(a-x) = f(x)$, $g(a-x) = -g(x)$ and $3h(x) - 4h(a-x) = 5$. Find $\int_0^a f(x)g(x)h(x)\, dx$.

P50. Let $f_e(x)$ and $f_o(x)$ be the even and odd parts of a function $f(x)$. Show that

$$\int_{-\infty}^{+\infty} f^2(x)dx = \int_{-\infty}^{+\infty} f_e^2(x)dx + \int_{-\infty}^{+\infty} f_o^2(x)dx$$

P51. Let $f(x)$ be a continuous function on $\mathbb{R}/\{0\}$ such that $\int_0^a f(x)\,dx$ exists for all $a > 0$. If $g(x) = \int_x^a \frac{f(t)}{t}\,dt$, find $\int_0^a (f(x) - g(x))\,dx$.

P52. Evaluate $\int_0^1 (\{2x\} - 1)(\{3x\} - 1)\,dx$, where $\{\cdot\}$ denotes the fractional part.

P53. Let $f(x)$ be a continuous function defined on $\mathbb{R}^+$, where $\int_0^x f(t)\,dt \to \infty$ as $x \to \infty$. Find the values of m for which $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$.

P54. Let $x > 0$. Evaluate $\int_0^x [t]dt$.

P55. Evaluate $\int_0^{\pi/2} \sin 2kx \cot x\,dx$.

P56. Evaluate

(a) $\int_{-1}^{1} \tan^{-1}(e^z)\,dz$ (b) $\int_{-1}^{1} \cot^{-1}(x^3 + \sqrt{1 + x^6})\,dx$

P57. What is the value of the expression $\frac{(5050)\int_0^1 (1-x^{50})^{100}\,dx}{\int_0^1 (1-x^{50})^{101}\,dx}$?

P58. Let $S_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx$ and $V_n = \int_0^{\pi/2} (\frac{\sin nx}{\sin x})^2\,dx$.

(a) Show that $V_{n+1} - V_n = S_{n+1} = S_n = \ldots = S_1 = \frac{\pi}{2}$.

(b) Hence find V_n.

P59. Evaluate the integral $\int_a^b \frac{f(\frac{x}{a}) - f(\frac{b}{x})}{x}\,dx$.

P60. Evaluate $\int_0^{\pi/2} \sin x \log(\sin x) dx$.

P61. Use $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$ to evaluate $I = \int_0^{\infty} t^{-1/2} e^{-2010\,(t+t^{-1})} dt$.

P62. Let I_m be an integral defined as

$$I_m = \int_0^{2\pi} \cos x \cos 2x \cos 3x \ldots \cos mx\,dx$$

m can take any value from the set $\{ 1, 2, 3, \ldots, 100 \}$ with equal likelihood. Find the probability that $I_m \neq 0$.

P63. Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

P64. Let $u_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$. It turns out that $u_1, u_2, u_3, \ldots$ form a GP. Find

(a) the ratio of the GP.

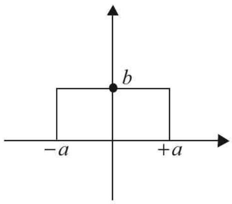
(b) u_n

P65. Let $I = \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\, dx$. Is I greater than $\frac{\pi}{6}$?

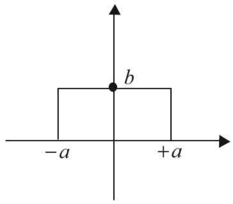
P66. Define the *convolution function* $I(t)$ of two given function $f(x)$ and $h(x)$ as

$$I(t) = \int_{-\infty}^{+\infty} f(x)h(t-x)\, dx$$

Determine the convolution function in the following case.



and



P67. Consider a function $f(x)$. We define a measure $R(t)$ of $f(x)$, called the *auto-correlation* and given by

$$R(t) = \int_{-\infty}^{+\infty} f(x)f(x+t)\, dx$$

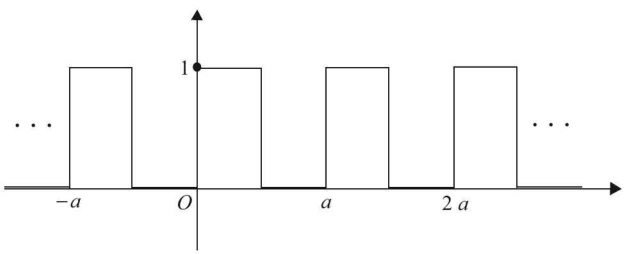
Therefore, $R(t)$ is related to $f(x)$ and '$f(x)$ shifted by t units'. Find $R(t)$ for

$$f(x) = \begin{Bmatrix} e^{-x}, & x \geq 0 \\ 0\ , & x < 0 \end{Bmatrix}$$

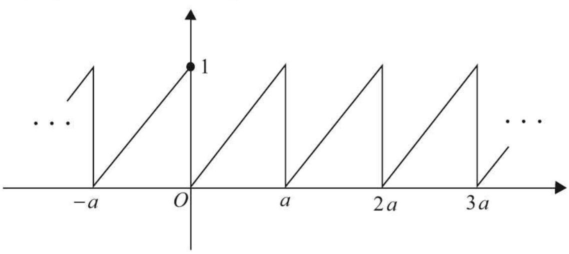
P68. Fourier analysis tells us that any periodic function $f(x)$ with period T (or frequency $\omega = \frac{2\pi}{T}$) can be written as a sum of sinusoids whose frequencies are multiples of ω.

$$f(x) = \sum_0^{\infty} a_k \cos k\omega x + \sum_0^{\infty} b_k \sin k\omega x$$

(a)Find a_k and b_k when $f(x)$ is the following function:



(b)Find a_k and b_k when $f(x)$ is the following function:



(c) Show that

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \sum_{k=0}^{\infty} (|a_k|^2 + |b_k|^2)$$

P69. Let f be defined on $\mathbb{R}^+ \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}^+$,

$$\int_1^{xy} f(t)dt = y\int_1^{x} f(t)dt + x\int_1^{y} f(t)dt$$

If $f(1) = 1$, find the function $f(x)$.

P70. (a) Evaluate $I = \int_0^\infty \frac{e^{-ax}\sin mx}{x}\, dx$.

(b) Hence evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$.

P71. Let $x_1, x_2, ..., x_n$ be distinct real numbers such that

$$\int \frac{x^{n-1}}{(x-x_1)(x-x_2)...(x-x_n)}\, dx = a_1 \log(x-x_1) + a_2 \log(x-x_2) + ... + a_n \log(x-x_n) + C$$

What is the value of $a_1 + a_2 + ... + a_n$?

P72. Find $f(x)$ such that $\int_0^x e^t f(t)\, dt = x + \int_x^1 \sin t f(t)\, dt$.

P73. Find a function $f:\mathbb{R}\to\mathbb{R}$ satisfying the following conditions:

(a) $f(x)$ is continuous on $[0,\infty)$.

(b) $f(x) > 0 \quad \forall\ x \in (0,\infty)$

(c) $\frac{1}{2}\int_0^x f^2(t)dt = \frac{1}{x}\left(\int_0^x f(t)dt\right)^2$

P74. Let $0 < \alpha < \beta$, and let

$$f(x) = 1 + \cos^2 x + \cos^4 x + \cos^6 x$$

If

$$\int_0^\alpha f(x)(ax^2+bx+c)\, dx = \int_0^\beta f(x)(ax^2+bx+c)\, dx = 0,$$

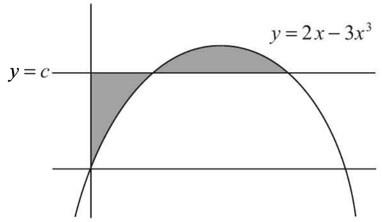
determine the location of the roots of the quadratic equation $ax^2 + bx + c = 0$.

P75. Find the area of the region $R = (x, y)$ such that

$$|x-1|-|y+1| \le 1$$
$$|x-1|+|y| \le 1$$

P76. Find the value of $[n]$ for which the area below the x-axis of the curve $y = x^n \log x$ is divided into two equal halves by the vertical line passing through the point of minimum of the curve.

P77.



Find c such that the two shaded regions have equal area.

(Figure not to scale.)

P78. Let $I_n = \int_0^\infty [ne^{-x}]\,dx$. Find the minimum value of n for which $[I_n]$ exceeds 2.

P79. Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

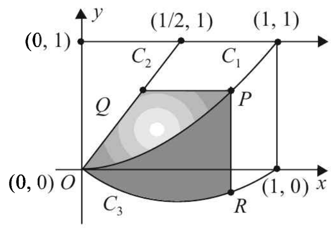
P80. Find the area bounded by the curves $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$, where $a, b > 0$.

P81. Find the area in the first quadrant bounded by $y^2 = 4ax$, $x^2 + y^2 = 2ax$ and $x - y = 2a$.

P82. A 2×3 rectangular plate has vertices at (0, 0), (2, 0), (0, 3) and (2, 3). It rotates 90° clockwise about the point (2, 0). It then rotates 90° clockwise about the point (5, 0), then 90° clockwise about (7, 0) and finally 90° clockwise about (10, 0).
(a) Plot the curve S traced by the point on the plate which was initially located at (1, 1).
(b) Find the area bounded between S and the x-axis.

P83. Let $b \ne 0$ and for $j = 0, 1, 2, \ldots, n$, let S_j be the area of the region bounded by the y-axis and the curve $xe^{ay} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$.

P84. Let C_1 and C_2 be the graphs of functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.



P85. Consider a square with vertices at (1, 1), (–1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.

P86. Let $f(x)$ be defined on [0, 1] such that $f(x) = \min |x - \frac{1}{2^n}|$ $n = 0, 1, 2, \ldots$. Find the area under $f(x)$.

P87. The parabola $y^2 = ax$ cuts the hyperbola $x^2 - y^2 = 2a^2$ at P and Q. The tangent at P to the hyperbola cuts the parabola again in R. Find the area of the curvilinear triangle PQR.

Integration

PART-D: Solutions to Advanced Problems

S1. The denominator is of the form sin P cos Q, where P and Q are variable, but notice an important fact: $P - Q$ is a constant. We use this fact as follows:

$$\int \frac{1}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-b)\cos(x-a)+\sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \{\cot(x-a)+\tan(x-b)\}dx$$

$$= \frac{1}{\cos(a-b)}\{\ln|\sin(x-a)| - \ln|\cos(x-b)|\} + C$$

$$= \frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$

The correct option is (B).

S2. A slight thought on the form of this expression will hint that a trigonometric substitution might help; recall that both $(1-\cos\theta)$ and $(1+\cos\theta)$ can be reduced to single terms. Therefore, we use the substitution:

$$\sqrt{x} = \cos\theta \quad \Rightarrow \quad x = \cos^2\theta \quad \Rightarrow \quad dx = -2\sin\theta\cos\theta d\theta$$

$$I = \int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\cdot\frac{1}{x}dx = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,\frac{1}{\cos^2\theta}\cdot(-2\sin\theta\cos\theta d\theta)$$

$$= -2\int\sqrt{\frac{2\sin^2(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})}}\tan\theta d\theta = -2\int\tan\left(\frac{\theta}{2}\right).\tan\theta d\theta = -4\int\frac{\sin^2(\frac{\theta}{2})}{\cos\theta}d\theta$$

$$= -2\int\frac{1-\cos\theta}{\cos\theta}d\theta = -2\int(\sec\theta - 1)\ d\theta = -2\{\ln|\sec\theta + \tan\theta| - \theta\} + C$$

$$= -2\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + 2\theta + C = -2\ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right) + 2\cos^{-1}\sqrt{x} + C$$

Therefore, the correct option is (A).

S3. We have to modify the given expression so that some substitution is possible. The first step that we could take is expand $\sin(x+a)$:

$$I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\,dx \quad = \int \frac{1}{\sqrt{\sin^3 x(\sin x \cos a + \cos x \sin a)}}\,dx$$

To proceed further, notice that there is a $\sin^3 x$ term in the denominator. What we now do is take out a common factor of $\sin x$ from the (inner) brackets so that $\sin^3 x$ becomes $\sin^4 x$:

$$I = \int \frac{1}{\sqrt{\sin^4 x(\cos a + \cot x \sin a)}}\,dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{\cos a + \cot x \sin a}}\,dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos a + \cot x \sin a}}\,dx$$

This is a form in which the numerator can be expressed as the derivative of the expression in the denominator. Thus, we make the following substitution:

$$\cos a + \cot x \sin a = t \quad \Rightarrow \quad -\sin a \operatorname{cosec}^2 x\,dx = dt$$

$$\Rightarrow \quad I = -\frac{1}{\sin a}\int \frac{dt}{\sqrt{t}} \quad = \frac{-2}{\sin a}\sqrt{t} + C \quad = \frac{-2\sqrt{\cos a + \cot x \sin a}}{\sin a} + C$$

The correct option is (D).

S4. Observe carefully the form of the function $f(x)$. It is in the form of an integral (of another function), with the lower limit being fixed and the upper limit being the variable x. As x varies, $f(x)$ will correspondingly vary. One approach that you might contemplate to solve this question is evaluate the anti-derivative $g(t)$ of $\frac{\ln t}{1+t}$ and then evaluate $g(x) - g(1)$ which will become $f(x)$. However, this will become unnecessarily cumbersome (try it!). We can, instead, proceed as follows:

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t}\,dt + \int_1^{1/x} \frac{\ln t}{1+t}\,dt$$

$$= \quad I_1 + I_2$$

Notice that the limits of integration of I_1 and I_2 are different. If they were the same, we could have added I_1 and I_2 easily. So we try to make them the same: in I_2, if we let $t = \frac{1}{y}$, and t varies from 1 to $\frac{1}{x}$, y will vary from 1 to x. This substitution will therefore make the limits of integration of I_2 the same as those of I_1:

$$t = \frac{1}{y} \text{ so that } dt = -\frac{1}{y^2}\,dy$$

$$t = 1 \quad \Rightarrow \quad y = 1$$

$$t = \frac{1}{x} \quad \Rightarrow \quad y = x$$

$$I_2 = \int_1^{1/x} \frac{\ln t}{1+t}\,dt \;= -\int_1^x \frac{\ln(\frac{1}{y})}{1+(\frac{1}{y})}\,\frac{1}{y^2}\,dy \;= \int_1^x \frac{\ln y}{y(1+y)}\,dy$$

I_1 and I_2 can now be easily added:

$$\Rightarrow \quad I_1 + I_2 = \int_1^x \left\{ \frac{\ln t}{1+t} + \frac{\ln t}{t(1+t)} \right\} dt = \int_1^x \frac{(1+t)\ln t}{t(1+t)}\, dt = \int_1^x \frac{\ln t}{t}\, dt$$

The final expression shows how simplified $I_1 + I_2$ has become. We let $\ln t = z \Rightarrow \frac{1}{t}\, dt = dz$, and the limits of integration become 0 to $\ln x$. We have

$$I_1 + I_2 = \int_0^{\ln x} z\, dz = \frac{1}{2}(\ln x)^2$$

Thus,

$$f(e) + f\left(\frac{1}{e}\right) = \left(\frac{\ln e}{2}\right)^2 = \frac{1}{2}$$

The correct option for part-(b) is (B).

S5. $J - I = \int \frac{e^{3x} - e^x}{1+e^{2x}+e^{4x}}\, dx$. The substitution $e^x = y$ reduces this to $\int \frac{y^2-1}{1+y^2+y^4}\, dy$, which is straightforward to evaluate, yielding

$$J - I = \frac{1}{2}\log\left| \frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right| + C$$

The reader is urged to verify the answer. The correct option is (A).

S6. We have

$$I = \int \frac{x^n - 1}{x^{n+1}\sqrt{2x^{2n} - 2x^n + 1}}\, dx = \int \frac{x^{-n-1} - x^{-2n-1}}{\sqrt{2 - 2x^{-n} + x^{-2n}}}\, dx$$

Now we use the substitution $2 - 2x^{-n} + x^{-2n} = t^2$:

$$\Rightarrow \quad 2n(x^{-n-1} - x^{-2n-1})dx = 2t\, dt$$

$$\Rightarrow \quad I = \frac{1}{n}\int dt = \frac{t}{n} + C = \frac{\sqrt{2 - 2x^{-n} + x^{-2n}}}{n} + C$$

The correct option is (B).

S7. By carrying out the composition of $f(x)$ n times with itself, we have

$$g(x) = \frac{\sin x}{(1 + n\sin^n x)^{1/n}}\, dx$$

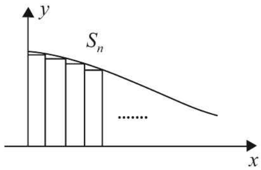
$$\Rightarrow \quad \int \sin^{n-2} x \cos x\, g(x) dx = \int \frac{\sin^{n-1} \cos x}{(1+n\sin^n x)^{1/n}}\, dx$$

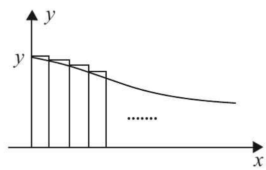
$$= \frac{1}{n^2}\int \frac{dt}{t^{1/n}}, \quad t = 1 + n\sin^n x$$

$$= \frac{t^{1-\frac{1}{n}}}{n^2 - n} + C = \frac{\left(1 + n\sin^n x\right)^{1-\frac{1}{n}}}{n^2 - n} + C$$

The correct option is (C).

S8. Note that as $n \to \infty, S_n$ and T_n represent the area under the curve $y = \frac{1}{1+x+x^2}$ from $x = 0$ to $x = 1$, but there is a minor difference, which you should observe from the following diagram:





This difference arises because in S_n, the index of summation k goes from 1 to n, while in T_n, the index k goes from 0 to $n-1$. This means that both S_n and T_n approach $\int_0^1 \frac{1}{1+x+x^2}\,dx = \frac{\pi}{3\sqrt{3}}$ as $n \to \infty$, but S_n approaches this value from the lower side, while T_n approaches this value from the higher side. Thus,

$$S_n < \frac{\pi}{3\sqrt{3}} \text{ and } T_n > \frac{\pi}{3\sqrt{3}}$$

Therefore, options (A) and (D) are correct.

S9. $f(kx)$ is periodic with period $\frac{a}{k}$. Also,

$$\int_{1+a}^{1+ka+a} f(kx)\,dx = \int_0^{ka} f(kx)\,dx = \int_0^{k^2a} f(t)\frac{dt}{k} = k\int_0^{a} f(t)dt \ = kI$$

$$\Rightarrow \quad I_1 = \sum_{k=1}^{n} kI = \frac{n(n+1)I}{2}$$

$$\Rightarrow \quad \lambda = \frac{n(n+1)}{2}$$

The correct option is (B).

S10. The integral can be split into two integrals:

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$$

The first of these integrals consists of an odd function, so its value is 0. The second integral is even. Thus,

$$I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 4\pi \times \frac{\pi}{2} = 2\pi^2 \quad \Rightarrow \quad I = \pi^2$$

The correct option is (C).

S11. We have

$$I = \int_0^{\pi/2} x^2 \operatorname{cosec}^2 x\,dx = -x^2 \cot x\Big|_0^{\pi/2} + 2\int_0^{\pi/2} x\cot x\,dx$$

$$= 2\left[x\log(\sin x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \log(\sin x)\, dx \right] = -2\int_0^{\pi/2} \log(\sin x)\, dx$$

$$= \pi \ln 2 \text{ (verify)}$$

The correct option is (B).

S12. Using $\cos x \to t$, so that $\sin x dx = -dt$, we have

$$I = \int_1^0 \frac{(1-t^2)(-dt)}{(1+t^2)\sqrt{1+t^2+t^4}} = \int_0^1 \frac{\left(\frac{1}{t^2}-1\right)}{(t+\frac{1}{t})\sqrt{(t+\frac{1}{t})^2-1}}\, dt$$

Using $t + \dfrac{1}{t} \to z$, so that $\left(1 - \dfrac{1}{t^2}\right)dt = dz$, we have

$$I = \int_\infty^2 \frac{-dz}{z\sqrt{z^2-1}} = \sec^{-1} z\Big|_2^\infty = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

The correct option is (A).

S13. Substituting $x \to (\lambda + (-\lambda) - x)$ or $x \to -x$, we have

$$I = \int_{-\lambda}^{\lambda} \frac{x^2}{(1+\sin^2 x^3)(1+e^{x^7})}\, dx = \int_{-\lambda}^{\lambda} \frac{x^2}{(1+\sin^2 x^3)(1+e^{-x^7})}\, dx = \int_{-\lambda}^{\lambda} \frac{x^2 e^{x^7}}{(1+\sin^2 x^3)(1+e^{x^7})}\, dx$$

$$\Rightarrow \quad 2I = \int_{-\lambda}^{\lambda} \frac{x^2}{(1+\sin^2 x^3)}\, dx$$

Substituting $x^3 \to t$, so that $x^2 dx = \dfrac{dt}{3}$, we have

$$I = \frac{1}{2}\cdot\frac{1}{3}\int_{-\lambda^3}^{\lambda^3} \frac{dt}{1+\sin^2 t} = \frac{1}{6}\int_{-\pi/4}^{\pi/4} \frac{\sec^2 t}{1+2\tan^2 t}\, dt$$

Substituting $\tan t \to y$, we have

$$I = \frac{1}{6}\int_{-1}^{1} \frac{dy}{1+2y^2} = \frac{1}{12}(\sqrt{2}\tan^{-1}\sqrt{2}y)\Big|_{-1}^{1} = \frac{\sqrt{2}}{6}\tan^{-1}\sqrt{2}$$

The correct option is (A).

S14. We have

$$I = \int_0^\pi \frac{x}{1-\cos\alpha\sin x}\, dx = \int_0^\pi \frac{\pi - x}{1-\cos\alpha\sin x}\, dx \Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \frac{1}{1-\cos\alpha\sin x}\, dx$$

We use $\tan\frac{x}{2}=t$ so that $dx=\frac{2dt}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$, and the integral I becomes:

$$I=\frac{\pi}{2}\int_0^{\infty}\frac{1}{1-\cos\alpha\left(\frac{2t}{1+t^2}\right)}\cdot\frac{2}{1+t^2}dt=\frac{\pi}{2}\int_0^{\infty}\frac{2}{(t-\cos\alpha)^2+\sin^2\alpha}dt$$

$$=\frac{\pi}{\sin\alpha}\left\{\tan^{-1}\left(\frac{t-\cos\alpha}{\sin\alpha}\right)\right\}\Bigg|_0^{\infty}=\frac{\pi}{\sin\alpha}(\pi-\alpha)$$

The correct option is (B).

S15. We note that

$$\int_0^x(t^2-8t+13)\,dx=\left(\frac{t^3}{3}-4t^2+13t\right)\Bigg|_0^x=\frac{x^3}{3}-4x^2+13x=\frac{x}{3}(x^2-12x+39)$$

$$=\frac{x}{3}((x-6)^2+3)\geq\frac{x}{3}\cdot 3=x$$

The right hand side has the maximum possible value as x (for any given value of x). Therefore, the only way the equality can hold is that

$$x-6=0\quad\text{and}\quad\sin\frac{a}{x}=1$$

$$\Rightarrow\quad x=6\quad\text{and}\quad\frac{a}{x}=\frac{\pi}{2}$$

$$\Rightarrow\quad a=3\pi$$

The correct option is (C).

S16. Using $e^t-1\to y^2$, so that $e^t dt=2ydy$, we have

$$dt=\frac{2ydy}{e^t}=\frac{2ydy}{1+y^2},$$

and the limits of integration become 1 to $\sqrt{e^x-1}$. The integral equation now becomes

$$\int_1^{\sqrt{e^x-1}}\frac{1}{y}\cdot\frac{2ydy}{1+y^2}=\frac{\pi}{6}$$

$$\Rightarrow\quad 2\tan^{-1}y\Big|_1^{\sqrt{e^x-1}}=\frac{\pi}{6}\quad\Rightarrow\quad\tan^{-1}\sqrt{e^x-1}-\frac{\pi}{4}=\frac{\pi}{12}$$

$$\Rightarrow\quad\tan^{-1}\sqrt{e^x-1}=\frac{\pi}{3}\quad\Rightarrow\quad\sqrt{e^x-1}=\tan\frac{\pi}{3}=\sqrt{3}$$

$$\Rightarrow\quad e^x=4\quad\Rightarrow\quad x=\ln 4$$

The correct option is (B).

S17. Consider only the function $f(x)=\frac{\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}$. Observe that $f(x)$ can be easily integrated. Using the substitution $\sin^2 x=t$, we get the integral as (verify):

$$\int f(x)dx=\frac{-1}{2(b^2-a^2)(a^2\cos^2 x+b^2\sin^2 x)}$$

Now, since we know the integral of $f(x)$, we use integration by parts on the original integral, which is nothing but:

$$I = \int_0^{\pi/2} x\, f(x) dx$$

Applying integration by parts, we obtain:

$$I = \left(x \int f(x)dx\right)\Big|_0^{\pi/2} - \int_0^{\pi/2} \left(\int f(x)\, dx\right) dx$$

$$= \frac{1}{2(b^2-a^2)}\left\{\frac{-x}{a^2\cos^2 x + b^2 \sin^2 x}\Bigg|_0^{\frac{\pi}{2}} + \int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x} dx\right\}$$

$$= \frac{1}{2(b^2-a^2)}\left\{\frac{-\pi}{2b^2} + \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x} dx\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{l}\text{To evaluate this integral, use the substitution } \tan x = t;\\ \text{this integral reduces to } \int_0^{\infty} \frac{dt}{a^2+b^2t^2} \text{ which is } \frac{\pi}{2ab}.\end{array}\right\}$$

$$= \frac{1}{2(b^2-a^2)}\left\{\frac{\pi}{2ab} - \frac{\pi}{2b^2}\right\} = \frac{\pi}{4ab^2(a+b)}1$$

The correct option is (D).

S18. The expression for I contains both $(1-x)$ and $(1+x)$; the substitution $x = \cos 2\theta$ could do the job.

$$x = \cos 2\theta \quad \Rightarrow \quad dx = -2\sin 2\theta d\theta \quad \text{and} \quad \begin{array}{l}\text{When } x = 0,\ \theta = \pi/4\\ \text{When } x = 1,\ \theta = 0\end{array}$$

Thus, I gets modified to

$$I = -2\int_{\pi/4}^{0} \ln\left(\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta}\right)\sin 2\theta d\theta$$

$$= 2\int_0^{\pi/4} \ln\{\sqrt{2}(\sin\theta + \cos\theta)\}\sin 2\theta d\theta$$

$$= 2\int_0^{\pi/4} (\ln\sqrt{2})\sin 2\theta d\theta + 2\int_0^{\pi/4} \ln(\underset{\text{First Function}}{\sin\theta + \cos\theta})\underset{\text{Second Function}}{\sin 2\theta}\, d\theta$$

$$= -\ln\sqrt{2}(\cos 2\theta)\Big|_0^{\pi/4} + 2\left\{\frac{-\ln(\sin\theta+\cos\theta)\cos 2\theta}{2}\Bigg|_0^{\pi/4} + \int_0^{\pi/4} \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\cdot\frac{\cos 2\theta}{2} d\theta\right\}$$

$$= \ln\sqrt{2} + 2\left\{0 + \frac{1}{2}\int_0^{\pi/4} (\cos\theta - \sin\theta)^2 d\theta\right\} = \ln\sqrt{2} + \int_0^{\pi/4} (\cos\theta - \sin\theta)^2 d\theta$$

$$= \ln\sqrt{2} + \int_0^{\pi/4} (1 - \sin 2\theta) d\theta = \ln\sqrt{2} + \frac{\pi}{4} + \frac{\cos 2\theta}{2}\Bigg|_0^{\pi/4} = \ln\sqrt{2} + \frac{\pi}{4} - \frac{1}{2}$$

The correct option is (B).

S19. In this example, instead of the direct application of any property, a non-trivial manipulation is first required to simplify I:

$$I = \int_0^1 \cot^{-1}(1+x^2-x)dx = \int_0^1 \tan^{-1}\left(\frac{1}{1+x(x-1)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)dx \text{ \{We wrote the numerator '1' as '}x-(x-1)\text{'\}}$$

$$= \int_0^1 \{\tan^{-1}x - \tan^{-1}(x-1)\}dx \quad \left\{\because \tan^{-1}\left(\frac{A-B}{1+AB}\right) = \tan^{-1}A - \tan^{-1}B\right\}$$

$$= \int_0^1 \tan^{-1}x\, dx - \int_0^1 \tan^{-1}(x-1)dx \tag{1}$$

Observe how simple the integral I has now become. A further simplification is possible by modifying the second integral in (1):

$$\int_0^1 \tan^{-1}(x-1) = \int_0^1 \tan^{-1}((1-x)-1)dx = -\int_0^1 \tan^{-1}x\, dx$$

Thus,

$$I = 2\int_0^1 \tan^{-1}x\, dx$$

We can now proceed to evaluate I using integration by parts:

$$I = 2\left\{x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right\}$$

$$= 2\left\{\frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{dt}{t}\right\} \quad \begin{pmatrix}\text{We used the substitution}\\ 1+x^2 = t\end{pmatrix}$$

$$= \frac{\pi}{2} - (\ln t)\Big|_1^2 = \frac{\pi}{2} - \ln 2$$

The correct option is (A).

S20. We use the substitution $x \to \pi - x$ in the function to be integrated:

$$I = \int_0^\pi \frac{(\pi-x)\sin(2\pi-2x)\sin(\frac{\pi}{2}\cos(\pi-x))}{2(\pi-x)-\pi}dx$$

$$= \int_0^\pi \frac{(\pi-x)\cdot(-\sin 2x)\cdot(-\sin(\frac{\pi}{2}\cos x))}{\pi-2x}dx$$

$$= \int_0^\pi \frac{(x-\pi)\sin 2x\sin(\frac{\pi}{2}\cos x)}{2x-\pi}dx$$

Adding the original and the modified forms of I, we obtain:

$$2I = \int_0^\pi \frac{(2x-\pi)\sin 2x\sin(\frac{\pi}{2}\cos x)}{2x-\pi}dx = \int_0^\pi \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx$$

We use the substitution $\frac{\pi}{2}\cos x = t$ in this simplified integral. The limits change from $(0 \text{ to } \pi)$ to $(\frac{\pi}{2} \text{ to } \frac{-\pi}{2})$:

$$2I = 2\int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx = \frac{-8}{\pi^2}\int_{\pi/2}^{-\pi/2} t \sin t \, dt$$

$$= \frac{8}{\pi^2}\int_{-\pi/2}^{\pi/2} t \sin t \, dt = \frac{8}{\pi^2}\left\{-t\cos t\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos t \, dt\right\}$$

$$= \frac{8}{\pi^2}\{0+2\} = \frac{16}{\pi^2}$$

$$\Rightarrow \quad I = \frac{8}{\pi^2}$$

The correct option is (C).

S21. We make the substitution $x \to 2\pi - x$ in the expression for I and then add the original and modified forms of I to obtain:

$$2I = 2\pi\int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

The function to be integrated is symmetric about π (verify this by the substitution $x \to 2\pi - x$). Therefore:

$$2I = 4\pi\int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

The function to be integrated is still symmetric, about $x = \frac{\pi}{2}$; verify this by the substitution $x \to \pi - x$. Therefore:

$$2I = 8\pi\int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \tag{1}$$

Now, we make the substitution $x \to \frac{\pi}{2} - x$. Thus,

$$2I = 8\pi\int_0^{\pi/2} \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \tag{2}$$

Adding (1) and (2), we obtain

$$4I = 8\pi\int_0^{\pi/2} \frac{\sin^{2n} x + \cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = 8\pi\int_0^{\pi/2} dx = 4\pi^2$$

$$\Rightarrow \quad I = \pi^2$$

The correct option is (A).

S22. Denote the given integral by I. Using $x \to \pi - x$, we have

$$I = -\int_0^{\pi} \frac{(\pi - x)^2 \sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi} dx$$

Adding this to the original integral yields

$$2I = \int_0^{\pi} \frac{(2\pi x - \pi^2)\sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi} dx$$

The part corresponding to 'π^2' is zero (why?), so we are left with

$$\frac{I}{\pi} = \int_0^{\pi} \frac{x \sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi} dx = -\int_0^{\pi} \frac{(\pi - x)\sin 2x \sin(\frac{\pi}{2}\cos x)}{2x - \pi} dx$$

Adding again yields

$$\frac{2I}{\pi} = \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx = 2\int_0^{\pi/2} \sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

$$= \frac{16}{\pi^2} \text{ (using substitution-verify this step)}$$

$$\Rightarrow \quad I = \frac{8}{\pi}$$

The correct option is (C).

S23. We have

$$I = \int_0^{\pi/2} f(\sin 2x)\sin x dx = \int_0^{\pi/2} f(\sin(\pi - 2x))\sin\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x) dx$$

Now, we observe that substituting $x \to \frac{\pi}{2} - x$ in the integral above leaves it unchanged. Thus,

$$2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\, dx$$

$$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right) dx$$

Finally, substituting $x \to \frac{\pi}{4} - x$ gives

$$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$$

The required ratio is thus $\sqrt{2}$. The correct option is (B).

S24. We start by solving the integral I_N in the numerator:

$$I_N = \int_0^{\pi} x^3 \log\sin x dx = \int_0^{\pi} (\pi - x)^3 \log\sin x dx$$

$$\Rightarrow \quad 2I_N = \int_0^{\pi} (\pi^3 - 3\pi^2 x + 3\pi x^2) \log \sin x dx$$

$$= \pi^3 \int_0^{\pi} \log \sin x dx - 3\pi^2 \int_0^{\pi} x \log \sin x dx + 3\pi \int_0^{\pi} x^2 \log \sin x dx$$

$$= \pi^3 I_1 - 3\pi^2 I_2 + 3\pi I_3 \tag{1}$$

Now,

$$I_1 = \int_0^{\pi} \log \sin x dx = 2\int_0^{\pi/2} \log \sin x dx = 2\int_0^{\pi/2} \log \cos x dx$$

$$\Rightarrow \quad 2I_1 = 2\left\{ \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx \right\} = 2\int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= 2\int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = 2\int_0^{\pi/2} \log \sin 2x dx - 2\int_0^{\pi/2} \log 2 dx$$

In the first integral above, we use $2x \to y$ so that

$$\int_0^{\pi/2} \log \sin 2x dx = \int_0^{\pi} \log \sin y \left(\frac{dy}{2}\right) = \frac{I_1}{2}$$

Thus,

$$2I_1 = I_1 - 2\int_0^{\pi/2} \log 2 dx = I_1 - \pi \log 2$$

$$\Rightarrow \quad I_1 = -\pi \log 2$$

We now evaluate I_2:

$$I_2 = \int_0^{\pi} x \log \sin x dx = \int_0^{\pi} (\pi - x) \log \sin x dx$$

$$\Rightarrow \quad 2I_2 = \pi \int_0^{\pi} \log \sin x dx = -\pi^2 \log 2$$

$$\Rightarrow \quad I_2 = -\frac{\pi^2}{2} \log 2$$

Using the values of I_1 and I_2 in (1), we have

$$2I_N = -\pi^4 \log 2 + \frac{3\pi^4}{2} \log 2 + 3\pi I_3$$

$$= \frac{\pi^4}{2} \log 2 + 3\pi \int_0^{\pi} x^2 \log \sin x dx$$

$$\Rightarrow \quad I_N = \frac{\pi^4}{4} \log 2 + \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sin x dx \tag{2}$$

We note that $\int_0^{\pi} x^2 dx = \frac{\pi^3}{3}$ and so $\frac{\pi^4}{4} \log 2$ be written as

$$\frac{\pi^4}{4}\log 2 = \frac{3\pi}{2}\cdot\left(\frac{\pi^3}{3}\right)\left(\frac{\log 2}{2}\right)$$

$$= \frac{3\pi}{2}\int_0^{\pi} x^2 \log\sqrt{2}\, dx$$

Using this in (2), we have

$$I_N = \frac{3\pi}{2}\int_0^{\pi} x^2 \log\sqrt{2}\, dx + \frac{3\pi}{2}\int_0^{\pi} x^2 \log\sin x dx$$

$$= \frac{3\pi}{2}\int_0^{\pi} x^2 \log(\sqrt{2}\sin x)\, dx$$

$$= \frac{3\pi}{2} I_D \quad (I_D \text{ is the integral in the denominator of the original expression})$$

$$\Rightarrow \frac{I_N}{I_D} = \frac{3\pi}{2}$$

The correct option is (D).

S25. Let $x + \sqrt{1+x^2} = t \Rightarrow \sqrt{1+x^2} = t - x$

$$\Rightarrow \quad x = \frac{t^2-1}{2t} = \frac{t}{2} - \frac{1}{2t} \quad \Rightarrow \quad dx = \left(\frac{1}{2} + \frac{1}{2t^2}\right)dt$$

$$\Rightarrow \quad I = \int_1^{\infty} \frac{1}{t^n}\cdot\left(\frac{1}{2} + \frac{1}{2t^2}\right)dt = \frac{1}{2}\left(\frac{t^{-n+1}}{-n+1} + \frac{t^{-n-1}}{-n-1}\right)\Bigg|_1^{\infty} = \frac{n}{n^2-1}$$

The correct option is (B).

S26. Use the substitution $x = \sin 4t$:

$$\Rightarrow \quad I = \int_{-\pi/8}^{\pi/8} \frac{1}{\sqrt{1+\sin 4t} + \sqrt{1-\sin 4t} + 2}(4\cos 4t\, dt)$$

$$= \int_{-\pi/8}^{\pi/8} \frac{4\cos 4t}{2(1+\cos 2t)}\, dt \quad \text{(why?)}$$

$$= 2\int_0^{\pi/8} \frac{\cos 4t}{\cos^2 t}\, dt$$

Now,

$$\cos 4t = 2\cos^2 2t - 1 = 2(2\cos^2 t - 1)^2 - 1 = 8\cos^4 t - 8\cos^2 t + 1$$

$$\Rightarrow \quad I = 2\int_0^{\pi/8} (8\cos^2 t - 8 + \sec^2 t)\, dt = 8\int_0^{\pi/8} (1+\cos 2t)\, dt - 16t\Big|_0^{\pi/8} + 2\tan t\Big|_0^{\pi/8}$$

$$= \pi + 4\sin 2t\Big|_0^{\pi/8} - 2\pi + 2\tan\left(\frac{\pi}{8}\right) = 2\sqrt{2} - \pi + 2(\sqrt{2}-1) = 4\sqrt{2} - \pi - 2$$

The correct option is (C).

S27. Putting $x \to \pi - x$, we have

$$I = \int_0^{\pi} e^{|\cos x|}\left\{-2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right\}\sin x dx$$

Adding with the original expression, we have

$$2I = 6\int_0^{\pi} e^{|\cos x|}\cos\left(\frac{1}{2}\cos x\right)\sin x dx \quad \Rightarrow \quad I = 6\int_0^{\pi/2} e^{|\cos x|}\cos\left(\frac{1}{2}\cos x\right)\sin x dx \tag{1}$$

Since $\cos x > 0$ for $x \in (0, \frac{\pi}{2})$, we can remove the modulus sign on $\cos x$ in (1). Substituting $\cos x \to 2t$, so that $\sin x dx = -2dt$, we have

$$I = -12\int_{1/2}^{0} e^{2t}\cos t\, dt$$

The integral $\int e^{2t}\cos t\, dt$ is a standard integral and can easily be evaluated:

$$\int e^{2t}\cos t\, dt = \frac{e^{2t}}{5}(2\cos t + \sin t)$$

Thus,

$$I = -\frac{12}{5}e^{2t}(2\cos t + \sin t)\Big|_{1/2}^{0} = \frac{24}{5}\left(e\cos\frac{1}{2} + \frac{e}{2}\sin\frac{1}{2} - 1\right)$$

The correct option is (B).

S28. We have

$$I_n = \int_0^{\infty} e^{-ax}x^n dx = \frac{x^n e^{-ax}}{-a}\Big|_0^{\infty} + \frac{n}{a}\int_0^{\infty} x^{n-1}\cdot e^{-ax} dx = \frac{n}{a}I_{n-1}$$

Also, since $I_0 = \frac{1}{a}$, $I_n = \frac{n!}{a^{n+1}}$. The correct option is (D).

S29. Let us treat I as a function of α. Therefore,

$$I(\alpha) = \int_0^1 \frac{x^{\alpha} - x^{\beta}}{\ln x} dx$$

Notice that

$$I(\beta) = \int_0^1 \frac{x^{\beta} - x^{\beta}}{\ln x} dx = 0$$

Now, differentiating under the integral sign, we have

$$\frac{dI(\alpha)}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^{\alpha} - x^{\beta}}{\ln x}\right) dx = \int_0^1 \frac{x^{\alpha}\ln x}{\ln x} dx = \frac{1}{\alpha + 1}$$

Thus, $$dI(\alpha) = \frac{d\alpha}{\alpha + 1}$$

$$\Rightarrow \quad I(\alpha) = \ln(\alpha + 1) + C$$

Using $I(\beta) = 0$ above, we obtain:

$$I(\beta) = 0 = \ln(\beta+1) + C$$

$$\Rightarrow \quad C = 0 - \ln(\beta+1) = -\ln(\beta+1)$$

Thus,

$$I(\alpha) = \ln(\alpha+1) - \ln(\beta+1) = \ln\left(\frac{\alpha+1}{\beta+1}\right)$$

The correct option is (C).

S30. Treat the given integral as a function of a:

$$I(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx \quad \Rightarrow \quad \frac{d\,I(a)}{da} = \int_0^\infty \frac{1}{(1+a^2x^2)(1+x^2)}\,dx$$

$$= \frac{\pi}{2(1+a)} \text{ (verify!)}$$

$$\Rightarrow \quad I(a) = \frac{\pi}{2}\ln(1+a)$$

The reader can verify that the constant of integration in the final step will be 0. The correct option is (C).

S31. (a) Representing the expression inside the limit by L, we have

$$\log L = \sum_{r=1}^{n} \frac{1}{r}\log\left(1+\left(\frac{r}{n}\right)^4\right) = \sum_{r=1}^{n}\frac{1}{n}\left(\frac{\log\left(1+\left(\frac{r}{n}\right)^4\right)}{\left(\frac{r}{n}\right)}\right)$$

As $n \to \infty$, this summation becomes a definite integral:

$$\log L = \int_0^1 \frac{\log(1+x^4)}{x}\,dx = \frac{1}{4}\int_0^1 \frac{\log(1+x^4)}{x^4}\,4x^3dx$$

Substituting $x^4 = t$, so that $4x^3dx = dt$, we have

$$\log L = \frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\,dt$$

Using the expansion series for $\log(1+t)$ and simplifying, we will obtain:

$$\log L = \frac{\pi^2}{48}$$

$$\Rightarrow \quad L = e^{\frac{\pi^2}{48}}$$

The correct option is (D).

(b) Once again, representing the expression inside the limit as L, we have

$$\log L = \sum_{r=1}^{n} \frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right)$$

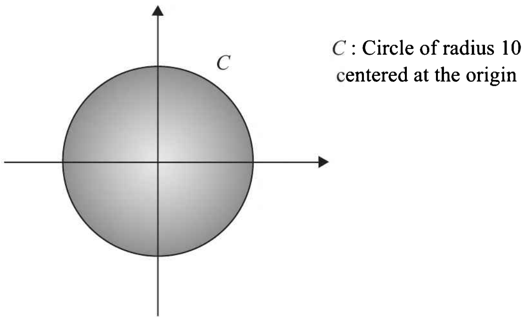
As $n \to \infty$, we have

$$\log L = \int_0^1 2x \log(1+x^2)dx = \int_1^2 \log t \, dt = \log \frac{4}{e}$$

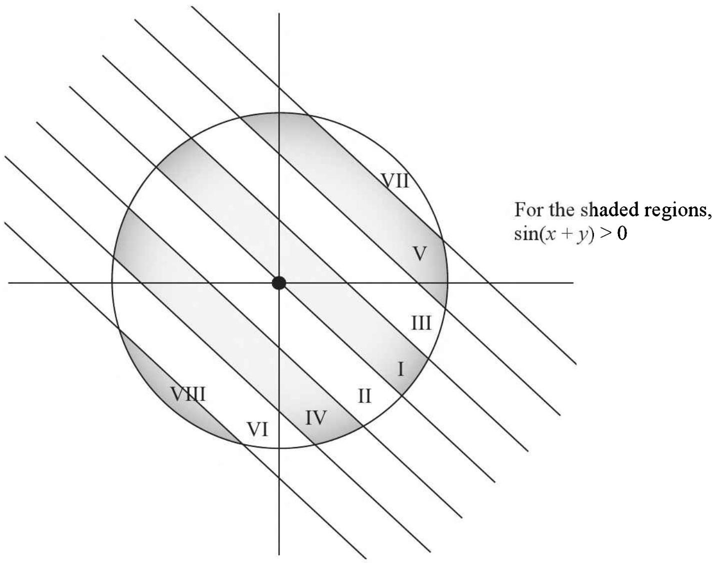
$$\Rightarrow \quad L = \frac{4}{e}$$

The correct option is (B).

S32. The expression $x^2 + y^2 \le 100$ represents the following region:

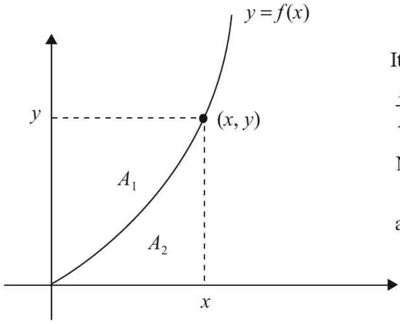


We now consider the region represented by $\sin(x+y) \ge 0$. Observe that $\sin(x+y) = 0$ when $x + y = k\pi, k \in \mathbb{Z}$. As k is varied over $\mathbb{Z}$, $x + y = k\pi$ will generate a set of parallel lines. In the regions between these parallel lines, the sign of $\sin(x+y)$ will alternate in successive regions. When superimposed on the circle $x^2 + y^2 = 100$, this is what we'll get:



Also, since $\sin(-x-y) = -\sin(x+y)$, the regions containing points (x, y) such that $\sin(x+y) > 0$ are symmetric with respect to the origin to the regions containing the points (x, y) such that $\sin(x+y) < 0$. This means, for example, that the areas of the regions I and II are the same, the areas of the regions III and IV are the same, and so on. Thus, the area of the shaded region is *exactly* half of the area of the total circle, that is, 50π. The correct option is (C).Note that no integration was required to calculate this area.

S33. The situation described in the question is graphically depicted below:



It is given that

$$\frac{A_1}{A_2} = \frac{m}{n}$$

Note that $A_1 + A_2 = xy$

and $A_2 = \int_0^x y\,dx$

From the given constraint, we have

$$\frac{xy - \int_0^x y\,dx}{\int_0^x y\,dx} = \frac{m}{n} \quad \Rightarrow \quad \int_0^x y\,dx = \frac{n}{m+n}(xy)$$

Differentiating both sides with respect to x, we obtain

$$y = \frac{n}{m+n}\left(\frac{xdy}{dx} + y\right) \quad \Rightarrow \quad my = nx\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{y} = \frac{m}{n}\frac{dx}{x}$$

$$\Rightarrow \quad \ln y = \frac{m}{n}\ln x + C = \frac{m}{n}\ln x + \ln k \text{ where } k \text{ is a constant}$$

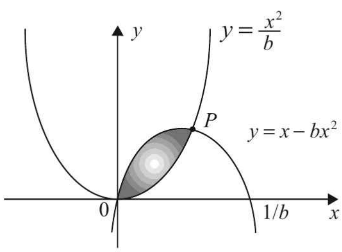
$$\Rightarrow \quad y = kx^{m/n}$$

The correct option is (B). Note that this curve passes through the origin.

S34. The first parabola has its zeroes given by

$$y = x - bx^2 = 0 \quad \Rightarrow \quad x = 0, \frac{1}{b}$$

This is a downwards facing parabola. The other parabola has its vertex at the origin.



The point of intersection P is given by:

$$\frac{x^2}{b} = x - bx^2$$

$$\Rightarrow \left\{ \begin{aligned} x &= \frac{b}{1+b^2} \\ y &= \frac{b}{(1+b^2)^2} \end{aligned} \right\}$$

The figure above tells us how to calculate the coordinates of the intersection point P:

$$P \equiv \left(\frac{b}{1+b^2}, \frac{b}{(1+b^2)^2}\right)$$

Now, the bounded area A is given by:

$$A = \int_0^{\frac{b}{1+b^2}} \left(x - bx^2 - \frac{x^2}{b} \right) dx = \left\{ \frac{x^2}{2} - \frac{1}{3}\left(b + \frac{1}{b} \right) x^3 \right\} \Bigg|_0^{\frac{b}{1+b^2}}$$

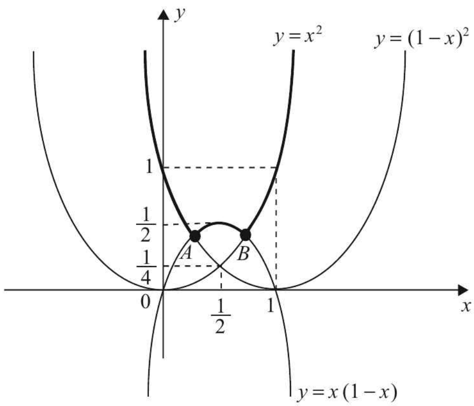
$$= \frac{b^2}{2(1+b^2)^2} - \frac{1}{3} \frac{b^2+1}{b} \cdot \frac{b^3}{(1+b^2)^3} = \frac{b^2}{6(1+b^2)^2}$$

We now maximize A (or more conveniently, and equivalently, $\sqrt{6A}$):

$$A_1 = \sqrt{6A} = \frac{b}{1+b^2} \quad \Rightarrow \quad \frac{dA_1}{db} = \frac{(1+b^2) - 2b^2}{(1+b^2)} = \frac{1-b^2}{1+b^2}$$

A_1 is maximum when $\frac{dA_1}{db} = 0$ *i.e.*, $b = 1$ (b is > 0 so it cannot be –1). Thus, the bounded area is maximum when $b = 1$. The correct option is (A).

S35. The technique to plot the curve for $f(x)$ will be as follows: we plot all the three curves $x^2, (1-x)^2$ and $2x(1-x)$ on the same axes, scan the x-axis from left to right, and at every point, pick out that graph which lies uppermost of all the three graphs. In the figure below, the heavyset curve is the curve for $f(x)$:



The intersection point A is given by $(1-x)^2 = 2x(1-x)$ $\Rightarrow x = \frac{1}{3}$

The intersection point B is given by $x^2 = 2x(1-x)$ $\Rightarrow x = \frac{2}{3}$

We can evaluate the required area, as is clear from the figure above, by dividing the integration interval [0, 1] into three sub-intervals:

$$A = \int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^{1} x^2 dx$$

$$= \frac{19}{18} + \frac{13}{81} + \frac{19}{81} \text{ (verify)}$$

$$= \frac{51}{81} = \frac{71}{27}$$

The correct option is (D).

SUBJECTIVE TYPE EXAMPLES

S36. We observe that $4x^2+8x+13=(2x+2)^2+9$, so the substitution $2x+2=3\tan\theta$ will work. This will reduce I to

$$I=\frac{3}{2}\int\theta\sec^2\theta d\theta=\frac{3}{2}(\theta\tan\theta-\int\tan\theta d\theta)$$

$$=\frac{3}{2}(\theta\tan\theta-\ln\sec\theta)+C,\text{where}\,\theta=\tan^{-1}\left(\frac{2x+2}{3}\right)$$

S37. We use the substitution $x+\sqrt{x^2+a^2}\to t$. This gives

$$\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)dx=dt$$

$$\Rightarrow\quad dx=\frac{\sqrt{x^2+a^2}}{t}dt$$

Also, we note that $\sqrt{x^2+a^2}-x=\frac{a^2}{t}$, and so

$$\sqrt{x^2+a^2}=\frac{1}{2}\left(t+\frac{a^2}{t}\right)=\frac{t^2+a^2}{2t}$$

$$\Rightarrow\quad dx=\left(\frac{t^2+a^2}{2t^2}\right)dt$$

In terms of t, I becomes

$$I=\frac{1}{2}\int(\sin^{-1}t)\left(1+\frac{a^2}{t^2}\right)dt$$

Applying integration by parts, we have

$$I=\frac{1}{2}\sin^{-1}t\left(t-\frac{a^2}{t}\right)-\frac{1}{2}\int\left(t-\frac{a^2}{t}\right)\frac{1}{\sqrt{1-t^2}}dt$$

$$=\frac{1}{2}\sin^{-1}t\left(t-\frac{a^2}{t}\right)-\frac{1}{2}\int\frac{t}{\sqrt{1-t^2}}dt+\frac{a^2}{2}\int\frac{dt}{t\sqrt{1-t^2}}$$

$$=\frac{1}{2}\sin^{-1}t\left(t-\frac{a^2}{t}\right)+\frac{1}{2}\sqrt{1-t^2}+\frac{a^2}{2}I_1$$

where

$$I_1=\int\frac{dt}{t\sqrt{1-t^2}}=-\int\frac{dy}{\sqrt{y^2-1}}\left(y\to\frac{1}{t}\right)$$

$$=-\ln|y+\sqrt{y^2-1}|+C_1$$

$$= \ln\left(\frac{t}{1+\sqrt{1-t^2}}\right) + C_1$$

$$\Rightarrow \quad I = \frac{1}{2}\sin^{-1} t\left(t - \frac{a^2}{t}\right) + \frac{1}{2}\sqrt{1-t^2} + \frac{a^2}{2}\ln\left(\frac{t}{1+\sqrt{1-t^2}}\right) + C, \text{ where } t = x + \sqrt{x^2+a^2}$$

S38. We have

$$I = \underbrace{3\int\sqrt{x-\sin x}\cdot x^{3/2}(1-\cos x)\,dx}_{I_1} \ \underbrace{+3\int\sqrt{x}\,(x-\sin x)^{3/2}\,dx}_{I_2}$$

We will proceed by applying integration by parts on the second integral, I_2:

$$I_2 = 3\left\{(x-\sin x)^{3/2}\frac{x^{3/2}}{\frac{3}{2}} - \frac{3}{2}\int(x-\sin x)^{1/2}\cdot(1-\cos x)\frac{x^{3/2}}{\frac{3}{2}}\,dx\right\}$$

$$= 2x^{3/2}(x-\sin x)^{3/2} - I_1 + C$$

Since I_1 is generated in this expression, it will now be clear to the reader why we chose to start by applying integration by parts on I_2. Thus,

$$I = I_1 + I_2 = 2x^{3/2}(x-\sin x)^{3/2} + C$$

S39. Multiplying the numerator and denominator by e^x, we have

$$I = \int\frac{e^x(x+1)}{xe^x(1+xe^x)^2}\,dx,$$

so the substitution $1+xe^x = t$ will reduce this to

$$I = \int\frac{dt}{t^2(t-1)} = \log\left|\frac{t-1}{t}\right| + \frac{1}{t} + C$$

The final result is obtained by substituting back $t = 1+xe^x$:

$$I = \log\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + C$$

S40. We first make the observation that

$$\frac{d}{d\theta}\left(\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right) = 2\sec 2\theta,$$

which means that it would be easy to integrate I by parts, taking $\cos 2\theta$ as the second function:

$$I = \frac{1}{2}\sin 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) + \frac{1}{2}\log(\cos 2\theta) + C$$

S41. Instead of directly expanding this expression using partial fractions, we first do the following rearrangement:

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)} = \frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}$$

$$= \frac{x}{(x+1)(x^2+1)} + \frac{2}{(x^2+1)^2}$$

$$= E_1 + E_2$$

We partially-expand E_1:

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad x = A(x^2+1) + (Bx + C)(x + 1)$$

Put $x = -1 \quad \Rightarrow \quad A = -\frac{1}{2}$

Put $x = 0 \quad \Rightarrow \quad A + C = 0$

$$\Rightarrow \quad C = \frac{1}{2}$$

Compare the coefficients of $x^2 \quad \Rightarrow \quad \mathrm{A + B = 0}$

$$\Rightarrow \quad B = \frac{1}{2}$$

The partial expansion of E_1 therefore is:

$$\frac{-\frac{1}{2}}{x+1} + \frac{(\frac{1}{2})x + \frac{1}{2}}{x^2+1}$$

The integral of E_1 is

$$I_1 = -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

To integrate E_2, we use the substitution $x = \tan\theta$. Thus, $dx = \sec^2\theta d\theta$:

$$I_2 = \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2} = \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \quad = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + C$$

We have evaluated the integrals of both E_1 and E_2 and thus the integral of $E_1 + E_2$.

S42.

$$I = \int (x-1)\sqrt{\frac{1+x}{1-x}}\,dx - \int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= -\int \sqrt{1-x^2}\,dx - \int \sqrt{\frac{1+x}{1-x}}\,dx$$

The first integral is straightforward to evaluate. To evaluate the second integral I_1, we can of course use the substitution $x = \cos 2\theta$. However, here we try a different substitution:

$$\frac{1+x}{1-x} = t^2 \qquad \Rightarrow \quad x = \frac{t^2-1}{t^2+1} \quad \Rightarrow \quad dx = \frac{4t}{(t^2+1)^2}dt$$

$$\Rightarrow \quad I_1 = \int t \cdot \frac{4t\,dt}{(t^2+1)^2} = 4\int \frac{t^2}{(t^2+1)^2}\,dt$$

$$= 4\left\{\int \frac{1}{t^2+1}\,dt \; - \; \int \frac{1}{(t^2+1)^2}\,dt\right\}$$

$$\begin{pmatrix}\text{substitute } t = \tan\theta \\ \Rightarrow dt = \sec^2\theta\, d\theta\end{pmatrix}$$

$$= 4\{\tan^{-1} t - \int \cos^2\theta\, d\theta\}$$

$$= 4\left\{\tan^{-1} t - \frac{1}{2}\tan^{-1} t - \frac{1}{4}\left(\frac{2t}{1+t^2}\right)\right\} + C$$

$$= 2\tan^{-1} t - \frac{2t}{1+t^2} + C$$

S43. We have

$$I = \int \frac{1}{(x^2-a^2)\{(x^4+a^2x^2+a^4)+a^2x^2\}}\,dx$$

$$= \int \frac{1}{(x^2-a^2)(x^4+2a^2x^2+a^4)}\,dx$$

$$= \int \frac{1}{(x^2-a^2)(x^2+a^2)^2}\,dx$$

We can split this into partial fractions; using $x^2 = t$ and $a^2 = p$, we have

$$\frac{1}{(t-p)(t+p)^2} = \frac{A}{t-p} + \frac{B}{t+p} + \frac{C}{(t+p)^2}$$

$$\Rightarrow \quad 1 = A(t+p)^2 + B(t^2-p^2) + C(t-p)$$

Substituting $t = p$, we have $A = \frac{1}{4p^2}$. Also, the coefficient of t^2 on the RHS is $A + B$ which must be 0; this gives $B = \frac{-1}{4p^2}$. Finally, the coefficient of t on the RHS is $2Ap + C$, which should be 0; this gives $C = -\frac{1}{2p}$. Thus,

$$I = \frac{1}{4a^4}\int \frac{1}{x^2-a^2}\,dx - \frac{1}{4a^4}\int \frac{1}{x^2+a^2}\,dx - \frac{1}{2a^2}\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$= \frac{1}{8a^5}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{4a^5}\tan^{-1}\frac{x}{a} - \frac{1}{2a^2}I_1 + C_1$$

To evaluate I_1, we use $x = a\tan\theta$:

$$I_1 = \int \frac{1}{(x^2+a^2)^2}\,dx = \int \frac{1}{a^4\sec^4\theta}\,a\sec^2\theta\, d\theta = \frac{1}{a^3}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{2a^3}\left(\theta + \frac{\sin 2\theta}{2}\right) + C_2 = \frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a} + \frac{ax}{a^2+x^2}\right) + C_2$$

$$\Rightarrow \quad I = \frac{1}{8a^5}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{4a^5}\tan^{-1}\frac{x}{a} - \frac{1}{4a^5}\left(\tan^{-1}\frac{x}{a} + \frac{ax}{a^2+x^2}\right) + C$$

$$= \frac{1}{4a^5}\left\{\frac{1}{2}\ln\left|\frac{x-a}{x+a}\right| - 2\tan^{-1}\frac{x}{a} - \frac{ax}{x^2+a^2}\right\} + C$$

S44. This is one of the best problems on indefinite integration we have seen in terms of the elegance of the solution. We can quickly conclude by looking at the integral expression that no rearrangement or substitution will really work. What will work is an appropriate integration by parts. However, we must admit that this realization is non-trivial.

We start with $f(x) = \frac{1}{x\sin x + \cos x}$, by differentiating it:

$$f'(x) = \frac{-x\cos x + \sin x - \sin x}{(x\sin x + \cos x)^2} = \frac{-x\cos x}{(x\sin x + \cos x)^2}$$

Thus,

$$\int \frac{x\cos x}{(x\sin x + \cos x)^2} = -f(x) \text{ (+ some integration constant)} \qquad (1)$$

This gives us our hint. We split the function in the original integral I into two parts, taking cue from (1), and apply integration by parts:

$$I = \int \underbrace{\frac{x}{\cos x}}_{\text{Ist function}} \cdot \underbrace{\frac{x\cos x}{(x\sin x + \cos x)^2}}_{\text{IInd function}} dx$$

$$= \frac{x}{\cos x} \cdot \frac{-1}{x\sin x + \cos x} - \int \frac{x\sin x + \cos x}{\cos^2 x} \cdot \frac{-1}{x\sin x + \cos x} dx$$

$$= \frac{-x}{\cos x(x\sin x + \cos x)} + \int \sec^2 x \, dx$$

$$= \frac{-x}{\cos x(x\sin x + \cos x)} + \tan x + C$$

$$= \frac{\sin x - x\cos x}{x\sin x + \cos x} + C$$

S45 The expression in the problem looks complicated, and yet the solution is quite simple. We just have to shift one x from the first bracket into the second bracket. Thus,

$$I = \int (x^{3m-1} + x^{2m-1} + x^{m-1})\,(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

Now, we simply substitute for the expression in the second bracket a new variable t:

$$2x^{3m} + 3x^{2m} + 6x^m = t \quad \Rightarrow \quad 6m(x^{3m-1} + x^{2m-1} + x^{m-1})\,dx = dt$$

Thus,

$$I = \frac{1}{6m}\int t^{1/m} dt = \frac{1}{6m}\frac{t^{\frac{1}{m}+1}}{\frac{1}{m}+1} + C = \frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{1}{m}+1}}{6(m+1)} + C$$

S46. (a) We have

$$I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = 2\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\, dx.$$

Now, assuming $\tan x = t^2$, we have $\sec^2 x dx = 2t dt$, *i.e.*,

$$dx = \frac{2tdt}{\sec^2 x} = \frac{2tdt}{1+\tan^2 x} = \frac{2tdt}{1+t^4}$$

Thus,

$$I = 2\int_0^1 \left(t + \frac{1}{t}\right)\cdot \frac{2t}{1+t^4}\, dt = 4\int_0^1 \frac{1+t^2}{1+t^4}\, dt$$

Now, we evaluate the following indefinite integral I_1:

$$I_1 = \int \frac{1+t^2}{1+t^4}\, dt = \int \frac{1+\frac{1}{t^2}}{(t-\frac{1}{t})^2+2}\, dt$$

Assuming $t - \frac{1}{t} = z$, we have $(1+\frac{1}{t^2})dt = dz$, so that

$$I_1 = \int \frac{dz}{z^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right)$$

$$\Rightarrow \quad I = 4I_1\Big|_0^1 = 2\sqrt{2} \quad (\tan^{-1} 0 - \tan^{-1}(-\infty)) = \sqrt{2}\pi$$

Strictly speaking, $\tan^{-1}(-\infty)$ is not the correct notation and what we should actually be writing is $\lim\limits_{x\to-\infty} \tan^{-1}(x)$.

(b) $$I = \int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}\, dx = \int_0^{\pi/2} (1+\sqrt{\tan x})^{-4} \sec^2 x\, dx$$

Using $\tan x = t^2$, so that $\sec^2 x dx = 2tdt$, we have

$$I = \int_0^{\infty} (1+t)^{-4}\cdot 2tdt = 2\int_0^{\infty} \frac{t}{(1+t)^4}\, dt = 2\int_0^{\infty} \left\{\frac{1}{(1+t)^3} - \frac{1}{(1+t)^4}\right\} dt$$

$$= 2\left\{\frac{(1+t)^{-2}}{-2} - \frac{(1+t)^{-3}}{-3}\right\}\Bigg|_0^{\infty} = \frac{1}{3}$$

S47. (a) We have $I + J = \int_0^a dx = a$, while

$$I - J = \int_0^a \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \ln|\sin x + \cos x|\Big|_0^a = \ln(\sin a + \cos a)$$

$$\Rightarrow \quad I = \frac{a}{2} + \frac{1}{2}\ln(\sin a + \cos a)$$

(b) Take $I = \int_0^{\pi/2} \frac{\cos x}{a\sin x + b\cos x}\, dx$ and $J = \int_0^{\pi/2} \frac{\sin x}{a\sin x + b\cos x}\, dx$. Consider the two expressions $bI + aJ$ and $aI - bJ$ and apply the approach in part-(a) to arrive at the value of I:

$$I = \frac{b}{a^2+b^2}\frac{\pi}{2} + \frac{a}{a^2+b^2}\ln\frac{a}{b}$$

(c) Take $I = \int_0^{\pi/2} \frac{\cos x+4}{3\sin x+4\cos x+25}\,dx$ and $J = \int_0^{\pi/2} \frac{\sin x+3}{3\sin x+4\cos x+25}\,dx$, and consider the expressions $4I+3J$ and $3I-4J$ to arrive at the value for I:

$$I = \frac{2\pi}{25} + \frac{3}{25}\ln\frac{28}{29}$$

S48. Rather than attempting to evaluate the integrals, we differentiate the given expression:

$$f'(x) = \sin^{-1}(\sqrt{\sin^2 x})(\sin^2 x)' + \cos^{-1}(\sqrt{\cos^2 x})(\cos^2 x)'$$

Since $x \in (0, \frac{\pi}{2})$, we have

$$\sqrt{\sin^2 x} = \sin x,\ \sqrt{\cos^2 x} = \cos x$$

Thus,

$$f'(x) = \sin^{-1}(\sin x)(2\sin x\cos x) + \cos^{-1}(\cos x)(-2\sin x\cos x)$$

Since $(0, \frac{\pi}{2})$ lies in the principle range of both $\sin^{-1}\theta$ and $\cos^{-1}\theta$, we have

$$\sin^{-1}(\sin x) = x,\ \cos^{-1}(\cos x) = x, \text{ for } x \in \left(0, \frac{\pi}{2}\right)$$

Therefore,

$$f'(x) = 0 \quad \Rightarrow \quad f(x) = k \text{ (constant)}$$

To evaluate k, we assume a particular value for x in the original expression. The choice which leads to the answer most easily is $x = \frac{\pi}{4}$, because then

$$f(x) = \int_0^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\,dt = \int_0^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt = \int_0^{1/2} \frac{\pi}{2}dt = \frac{\pi}{4}$$

Thus,

$$f(x) = \frac{\pi}{4}$$

S49. We have

$$I = \int_0^a f(x)g(x)h(x)dx = \int_0^a f(a-x)g(a-x)h(a-x)dx = \int_0^a f(x)(-g(x))\left(\frac{3h(x)-5}{4}\right)dx$$

$$= -\frac{3}{4}\int_0^a f(x)g(x)h(x)\,dx + \frac{5}{4}\int_0^a f(x)g(x)\,dx = -\frac{3}{4}I + \frac{5}{4}I_1$$

Now,

$$I_1 = \int_0^a f(x)g(x)\,dx = \int_0^a f(a-x)g(a-x)\,dx = -\int_0^a f(x)g(x)\,dx = -I_1$$

$$\Rightarrow \quad I_1 = 0$$

This gives $I = 0$.

S50. We have $f_e(x) = \dfrac{f(x)+f(-x)}{2}$ and $f_o(x) = \dfrac{f(x)-f(-x)}{2}$, so that

$$\int_{-\infty}^{+\infty} f_e^2(x)dx + \int_{-\infty}^{+\infty} f_o^2(x)dx = \frac{1}{4}\int_{-\infty}^{+\infty}\{f^2(x)+f^2(-x)+2f(x)f(-x)\}\,dx$$

$$+$$

$$\frac{1}{4}\int_{-\infty}^{+\infty}\{f^2(x)+f^2(-x)-2f(x)f(-x)\}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\{f^2(x)+f^2(-x)\}\,dx \qquad (1)$$

However,

$$\int_{-\infty}^{+\infty} f^2(-x)dx = \int_{+\infty}^{-\infty} f^2(t)(-dt) \quad (\text{Using } t \to -x)$$

$$= \int_{-\infty}^{+\infty} f^2(t)dt = \int_{-\infty}^{+\infty} f^2(x)dx$$

Thus, (1) equals $\int_{-\infty}^{+\infty} f^2(x)dx$, and so the proof is complete.

S51. We note that $g'(x) = -\frac{f(x)}{x}$, so that $f(x)+xg'(x)=0$. Thus,

$$\int_0^a (f(x)+xg'(x))\,dx = 0 \quad \Rightarrow \quad \int_0^a f(x)dx + xg(x)\Big|_0^a - \int_0^a g(x)dx = 0$$

Noting that $g(a)=0$, we are thus left with

$$\int_0^a f(x)dx - \int_0^a g(x)dx = \int_0^a (f(x)-g(x))\,dx = 0$$

S52. In (0, 1),

$$\{2x\} = \begin{cases} 2x, & 0 < x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x < 1 \end{cases}, \quad \{3x\} = \begin{cases} 3x, & 0 < x < \frac{1}{3} \\ 3x-1, & \frac{1}{3} \le x < \frac{2}{3} \\ 3x-2, & \frac{2}{3} \le x < 1 \end{cases}$$

Thus,

$$(\{2x\}-1)(\{3x\}-1) = \begin{cases} (2x-1)(3x-1) & 0 < x < \frac{1}{3} \\ (2x-1)(3x-2) & \frac{1}{3} \le x < \frac{1}{2} \\ (2x-2)(3x-2) & \frac{1}{2} \le x < \frac{2}{3} \\ (2x-2)(3x-3) & \frac{2}{3} \le x < 1 \end{cases}$$

$$\Rightarrow \quad I = \int_0^{1/3}(2x-1)(3x-1)dx + \int_{1/3}^{1/2}(2x-1)(3x-2)dx + \int_{1/2}^{2/3}(2x-2)(3x-2)dx + \int_{2/3}^{1}(2x-2)(3x-3)dx$$

$$= \int_0^{1}(6x^2-5x+1)dx + \int_{1/3}^{1/2}(1-2x)dx + \int_{1/2}^{2/3}(3-5x)\,dx + \int_{2/3}^{1}(5-7x)\,dx$$

$$= \frac{1}{2}+\frac{1}{36}+\frac{1}{72}-\frac{5}{18} = \frac{19}{72}$$

S53. The line $y = mx$ intersects the given curve if there exists at least one value of $x \in \mathbb{R}^+$ for which

$$m^2x^2 + \int_0^x f(x)dt = 2$$

Consider the function

$$h(x) = m^2x^2 + \int_0^x f(t)dt$$

For every $m \in \mathbb{R}$, we note that

(i) $h(x)$ is continuous (ii) $h(0) = 0$ (iii) $h(x) \to \infty$ as $x \to \infty$

By the intermediate value theorem, there must exist some $x \in \mathbb{R}^+$ for which $h(x) = 2$. Thus,

$$m \in \mathbb{R}$$

S54. The function to be integrated is discontinuous at all integer points. Therefore, we integrate it piecewise:

$$\int_0^x [t]dt = \int_0^1 [t]dt + \int_1^2 [t]dt + \ldots + \int_{[x]-1}^{[x]} [t]dt + \int_{[x]}^x [t]dt$$

$$= 0\cdot\int_0^1 dt + 1\cdot\int_1^2 dt + 2\cdot\int_2^3 dt + \ldots + ([x]-1)\cdot\int_{[x]-1}^{[x]} dt + \int_{[x]}^x [t]dt$$

$$= 0+1+2+\cdots+([x]-1)+(x-[x])([x]) \quad \left(\begin{array}{l}\text{Verify that the last}\\ \text{term is correct}\end{array}\right)$$

$$= \frac{[x]([x]-1)}{2} + [x](x-[x])$$

S55. We have

$$I = \int_0^{\pi/2} \sin 2kx \cot x dx = \int_0^{\pi/2} \frac{\sin 2kx}{\sin x}\cos x dx$$

$$= \int_0^{\pi/2} 2(\cos x + \cos 3x + \cdots + \cos(2k-1)x)\cos x dx$$

$$= \int_0^{\pi/2} \{(1+\cos 2x)+(\cos 2x+\cos 4x)+\cdots+\cos(2k-2)x+\cos 2kx\}dx$$

$$= \int_0^{\pi/2} \{1 + 2\cos 2x + 2\cos 4x + \cdots + 2\cos(2k-2)x + \cos 2kx\}\, dx$$

$$= \frac{\pi}{2} \quad \left(\begin{array}{l}\text{Only the first term, } i.e., 1, \text{ has a non-zero} \\ \text{contribution to the integral.}\end{array}\right)$$

S56. **(a)** $I = \int_{-1}^{1} \tan^{-1}(e^z) dz = \int_{-1}^{1} \tan^{-1}(e^{-z}) dz = \int_{-1}^{1} \cot^{-1}(e^z)\, dz$

$$= \int_{-1}^{1} \left(\frac{\pi}{2} - \tan^{-1}(e^z)\right) dz = \pi - I$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

(b) $I = \int_{-1}^{1} \cot^{-1}(x^3 + \sqrt{1+x^6})\, dx = \int_{-1}^{1} \cot^{-1}(-x^3 + \sqrt{1+x^6}) dx$

$$= \int_{-1}^{1} \left(\frac{\pi}{2} - \cot^{-1}(x^3 + \sqrt{1+x^6})\right) dx \quad \text{(how?)}$$

$$= \pi - I$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

S57. Let us represent the integrals in the numerator and denominator by I_n and I_d respectively.

$$I_d = \int_0^1 (1 - x^{50})^{101} dx = x(1 - x^{50})^{101}\Big|_0^1 + 5050 \int_0^1 x^{50} (1 - x^{50})^{100} dx$$

$$= -5050 \left\{ \int_0^1 (1 - x^{50} - 1)\, (1 - x^{50})^{100} dx \right\}$$

$$= 5050 I_n - 5050 I_d$$

$$\Rightarrow \quad \frac{5050\, I_n}{I_d} = 5051$$

S58. (a) First we show that S_n is independent of n:

$$S_{n+1} - S_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\, dx = 2 \int_0^{\pi/2} \cos 2\, nx\, dx = 0$$

Also, $S_1 = \int_0^{\pi/2} dx = \frac{\pi}{2}$. Therefore,

$$S_1 = S_2 = S_3 = \cdots = \frac{\pi}{2}$$

Now,

$$V_{n+1} - V_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx = \int_0^{\pi/2} \frac{\sin(2n+1)x \sin x}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx = S_{n+1}$$

(b) Thus, $\{V_i\}$ is an AP with common difference equal to $\frac{\pi}{2}$, and V_1 equals $\frac{\pi}{2}$.

$$\Rightarrow \quad V_n = \frac{n\pi}{2}$$

S59. Using $x \to \frac{ab}{t}$ so that $dx = \frac{-ab}{t^2}$, the integral I becomes

$$I = \int_b^a \frac{f(\frac{b}{t}) - f(\frac{t}{a})}{\frac{ab}{t}} \left(\frac{-ab}{t^2} \right) dt$$

$$= \int_a^b \frac{f(\frac{b}{x}) - f(\frac{x}{a})}{x} dx \quad \left\{ \begin{array}{l} \text{Using } x \text{ instead of } t\text{; note carefully} \\ \text{why this can be done} \end{array} \right\}$$

The original expression for I was

$$I = \int_a^b \frac{f(\frac{x}{a}) - f(\frac{b}{x})}{x} dx$$

Clearly, I is an odd integral, so $I = 0$.

S60. We first evaluate the indefinite integral $g(x)$ using integration by parts (taking $\sin x$ as the second function):

$$g(x) = \int \sin x \log(\sin x) dx$$

$$= -\cos x \log(\sin x) + \int \frac{\cos^2 x}{\sin x} dx$$

$$= -\cos x \log(\sin x) + \int (\operatorname{cosec} x - \sin x)\, dx$$

$$= -\cos x \log(\sin x) + \log(\operatorname{cosec} - \cot x) + \cos x$$

$$= \cos x - \cos x \log(\sin x) + \log\left(\tan \frac{x}{2} \right) \quad \text{(how ?)}$$

$$= \cos x + \log\left(\frac{\tan(\frac{x}{2})}{(\sin x)^{\cos x}} \right) \quad \left(\begin{array}{l} \text{Take the constant of integration} \\ \text{to be 0; it does not matter here} \end{array} \right)$$

Now, if we apply the integration limits, we have $g(\frac{\pi}{2}) = 0$ while

$$g(0) = \lim_{h \to 0} g(h) = 1 + \log \underbrace{\left\{ \lim_{h \to 0} \left(\frac{\tan(\frac{h}{2})}{(\sin h)^{\cos h}} \right) \right\}}_{L}$$

We note that the limit L is of the form $\frac{0}{0}$, and thus we apply the LH rule:

$$L = \lim_{h\to 0} \frac{\frac{1}{2}\sec^2 \frac{h}{2}}{(\sin h)^{\cos h}\left(\frac{\cos^2 h}{\sin h} - \sin h \log(\sin h)\right)}$$

$$= \underbrace{\lim_{h\to 0}\left(\frac{\frac{1}{2}\sec^2 \frac{h}{2}}{\cos^2 h - \sin^2 h \log(\sin h)}\right)}_{L_1} \underbrace{\lim_{h\to 0}((\sin h)^{1-\cos h})}_{L_2}$$

We observe that since $\lim\limits_{h\to 0}(\sin^2 h \log(\sin h)) = 0$ (why?), we must have $L_1 = \frac{1}{2}$. Now, we evaluate L_2:

$$\log L_2 = \lim_{h\to 0}(1-\cos h)\log(\sin h) = \lim_{h\to 0}\left(2\sin^2 \frac{h}{2}\right)\log(\sin h) = 0 \text{(why?)}$$

$$\Rightarrow \quad L_2 = 1$$

Thus, $$L = L_1 L_2 = \frac{1}{2} \Rightarrow g(0) = 1 + \log\left(\frac{1}{2}\right) = 1 - \log 2$$

Finally, the value of the definite integral I is

$$I = g\left(\frac{\pi}{2}\right) - g(0) = \log 2 - 1 = \log\left(\frac{2}{e}\right)$$

S61. The trick is to split the range of integration $(0, \infty)$ for I into (0, 1) and $(1, \infty)$:

$$I = \int_0^1 t^{-1/2} e^{-2010(t+t^{-1})} dt + \int_1^\infty t^{-1/2} e^{-2010(t+t^{-1})} dt = I_1 + I_2 \text{ (say)}$$

In I_1, we substitute $t \to \frac{1}{t}$. Thus, I_1 becomes

$$I_1 = \int_1^\infty t^{-3/2} e^{-2010(t+t^{-1})} dt \quad \Rightarrow \quad I = I_1 + I_2 = \int_1^\infty \left(\frac{1}{t^{1/2}} + \frac{1}{t^{3/2}}\right) e^{-2010(t+t^{-1})} dt$$

Substitute $\sqrt{2010}(\sqrt{t} - \frac{1}{\sqrt{t}}) \to u$ to obtain

$$I = \frac{2}{\sqrt{2010}} \int_0^\infty e^{-u^2 - (2\times 2010)} du = \frac{e^{-4020}}{\sqrt{2010}} \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\frac{\pi}{2010}}\, e^{-4020}$$

S62. First substitute $x \to x - \pi$ to center the integral aroung $x = \pi$:

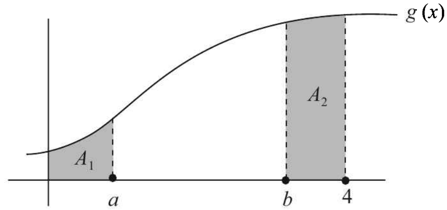
$$I_m = (-1)^m \int_{-\pi}^{\pi} \cos x \cos 2x \ldots \cos mx \, dx = 2(-1)^m \int_0^\pi \cos x \cos 2x \ldots \cos mx \, dx$$

Now, substitute $x \to x - \frac{\pi}{2}$ to center the integral around $x = \frac{\pi}{2}$:

$$I_m = 2(-1)^m \int_{-\pi/2}^{\pi/2} \sin x \cdot \cos 2x \cdot (-\sin 3x) \cdot (-\cos 4x) \cdot \sin 5x \cdot \cos 6x \cdot (-\sin 7x) \cdot (-\cos 8x) \ldots dx$$

Note that $I_m = 0$ when m = 1, 2, 5, 6, 9, 10,..., that is, $I_m = 0$ whenever m is of form $4k+1, 4k+2$ (why?). In {1, 2, 3,..., 100}, there are 50 such values of m. Thus,

$$P(I_m \neq 0) = \frac{1}{2}$$



S63. Let $a = 2 - y, b = 2 + y$.
Thus, the sum of the integrals A given to us can be specified as a function of y:

$$A(y) = \int_0^{2-y} g(x)dx + \int_0^{2+y} g(x)dx$$

Also, $b - a = 2y$.
Thus, to show that A increases with $b - a$, we show that $\frac{dA}{dy} > 0$:

$$\frac{dA}{dy} = -g(2-y) + g(2+y) = g(b) - g(a)$$

But this difference is positive, since g is an increasing function. Hence, A increases as $b - a$ increases. Graphically, $A = \int_0^4 g(x)dx - A_2 + A_1$ (make sure you understand this). Proving that A increases as $b - a$ increases is thus equivalent to proving that $A_2 - A_1$ decreases as $b - a$ increases.

S64. We first use integration by parts to show that

$$u_n = \int_0^{\pi/2} \cos^{n-1} x \sin x \sin nx \, dx,$$

and then subsequently use the identity

$$\sin nx \sin x = \cos(n-1)x - \cos nx \cos x$$

Thus,

$$u_n = \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^n x \cos nx \, dx = u_{n-1} - u_n$$

$$\Rightarrow \quad \frac{u_n}{u_{n-1}} = \frac{1}{2}$$

(a) The common ratio of the GP is $\frac{1}{2}$.
(b) We have $u_1 = \frac{\pi}{4}$, so that

$$u_n = \frac{1}{2^{n-1}} \cdot \frac{\pi}{4} = \frac{\pi}{2^{n+1}}$$

S65. We will try to find an upper and a lower bound for I. In the interval (0, 1), we note that $x^3 < x^2$, so that

$$4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$$

Thus,

$$\frac{1}{\sqrt{4-2x^2}} > \frac{1}{\sqrt{4-x^2-x^3}} > \frac{1}{\sqrt{4-x^2}}$$

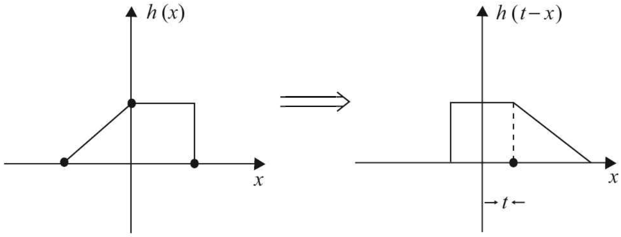
$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{4-2x^2}} dx > I > \int_0^1 \frac{1}{\sqrt{4-x^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} \left(\sin^{-1} \frac{x}{\sqrt{2}} \right) \Bigg|_0^1 > I > \left(\sin^{-1} \frac{x}{2} \right) \Bigg|_0^1$$

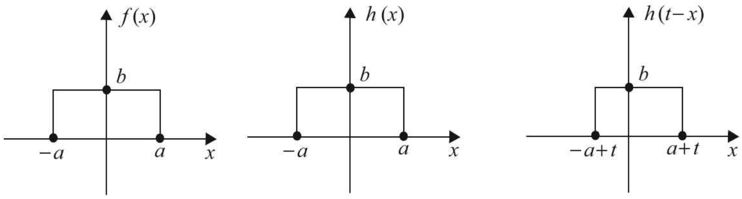
$$\Rightarrow \quad \frac{\pi}{4\sqrt{2}} > I > \frac{\pi}{6}$$

Therefore, I is greater than $\frac{\pi}{6}$.

S66. When evaluating the convolution function $I(t)$ of two functions $f(x)$ and $h(x)$, note that one of the functions in the integral is $h(t-x)$, *i.e.*, a flipped and *time delayed* version of $h(x)$:



So what we are actually doing while evaluating the convolution is trying to find the 'overlap' between $f(x)$ and a flipped version of $h(x)$, for every time delay t. It might take you some time and effort to fully understand this operation, but working out on your own the example in this question will surely contribute to that understanding



Note that the direction of the shift in $h(x)$ will depend on the sign of t. For example, if t is extremely negative, then the shift will be so much to the left that there will be no overlap between $f(x)$ and $h(t-x)$, *i.e.*, their product will be zero everywhere. Try this out yourself and make sure you understand it. Similarly, understand each of the following cases carefully:

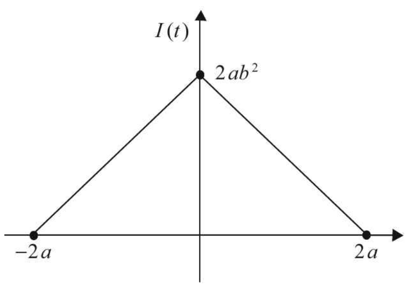
$t < -2a \qquad I(t) = 0$

$-2a < t < 0 \qquad I(t) = \int_{-a}^{t+a} b^2 dx = b^2(t+2a)$

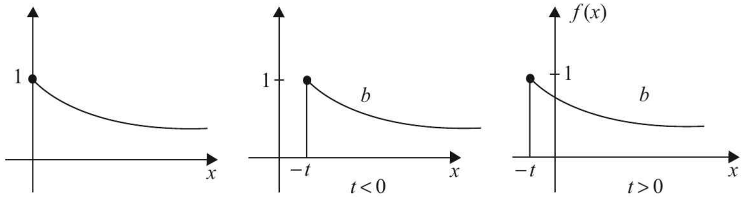
$0 < t < 2a \qquad I(t) = \int_{t-a}^{a} b^2 dx = b^2(2a-t)$

$t > 2a \qquad I(t) = 0$

Now, we plot the graph of $I(t)$:



S67. The plots of the given function $f(x)$ and $f(x)$ shifted by t units are shown below:



For $t < 0$,

$$R(t) = \int_{-t}^{\infty} e^{-x} e^{-(x+t)} dx = \frac{1}{2} e^{t}$$

For $t > 0$,

$$R(t) = \int_{0}^{\infty} e^{-x} e^{-(x+t)} dx = \frac{1}{2} e^{-t}$$

$$R(t) = \frac{1}{2} e^{-|t|}$$

The reader is urged to plot $R(t)$.

S68. We make use of the *orthogonality property* of sinusoids for this problem:

$$\int_0^T \sin(k_1\, \omega x)\sin(k_2\, \omega x) = \frac{1}{2}\int_0^T (\cos(k_1 - k_2)\omega x - \cos(k_1 + k_2)\omega x) dx = 0$$

Similarly,

$$\int_0^T \cos(k_1\, \omega x)\cos(k_2\, \omega x)\, dx = 0 \quad \text{and} \quad \int_0^T \sin(k_1 \omega x)\cos(k_2\, \omega x)\, dx = 0$$

On the other hand,

$$\int_0^T \cos(k\, \omega x)\, \cos(k\, \omega x)\, dx = \frac{1}{2}\int_0^T (1 + \cos 2k\, \omega x)\, dx = \frac{T}{2}$$

$$\int_0^T \sin(k\, \omega x)\, \sin(k\, \omega x)\, dx = \frac{1}{2}\int_0^T (1 - \cos 2k\, \omega x)\, dx = \frac{T}{2}$$

This means that for two sinusoids S_1 and S_2 picked from a set of sinusoids all with frequencies as multiples of some ω, if S_1 and S_2 have different frequencies, their product integral over one time period is 0. If we represent the product integral by $\otimes$, then

$$S_1 \otimes S_2 = 0 \quad \text{while} \quad S_1 \otimes S_1 = \frac{T}{2}$$

This may remind one of orthogonal vectors:

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0, \text{ while } \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

That is why this property of sinusoids is named orthogonality. Note carefully how we use the orthogonality property to evaluate a_k and b_k.

(a)
$$a_k = \frac{2}{a}\int_0^a f(x)\cos k\omega x\,dx = \frac{2}{a}\int_0^{a/2}\cos\frac{2\pi kx}{a}dx = 0$$

$$b_k = \frac{2}{a}\int_0^a f(x)\sin k\omega x\,dx = \frac{2}{a}\int_0^{a/2}\sin\frac{2\pi kx}{a}dx = \begin{cases}\dfrac{2}{k\pi}, & \text{if } k \text{ is odd}\\ 0, & \text{if } k \text{ is even}\end{cases}$$

Also,

$$\int_0^a f(x)\,dx = \frac{a}{2} = aa_0$$

so that we can take

$$a_0 = \frac{1}{2} \text{ and } b_0 = 0$$

Thus, $f(x)$ can now be expressed as a sum of sinusoids as follows:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2m-1)\omega x}{2m-1}, \text{ where } \omega = \frac{2\pi}{a}$$

(b) $a_k = \dfrac{2}{a}\displaystyle\int_0^a \frac{x}{a}\cos\frac{2\pi kx}{a}dx = 0$ {you will have to use integration by parts}

$$b_k = \frac{2}{a}\int_0^a \frac{x}{a}\sin\frac{2\pi kx}{a}dx = -\frac{1}{k\pi}$$

Also,

$$a_0 = \frac{1}{2}$$

Thus,

$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{2\pi mx}{a}\right)$$

(c) We will only provide an outline, the reader is urged to fill in the details: Writing:

$$\cos k\omega x = \frac{e^{ik\omega x} + e^{-ik\omega x}}{2}, \sin k\omega x = \frac{e^{ik\omega x} - e^{-ik\omega x}}{2i},$$

we have

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty}\{(a_k - ib_k)e^{ik\omega x} + (a_k + ib_k)e^{-ik\omega x}\}$$

Thus,

$$\overline{f(x)} = \frac{1}{2}\sum_{j=0}^{\infty}\{(a_j + ib_j)e^{-ij\omega x} + (a_j - ib_j)e^{ij\omega x}\}$$

Now, we multiply and obtain the expression for $|f(x)|^2$. If we integrate this over one time period, the only non-zero contribution on the RHS will come when $j = k$, and summing all these contributions, we will obtain

$$\int_0^T |f(x)|^2 dx = \sum_{k=0}^{\infty}(|a_k|^2 + |b_k|^2)$$

S69. The given (integral) equation consists of two variables. We need to isolate a single variable to be able to obtain the expression for f. Differentiating the given expression on both sides with respect to y, we have

$$xf(xy) = \int_1^x f(t)dt + xf(y)$$

Substituting $y = 1$, we have

$$xf(x) = \int_1^x f(t)\,dt + xf(1) \quad \Rightarrow \quad xf(x) = \int_1^x f(t)\,dt + x$$

Now we differentiate with respect to x :

$$xf'(x) + f(x) = f(x) + 1 \quad \Rightarrow \quad f'(x) = \frac{1}{x} \quad \Rightarrow \quad f(x) = \ln x + C$$

Since $f(1) = 1$, $C = 1$, and thus

$$f(x) = 1 + \ln x = \ln(ex)$$

S70. **(a)** We treat I as a function of the 'variable' a, treating m as a constant:

$$I(a) = \int_0^\infty \frac{e^{-ax}\sin mx}{x}\,dx, \quad a > 0$$

We now differentiate this *with respect to a*:

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-ax}\cdot(-x)\sin mx}{x}\,dx = -\int_0^\infty e^{-ax}\sin mx\,dx$$

This integral is pretty standard, and if the reader does not remember the result, she can derive it quickly :

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

Thus,

$$\frac{dI}{da} = -\frac{e^{-ax}}{a^2+m^2}(-a\sin mx - m\cos mx)\Big|_0^\infty$$

Note that as $x \to \infty$, $e^{-ax} \to 0$, and so

$$\frac{dI}{da} = -\frac{m}{a^2+m^2} \quad \Rightarrow \quad I(a) = -\tan^{-1}\frac{a}{m} + C$$

Now, in the original integral, we note that if $a \to \infty$, then $e^{-ax} \to 0$, and so $I(a) \to 0$. Thus,

$$I(\infty) = 0 = -\tan^{-1}\left(\frac{\infty}{m}\right) + C = -\frac{\pi}{2} + C \quad \Rightarrow \quad C = \frac{\pi}{2}$$

We thus obtain:

$$I = \frac{\pi}{2} - \tan^{-1}\frac{a}{m}$$

(b) For $\int_0^\infty \frac{\sin x}{x}\,dx$, we have $a = 0$ and $m = 1$, and so

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

S71. Differentiating, we have

$$\frac{x^{n-1}}{(x-x_1)(x-x_2)\cdots(x-x_n)} = \frac{a_1}{x-x_1} + \frac{a_2}{x-x_2} + \cdots + \frac{a_n}{x-x_n}$$

Now, to determine the sum $a_1 + a_2 + \cdots + a_n$, you can do either of the following:

(a) Multiply both sides by x and apply the limit $x \to \infty$.
(b) Take the LCM on the RHS and compare the coefficients of x^{n-1} in the numerator on both sides.

The required answer will be 1.

S72. Differentiating both sides (under the integral sign) yields the desired answer: $f(x) = \frac{1}{e^x + \sin x}$.

S73. Differentiating (c), we have

$$\frac{1}{2} f^2(x) = \frac{2}{x}\left(\int_0^x f(t)dt\right) \cdot f(x) - \frac{1}{x^2}\left(\int_0^x f(t)dt\right)^2 \tag{1}$$

Using $\int_0^x f(t)dt = y$, we have

$$\frac{1}{2} f^2(x) = \frac{2f(x)}{x} y - \frac{1}{x^2} y^2$$

$$\Rightarrow \quad y^2 - 2xf(x)y + \frac{x^2 f^2(x)}{2} = 0$$

$$\Rightarrow \quad y = \frac{2xf(x) \pm \sqrt{4x^2 f^2(x) - 2x^2 f^2(x)}}{2} = \left(1 \pm \frac{1}{\sqrt{2}}\right) x\, f(x)$$

$$\Rightarrow \quad \int_0^x f(t)dt = \lambda x\, f(x), \text{ where } \lambda = 1 \pm \frac{1}{\sqrt{2}} \tag{2}$$

Differentiating this again, we have

$$f(x) = \lambda x\, f'(x) + \lambda f(x) \quad \Rightarrow \quad (1-\lambda) f(x) = \lambda x f'(x)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{\left(\frac{1-\lambda}{\lambda}\right)}{x}$$

$$\Rightarrow \quad f(x) = Cx^{\left(\frac{1-\lambda}{\lambda}\right)}, \text{ where } \lambda = 1 \pm \frac{1}{\sqrt{2}}$$

S74. We construct a new function $g(x)$ given by

$$g(x) = \int_0^x f(t)(at^2 + bt + c)\, dt$$

Thus,

$$g(0) = g(\alpha) = g(\beta) = 0$$

By Rolle's theorem,

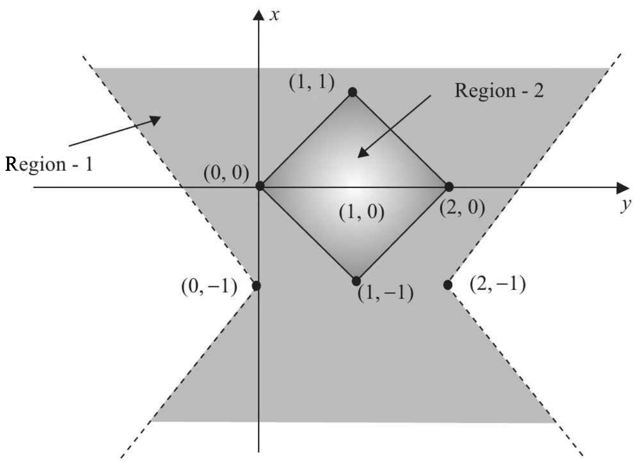
$$\begin{cases} g'(c_1) = 0 & \text{for at least one } c_1 \in (0, \alpha) \\ g'(c_2) = 0 & \text{for at least one } c_2 \in (\alpha, \beta) \end{cases}$$

However, since $g'(x) = f(x)(ax^2 + bx + c)$, and $f(x) \neq 0 \quad \forall\, x \in \mathbb{R}$, we have

$$ax^2 + bx + c = \begin{cases} 0 & \text{for at least one } c_1 \in (0, \alpha) \\ 0 & \text{for at least one } c_2 \in (\alpha, \beta) \end{cases}$$

Since $ax^2 + bx + c$ is a quadratic expression, it must have *exactly two* zeroes. This means that $ax^2 + bx + c = 0$ has one of its roots in $(0, \alpha)$ and the other root in (α, β).

S75. We simply plot the regions which the two expressions represent. Deducing them is left to the reader as an exercise



The area bounded is the area of region – 2, *i.e.*, 2 sq units.

S76. The minimum for the curve occurs at $e^{-1/n}$. Thus,

$$\int_0^{e^{-1/n}} x^n \log x \, dx = \int_{e^{-1/n}}^{1} x^n \log x \, dx$$

$$\Rightarrow \quad \left(\frac{1}{n+1}\right)^2 = 2\left(\frac{1}{n(n+1)} + \frac{1}{(n+1)^2}\right) e^{-\frac{n+1}{n}}$$

$$\Rightarrow \quad 4 + \frac{2}{n} = e^{1+\frac{1}{n}} \quad \Rightarrow \quad 1 < n < 2$$

$$\Rightarrow \quad [n] = 1$$

S77. We consider the function $f(x) = 2x - 3x^3 - c$. If (b, c) is the (right) point of intersection of the line and the curve, then the area under $f(x)$ from $x = 0$ to $x = b$ must be zero:

$$\int_0^b (2x-3x^3-c)\,dx = 0 \;\Rightarrow\; b^2 - \frac{3b^4}{4} - cb = 0$$

$$\Rightarrow\; c = b - \frac{3b^3}{4} = 2b - 3b^3 \;\{(b,\,c) \text{ lies an the curve}\}$$

$$\Rightarrow\; b = \frac{2}{3} \;\Rightarrow\; c = \frac{4}{9}$$

S78. The value of ne^{-x} crosses an integer whenever x takes on the appropriate value:

$$ne^{-x} = k$$

where $k = n$ for $x = 0$,and takes the integer values $n-1,\, n-2,\, n-3, \ldots, 1$ and finally, as $x \to \infty,\, k \to 0.$ Thus, if $ne^{-x} \in (k, k+1),$ then

$$x \in \left(\ln\left(\frac{n}{k+1}\right), \ln\frac{n}{k}\right)$$

$$\Rightarrow\; [ne^{-x}] = \begin{cases} n-1, & x \in \left(0, \ln\left(\frac{n}{n-1}\right)\right] \\ n-2, & x \in \left(\ln\left(\frac{n}{n-1}\right), \ln\left(\frac{n}{n-2}\right)\right] \\ & \vdots \\ n-r, & x \in \left(\ln\left(\frac{n}{n-r+1}\right), \ln\left(\frac{n}{n-r}\right)\right] \\ & \vdots \\ 0, & x \in (\ln n, \infty) \end{cases}$$

We therefore have

$$\int_0^\infty [ne^{-x}]\,dx = (n-1)\ln\left(\frac{n}{n-1}\right) + (n-2)\ln\left(\frac{n-1}{n-2}\right) + \cdots + (n-r)\ln\left(\frac{n-r+1}{n-r}\right) + \cdots$$

$$= \ln\left\{\left(\frac{n}{n-1}\right)^{n-1} \cdot \left(\frac{n-1}{n-2}\right)^{n-2} \cdots \left(\frac{2}{1}\right)^1\right\}$$

$$= \ln\left(\frac{n^n}{n!}\right)$$

Thus, $[I_n] > 2$ implies that $\frac{n^n}{n!} \geq e^3$. It can be easily verified that this first happens for $n = 4$.

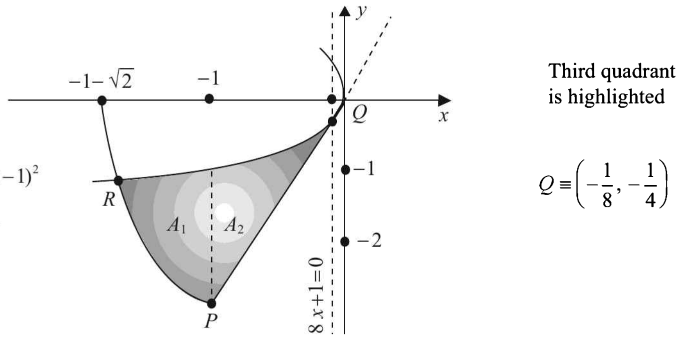
S79. Enforcing continuity at $x = 1$ and $x = -1$ respectively gives

$$2 = 1 + a + b, \quad 1 - a + b = -2$$

$$\Rightarrow\; a = 2, \; b = -1$$

$$\Rightarrow\; f(x) = \begin{cases} 2x, & |x| \leq 1 \\ x^2 + 2x - 1, & |x| > 1 \end{cases}$$

We now draw the graph of $f(x)$



R is given by

$$y^2 = -\frac{x}{2} = (x^2+2x-1)^2$$

$$\Rightarrow \quad x=-2,\ y=-1$$

$$\Rightarrow \quad R \equiv (-2,-1)$$

$$Q \equiv \left(-\frac{1}{8}, -\frac{1}{4}\right)$$

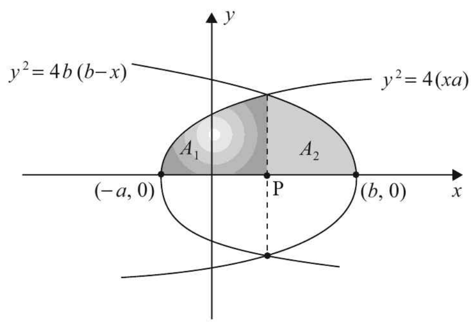
The required area A is given by

$$A = A_1 + A_2 = \int_{-2}^{-1}\left(-\sqrt{\frac{-x}{2}} - (x^2+2x-1)\right)dt + \int_{-1}^{-1/8}\left(-\sqrt{\frac{-x}{2}} - 2x\right)dx$$

$$= \int_{-2}^{-1/8} -\sqrt{\frac{-x}{2}}dx - \left\{\int_{-2}^{-1}(x^2+2x-1)\,dx + \int_{-1}^{-1/8} 2x\,dx\right\} = -\frac{21}{16} - \left\{-\frac{5}{3} - \frac{63}{64}\right\} = \frac{257}{192}$$

S80. Note that

$$P \equiv (b-a, 0)$$



If A is the required bounded area, then

$$A = 2(A_1 + A_2)$$

We have

$$A = 2\left\{\int_{-a}^{b-a}\sqrt{4a(x+a)}\,dx + \int_{b-a}^{b}\sqrt{4b(b-x)}\,dx\right\}$$

$$= 4\left\{\left.\frac{(a(x+a))^{3/2}}{\frac{3}{2}a}\right|_{-a}^{b-a} + \left.\frac{(b(b-x))^{3/2}}{-\frac{3}{2}b}\right|_{b-a}^{b}\right\}$$

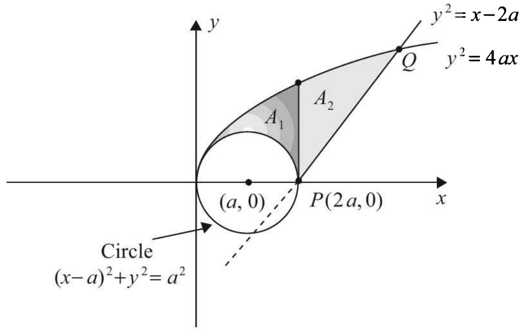
$$= \frac{8}{3}\sqrt{ab}\,(a+b)$$

S81. Note that Q is given by $(x-2a)^2 = 4ax$:

$$\Rightarrow \quad x = (4+2\sqrt{3})a$$

If the total bounded area is A, we have

$$A = A_1 + A_2$$



Thus,

$$A = \underbrace{\int_0^{2a}\left(\sqrt{4ax}-\sqrt{2ax-x^2}\right)dx}_{A_1} + \underbrace{\int_{2a}^{(4+2\sqrt{3})a}(\sqrt{4ax}-(x-2a))dx}_{A_2}$$

However, A_1 can be evaluated more simply as follows:

$$A_1 = \int_0^{2a}\sqrt{4ax}\,dx - (\text{area of semi-circle}) = \frac{2}{3}\sqrt{4a}(x^{3/2})\Big|_0^{2a} - \frac{1}{2}\pi a^2$$

$$= \frac{8\sqrt{2}}{3}a^2 - \frac{\pi}{2}a^2 = \left(\frac{8\sqrt{2}}{3}-\frac{\pi}{2}\right)a^2$$

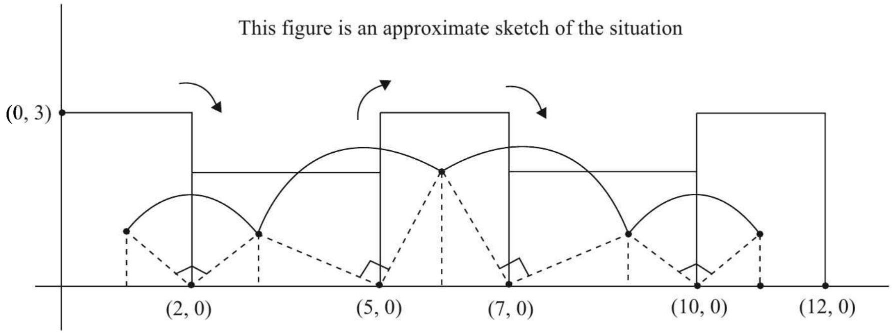
Now,

$$A_2 = \frac{2}{3}\sqrt{4a}\left(x^{\frac{3}{2}}\right)\Bigg|_{2a}^{(4+2\sqrt{3})a} - \left(\frac{x^2}{2}-2ax\right)\Bigg|_{2a}^{(4+2\sqrt{3})a} = \frac{8a^2}{3}(5+3\sqrt{3}-\sqrt{2}) - a^2(8+4\sqrt{3})$$

$$= \left(\frac{16}{3}+4\sqrt{3}-\frac{8\sqrt{2}}{3}\right)a^2$$

$$\Rightarrow \quad A = A_1 + A_2 = \left(\frac{16}{3}+4\sqrt{3}-\frac{\pi}{2}\right)a^2 \text{ sq units}$$

S82.



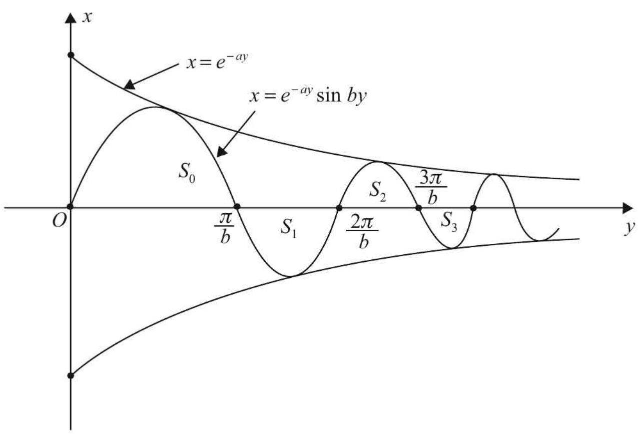
The required area A can be written as a sum of areas of right triangles and circular segments.

$$A = \left\{ \begin{array}{c} A = \left(\text{Area of 4 right triangles of area } \dfrac{1}{2} \right) \\ + \\ (\text{Area of 4 right triangles of area 1}) \\ + \\ \left(\text{2 quarter circles of area } \dfrac{\pi}{4}(\sqrt{2})^2 \right) \\ + \\ \left(\text{2 quarter circles of area } \dfrac{\pi}{4}(\sqrt{5})^2 \right) \end{array} \right\} = 6 + \frac{7\pi}{2}$$

S83. We plot the (approximate) graph of $x = e^{-ay} \sin by$, taking y on the horizontal axis and x on the vertical axis:

Thus,

$$S_j = \int_{\frac{j\pi}{b}}^{\frac{(j+1)\pi}{b}} e^{-ay} \sin by \, dy \tag{1}$$



We first evaluate the indefinite integral $I = \int e^{-ay} \sin by \, dy$, using integration by parts:

$$I = \frac{-e^{ay}\cos by}{b} - \frac{a}{b}\int e^{-ay}\cos by \, dy = \frac{-e^{-ay}\cos by}{b} - \frac{a}{b}\left\{\frac{e^{-ay}\sin by}{b} + \frac{a}{b}\int e^{-ay}\sin by \, dy\right\}$$

$$= \frac{-e^{-ay}}{b}\left(\cos by + \frac{a}{b}\sin by\right) - \frac{a^2}{b^2} I$$

$$\Rightarrow \quad I = \frac{-e^{-ay}(a \sin by + b \cos by)}{a^2 + b^2} \tag{2}$$

Using the integration limits in (1) and simplifying, we have

$$S_j = \underbrace{\left\{\frac{b}{a^2+b^2}\cdot\left(1 + e^{\frac{-a\pi}{b}}\right)\right\}}_{\text{Call this } \lambda} (-1)^j e^{\frac{-aj\pi}{b}} = \lambda(-1)^j e^{\frac{-aj\pi}{b}} \tag{3}$$

Thus,

$$S_{j+1} = \lambda(-1)^{j+1} e^{\frac{-a(j+1)\pi}{b}} \tag{4}$$

Dividing (4) by (3), we have

$$\frac{S_{j+1}}{S_j} = -e^{\frac{-a\pi}{b}}$$

Thus, the sequence $\{S_j\}_{j=0}^{\infty}$ is a GP. The sum S is given by

$$S = S_0 + S_1 + S_2 + \cdots + \infty = \frac{S_0}{1 + e^{\frac{-a\pi}{b}}} = \frac{b}{a^2 + b^2}$$

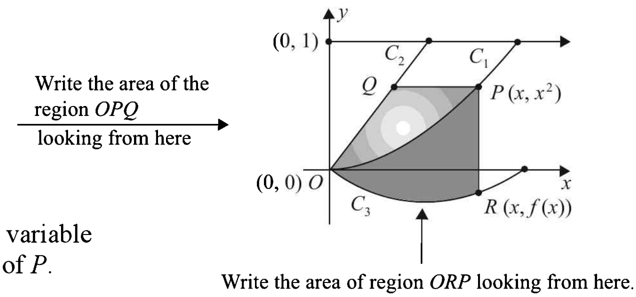
For $\quad a = -1$ and $b = \pi$, this equals $\dfrac{\pi}{1+\pi^2}$.

S84. Note (very!) carefully that the area of region OPQ can be written as

$$\Delta_1 = \int_0^{x^2}\left(\sqrt{y} - \frac{y}{2}\right) dy$$

The area of region ORP can be written as

$$\Delta_2 = \int_0^{x}(x_1^2 - f(x_1))\, dx_1$$



where x_1 has been taken as the integration variable to differentiate it from x, the x-coordinate of P. Since $\Delta_1 = \Delta_2$, we have

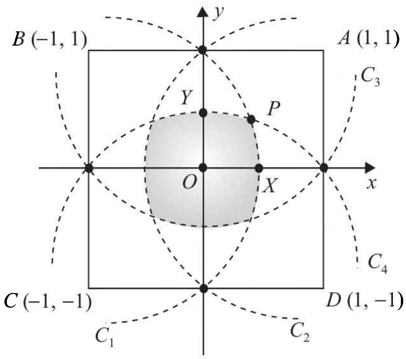
$$\int_0^{x^2}\left(\sqrt{y} - \frac{y}{2}\right) dy = \int_0^{x}(x_1^2 - f(x_1))dx_1$$

Differentiating this with respect to x on both sides, we have

$$\left(x-\frac{x^2}{2}\right)\cdot 2x = x^2 - f(x)$$

$$\Rightarrow \quad f(x) = x^3 - x^2$$

S85. Consider the curve C_1. It is the parabola whose focus is O and whose directrix is AD. Thus, all points on C_1 are equidistant from O and AD, while all points to the left of C_1 are closer to O than to AD.



Similarly,

Points to the right of C_2 are closer to O than to BC.
Points above C_3 are closer to O than to CD.
Points below C_4 are closer to O than to AB.

From these, we deduce that the shaded region contains those points that are closer to O than to any side. To find its area, we analyze it more closely in the first quadrant.

We note that $C_1 \equiv y^2 = 1-2x$ and $C_4 \equiv x^2 = 1-2y$.

By symmetry, the x and y coordinates of P must be equal; say $P \equiv (\lambda, \lambda)$. Then,

$$\lambda^2 = 1-2\lambda$$

$$\Rightarrow \lambda = \sqrt{2}-1$$

Thus,

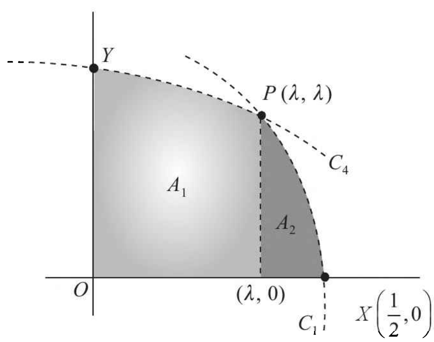
$$A = \underbrace{\int_0^{\lambda}\left(\frac{1-x^2}{2}\right)dx}_{A_1} + \underbrace{\int_{\lambda}^{1/2}\sqrt{1-2x}\,dx}_{A_2}$$

$$= \frac{\sqrt{2}}{3}(\sqrt{2}-1) + \frac{1}{3}(\sqrt{2}-1)^3$$

$$= \frac{4\sqrt{2}-5}{3}$$

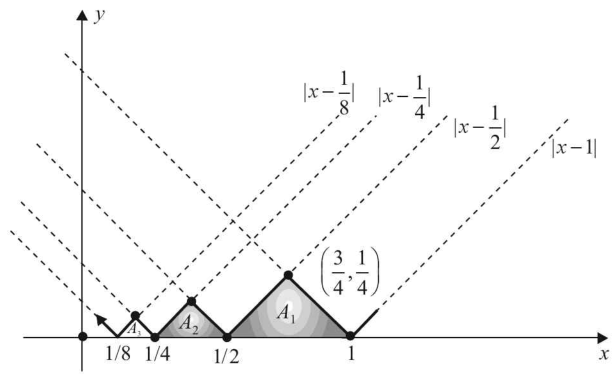
The total area Δ is 4 times A:

$$\Delta = \frac{4}{3}(4\sqrt{2}-5) \text{ sq units}$$



Let the area in the first quadrant be A.
We note that $A = A_1 + A_2$,

S86.



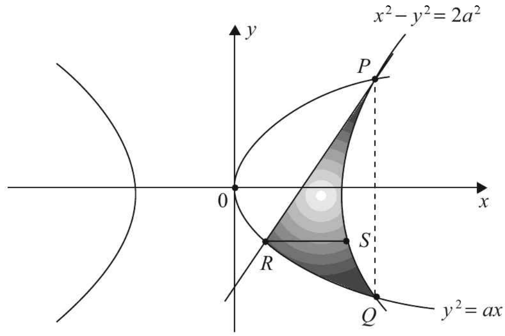
It can easily be deduced that $A_1, A_2, A_3, \ldots$ form an infinite GP with common ratio $\frac{1}{4}$, since the base length and height of each successive triangle is half of the previous triangle. Also, $A_1 = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{16}$. Thus, the total area A is

$$A = \frac{1}{16}\left(1+\frac{1}{4}+\frac{1}{4^2}+\cdots\infty\right) = \frac{\frac{1}{16}}{1-\frac{1}{4}} = \frac{1}{12}\text{sq units}$$

S87. Note that P is the point $(2a, \sqrt{2}a)$.

The tangent at P has the equation $\sqrt{2}x - y = \sqrt{2}a$ and this intersects the parabola again at

$$R \equiv \left(\frac{a}{2}, \frac{-a}{\sqrt{2}}\right)$$



We observe that referred to the y-axis, the hyperbolic segment PQ, the parabolic segment QR and the segment PR have the equations.

$$x = \sqrt{y^2+2a^2},\ x = \frac{y^2}{a},\ x = \frac{1}{\sqrt{2}}y + a$$

Thus, the shaded area Δ can be expressed as

$$\Delta = \text{Area(Region } PSR) + \text{Area(Region } SRQ)$$

$$= \int_{\frac{-a}{\sqrt{2}}}^{\sqrt{2}a}\left\{\sqrt{y^2+2a^2} - \left(\frac{1}{\sqrt{2}}y + a\right)\right\}dy + \int_{-\sqrt{2}a}^{-\frac{a}{\sqrt{2}}}\left(\sqrt{y^2+2a^2} - \frac{y^2}{a}\right)dy$$

$$= a^2\left\{2(\sqrt{2}+\ln(1+\sqrt{2})) - \frac{59}{12\sqrt{2}}\right\}$$

upon simplification. The reader is urged to verify these calculations.

Differential Equation

PART-A: Summary of Important Concepts

1. Fundamentals of Differential Equations

A differential equation can simply be said to be an equation involving derivatives of an unknown function. For example, consider the equation

$$\frac{dy}{dx} + xy = x^2$$

This is a differential equation since it involves the derivative of the function $y(x)$ which we may wish to determine. The following is an example demonstrating how a differential equation arises in a practical scenario. A body is released at rest from a height *h*. How do we described the motion of this body? The height *x* of the body is a function of time. Since the acceleration of the body is *g*, we have

$$\frac{d^2x}{dt^2} = -g$$

This is the differential equation describing the motion of the body. Along with the initial condition $x(0) = h$, it completely describes the motion of the body at all instants after the body starts falling. In this differential equation, there is only one independent variable (the time *t*). Such equations are termed *ordinary differential equations*. We might have equations involving more than one independent variable:

$$\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = x^2$$

where the notation $\frac{\partial}{\partial x}$ stands for the partial derivative, *i.e.*, the term $\frac{\partial f}{\partial x}$ would imply that we differentiate the function *f* with respect to the independent variable *x* as the variable (while treating the other independent variable *y* as a constant). A similar interpretation can be attached to $\frac{\partial}{\partial y}$. Such equations are termed *partial differential equations* but we'll not be concerned with them in this chapter.
Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 = c$$

The order of the highest derivative present in this equation is two; thus, we'll call it a *second order* differential equation (DE, for convenience). The *order* of DE is the order of the highest derivative that occurs in the equation. Again, consider the DE

$$\left(\frac{d^3y}{dx^3}\right)^2+\frac{dy}{dx}=x^2y^2$$

The degree of the highest order derivative in this DE is two, so this is a DE of *degree two* (and order three). The *degree* of a DE is the degree of the highest order derivative that occurs in the equation, when all the derivatives in the equation are made of free of fractional powers. For example, the DE

$$\sqrt{\left(\frac{dy}{dx}\right)^2-1}+x\left(\frac{d^2y}{dx^2}\right)^2=k$$

is not of degree two. When we make this equation free of fractional powers, by the following rearrangement,

$$\left(\frac{dy}{dx}\right)^2-1=\left\{k-x\left(\frac{d^2y}{dx^2}\right)^2\right\}^2,$$

we see that the degree of the highest order derivative will become four. Thus, this is a DE of degree four (and order two). Finally, an *n*th *linear DE* (degree one) is an equation of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=b$$

where the a_i's and b are functions of x. Solving an nth order DE to evaluate the unknown function will essentially consist of doing n integrations on the DE. Each integration step will introduce an arbitrary constant. Thus, you can expect in general that the *solution of an nth order DE will contain n independent arbitrary constants*. By n independent constants, we mean to say that the most general solution of the DE cannot be expressed in fewer than n constants. The following example illustrates the formation of a differential equation corresponding to a specific curve.

Illustration 1: Find the DE associated with the family of circles of a fixed radius r.

Working: The circles are of a fixed radius but their centres are not. Let the centre be denoted by the variable point (h,k). Then, the equation of an arbitrary circle of the family is

$$(x-h)^2+(y-k)^2=r^2 \tag{1}$$

This contains two arbitrary constants and therefore will give rise to a second-order DE. Differentiating (1), we have

$$(x-h)+(y-k)\frac{dy}{dx}=0 \tag{2}$$

Differentiating (2) again, we have

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \tag{3}$$

$$\Rightarrow\quad (y-k)=-\frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}} \tag{4}$$

Using (4) in (2), we have

$$(x-h)=\frac{\{1+(\frac{dy}{dx})^2\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \tag{5}$$

Using (4) and (5) in (1), and simplifying, we have the required DE as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which as expected, is second order.

2. Solving Differential Equations

We will be considering only first order and first degree DEs. Note that any such DE can be written in the general form

$$\boxed{M(x,y)dx+N(x,y)dy=0}$$

where M and N are functions of x and y.

2.1 Variable-separable Form

This is by and large the simplest type of DE that we will encounter. As the name suggests, in such an equation, M is a function of x only and N is a function of y only. Thus, such a DE is of the form

$$f(x)dx+g(y)dy=0$$

which can be solved by straightforward integration to obtain

$$\int f(x)dx+\int g(y)dy=C,$$

where C is an arbitrary constant. Observe how the *variables are separated* in this type of DE and its general solution. As a simple example, consider the DE

$$xdx+y^2dy=0$$

This is obviously in variable-separable form. Integrating, we obtain

$$\int xdx+\int y^2dy=C \quad\Rightarrow\quad \frac{x^2}{2}+\frac{y^3}{3}=C$$

This is the required general solution of the DE. Sometimes, the DE might not be in the variable-separable (VS) form; however, some manipulations might be able to transform it to a VS form. Lets see how this can be done. Consider the DE

$$\frac{dy}{dx}=\cos(x+y)$$

This is obviously not in VS form. Observe what happens if we use the following substitution in this DE:

$$x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx}$$

Thus, the DE transforms to

$$\frac{dv}{dx}-1=\cos v \quad\Rightarrow\quad \frac{dv}{dx}=1+\cos v \quad\Rightarrow\quad \frac{dv}{1+\cos v}=dx$$

which is clearly in VS form. Integrating both sides, we obtain

$$\int\frac{dv}{1+\cos v}=\int dx \quad\Rightarrow\quad \frac{1}{2}\int\sec^2\frac{v}{2}\,dv=\int dx$$

$$\Rightarrow\quad \tan\frac{v}{2}=x+C \quad\Rightarrow\quad \tan\left(\frac{x+y}{2}\right)=x+C$$

This is the required general solution to the DE. Any DE of the form

$$\frac{dy}{dx}=f(ax+by+c)$$

is reducible to a VS form using the technique described.

2.2 Homogenous Differential Equations

By definition, a homogeneous function $f(x,y)$ of degree n satisfies the property

$$f(\lambda x,\lambda y)=\lambda^n f(x,y)$$

For example, the functions

$$f_1(x,y)=x^3+y^3$$
$$f_2(x,y)=x^2+xy+y^2$$
$$f_3(x,y)=x^3e^{x/y}+xy^2$$

are all homogeneous functions, of degrees three, two and three respectively (verify this). Observe that any homogeneous function $f(x,y)$ of degree n can be equivalently written as follows:

$$f(x,y)=x^n f\left(\frac{y}{x}\right)=y^n f\left(\frac{x}{y}\right)$$

For example,

$$f_1(x,y)=x^3+y^3=x^3\left(1+\left(\frac{y}{x}\right)^3\right)=y^3\left(1+\left(\frac{x}{y}\right)^3\right)$$

We define *homogeneous DEs* as follows. Any DE of the form $M(x,y)\,dx+N(x,y)\,dy=0$ or $\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree. What is so special about homogeneous DEs? Well, it turns out that they are extremely simple to solve. To see how, we express both $M(x,y)$ and $N(x,y)$ as, say $x^n M(\frac{y}{x})$ and $x^n N(\frac{y}{x})$. This can be done since $M(x,y)$ and $N(x,y)$ are both homogeneous functions of degree n. Doing this reduces our DE to

$$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}=-\frac{x^n M(\frac{y}{x})}{x^n N(\frac{y}{x})}=-\frac{M(\frac{y}{x})}{N(\frac{y}{x})}=P(\tfrac{y}{x})$$

The function $P(t)$ stands for $\frac{-M(t)}{N(t)}$. Now, the simple substitution $y = vx$ reduces this DE to a VS form:

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus, $\frac{dy}{dx} = P(\frac{y}{x})$ transforms to

$$v + x\frac{dv}{dx} = P(v) \quad \Rightarrow \quad \frac{dv}{P(v) - v} = \frac{dx}{x}$$

This can now be integrated directly since it is in VS form.

Illustration 2: Solve the DE $\dfrac{dy}{dx} = \dfrac{2x - y}{x + y}$.

Working: This is a homogeneous DE of degree one since the RHS can be written as

$$\frac{2x - y}{x + y} = \frac{x \cdot (2 - \frac{y}{x})}{x \cdot (1 + \frac{y}{x})} = \frac{2 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Using the substitution $y = vx$ reduces this DE to

$$v + x\frac{dv}{dx} = \frac{2 - v}{1 + v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2 - v}{1 + v} - v = \frac{2 - 2v - v^2}{1 + v} = \frac{3 - (1 + v)^2}{1 + v}$$

$$\Rightarrow \quad \frac{(1 + v)}{3 - (1 + v)^2}dv = \frac{dx}{x}$$

Using $t = 1 + v$ above, we have

$$\frac{t}{3 - t^2}dt = \frac{dx}{x}$$

Integrating, we have

$$\int \frac{t}{3 - t^2}dt = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{2}\ln|3 - t^2| = \ln x + \ln C_1 \quad \Rightarrow \quad \ln(x^2(3 - t^2)) = C_2$$

$$\Rightarrow \quad x^2(3 - t^2) = C \quad \Rightarrow \quad x^2(3 - (1 + v)^2) = C$$

$$\Rightarrow \quad x^2(2 - 2v - v^2) = C$$

Substituting $\frac{y}{x}$ for v, we finally obtain the required general solution to the DE:

$$2x^2 - 2xy - y^2 = C$$

Many a times, the DE specified may not be homogeneous but some suitable manipulation might reduce it to a homogeneous form. Generally, such equations involve a function of a rational expression whose numerator and denominator are linear functions of the variable, *i.e.*, of the form

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+cy+f}\right) \tag{1}$$

Note that the presence of the constant c and f causes this DE to be non-homogeneous. To make it homogeneous, we use the substitutions

$$x \to X + h$$
$$y \to Y + k$$

and select h and k so that

$$\left.\begin{aligned} ah + bk + c &= 0 \\ dh + ek + f &= 0 \end{aligned}\right\} \tag{2}$$

This can always be done (if $\frac{a}{b} \neq \frac{d}{e}$). The RHS of the DE in (1) now reduces to

$$f\left(\frac{a(X+h)+b(Y+k)+c}{d(X+h)+e(Y+k)+f}\right) = f\left(\frac{aX+bY+(ah+bk+c)}{dX+eY+(dh+ek+f)}\right)$$

$$= f\left(\frac{aX+bY}{dX+eY}\right) \quad \text{(Using (2))}$$

This expression is clearly homogeneous. The LHS of (1) is $\frac{dy}{dx}$ which equals $\frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx}$. Since $\frac{dy}{dY}\cdot\frac{dx}{dX} = 1$, the LHS $\frac{dy}{dx}$ equals $\frac{dY}{dX}$. Thus, our equation becomes

$$\frac{dY}{dX} = f\left(\frac{aX+bY}{dX+eY}\right) \tag{3}$$

We have thus succeeded in transforming the non-homogeneous DE in (1) to the homogeneous DE in (3). This can now be solved as described earlier.

Illustration 3: Solve the DE $\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$.

Working: We substitute $x \to X + h$ and $y \to Y + k$ where h, k need to be determined:

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{(2Y-X)+(2k-h-4)}{(Y-3X)+(k-3h+3)}$$

h and k must be chosen so that

$$2k - h - 4 = 0$$
$$k - 3h + 3 = 0$$

This gives $h = 2$ and $k = 3$. Thus,

$$x = X + 2$$
$$y = Y + 3$$

Our DE now reduces to

$$\frac{dY}{dX} = \frac{2Y - X}{Y - 3X}$$

Using the substitution $Y = vX$, and simplifying, we have (verify)

$$\frac{v-3}{v^2 - 5v + 1} dv = \frac{-dX}{X}$$

We now integrate this DE which is in VS form; finally, we substitute $v = \frac{Y}{X}$ and

$$X = x - 2$$
$$Y = y - 3$$

to obtain the general solution.
Note that the system

$$ah + bk + c = 0$$

$$dh + ek + f = 0$$

will not yield a solution if $\frac{a}{b} = \frac{d}{e}$. How do we reduce the DE to a homogeneous one in such a case? Let $\frac{a}{d} = \frac{b}{e} = \lambda$ (say). Thus,

$$\frac{ax + by + c}{dx + ey + f} = \frac{\lambda(dx + ey) + c}{dx + ey + f}$$

This suggests the substitution $dx + ey = v$, as in the following example.

Illustration 4: Solve the DE $\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1}$.

Working: Note that h, k do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

$$x + 2y = v \quad \Rightarrow \quad 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

Thus, our DE becomes

$$\frac{1}{2}\left(\frac{dv}{dx} - 1\right) = \frac{v-1}{v+1} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{2v-2}{v+1} + 1 = \frac{3v-1}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{3v-1} dv = dx \quad \Rightarrow \quad \frac{1}{3}\left(1 + \frac{4}{3v-1}\right) dv = dx$$

Integrating, we have

$$\frac{1}{3}\left(v + \frac{4}{3}\ln(3v-1)\right) = x + C_1$$

Substituting $v = x + 2y$, we have

$$x + 2y + \frac{4}{3}\ln(3x + 6y - 1) = 3x + C_2 \quad \Rightarrow \quad y - x + \frac{2}{3}\ln(3x + 6y - 1) = C$$

2.3 First Order Differential Equations

One of the most important classes of DEs is first-order linear DEs, their importance arising from the fact that many natural phenomena can be described using such DEs. First order linear DEs take the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x alone. To solve such DEs, we proceed as follows: we multiply both sides of the DE by a quantity called the *integrating factor* (IF) where

$$\text{IF}=e^{\int Pdx}$$

Why this is chosen as the IF will soon become clear when we see what the IF actually does:

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=e^{\int Pdx}\cdot Q$$

The left hand side now becomes exact, in the sense that it can be expressed as the exact differential of some expression:

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=\frac{d}{dx}\left(ye^{\int Pdx}\right)$$

Now our DE becomes

$$\frac{d}{dx}\left(ye^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

This can now easily be integrated to yield the required general solution:

$$ye^{\int Pdx}=\int\left(Qe^{\int Pdx}\right)dx+C$$

Here's an example:

Illustration 5: Solve the DE $\dfrac{dy}{dx}+y\tan x=\cos x$.

Solution: Comparing this DE with the standard form of the linear DE $\frac{dy}{dx}+Py=Q$, we see that

$$P(x)=\tan x,\quad Q(x)=\cos x$$

Thus, the IF is

$$\text{IF}=e^{\int\tan x dx}=e^{\ln(\sec x)}=\sec x$$

Multiplying by $\sec x$ on both sides of the given DE, we obtain

$$\sec x\frac{dy}{dx}+y\tan x\sec x=1$$

The left hand side is an exact differential:

$$\frac{d}{dx}(y\sec x)=1\quad\Rightarrow\quad d(y\sec x)=dx$$

Integrating both sides, we obtain the solution to our DE as

$$y \sec x = x + C$$

Suppose that we have a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

We can reduce it to a first-order linear DE form by dividing by y^n on both sides:

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$$

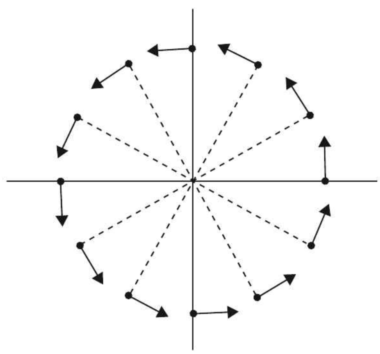
and then substituting $y^{-n+1} = v$.

IMPORTANT IDEAS AND TIPS

1. *Physical Significance of DEs:* Think of the DE of a curve as the equation which the slope of the curve satisfies as different points on the curve are taken. For example, lets interpret the DE

$$\frac{dy}{dx} = -\frac{x}{y}$$

geometrically, without solving the DE explicitly. The equation says that there is a curve, at each of whose point (x, y), the slope has the value $-x/y$. Note that the slope of the line joining the point (x, y) to the origin is y/x. Therefore, what the DE says is that the tangent to the curve at each point is perpendicular to the line joining that point to the origin. Geometrically, we could represent this fact by the following diagram, where for different points on the curve, the tangent vectors are all perpendicular to the lines joining the respective points to the origin.



It is visually obvious that this curve must be a circle. Verify this fact analytically. Note that the radius of the circle does not figure anywhere. Every circle centered at the origin will satisfy this DE. We need an additional constraint to be able to fix the radius. The *vector* diagram which we have drawn above can in principle be drawn for any DE. Try doing the same for the following DEs, and intuitively determining the curves which satisfy them (without solving them explicitly):

$$\frac{dy}{dx} = k \text{ (a constant)}, \quad \frac{dy}{dx} = \frac{y}{x}, \quad \frac{dy}{dx} = -\frac{x-1}{y-2}$$

2. *First Order Linear DEs:* We have seen that a first order linear DE is of the form $y' + Py = Q$, and we have seen how to solve it by multiplying both sides with the integrating factor $e^{\int P dx}$. A very important point that must always be remembered is that P and Q must be functions of x (the independent variable) alone. For example, the DE

$$\frac{dy}{dx} + xy = \frac{\sqrt{x}}{y}$$

is *not* a first order linear DE, as the right hand side (Q) is not a function of x alone.

3. *Mathematical Modeling:* The most important skill in this chapter is to be able to represent physical situations in terms of differential equations. For that, the concept of rate of change and its physical implications must be absolutely clear to you. For example, you should be able to immediately transform a statement like '*the rate of population growth is inversely proportional to the current population*' into an equivalent mathematical model in terms of differential equations. This skill gets developed with a lot of problem-solving.

Differential Equations

PART-B: Illustrative Examples

OBJECTIVE TYPE EXAMPLES

Example 1

Which of the following is the DE associated with the family of straight lines, each of which is at a constant distance p from the origin?

(A) $(xy' + y)^2 = p^2(1-(y')^2)$ (C) $(y' - xy)^2 = p^2(1+(y')^2)$

(B) $(xy' - y)^2 = p^2(1+(y')^2)$ (D) None of these

Solution: Any such line has the equation

$$x\cos\alpha + y\sin\alpha = p \tag{1}$$

where α is a variable. Different values of α give different lines belonging to this family. Since the equation representing this family contains only one arbitrary constant, its corresponding DE will be first order. Differentiating(1), we have

$$\cos\alpha + y'\sin\alpha = 0$$

$$\Rightarrow \quad \tan\alpha = -\frac{1}{y'}$$

$$\Rightarrow \quad \sin\alpha = \frac{-1}{\sqrt{1+(y')^2}}, \quad \cos\alpha = \frac{y'}{\sqrt{1+(y')^2}} \tag{2}$$

Using (2) in (1), we have

$$\frac{xy'}{\sqrt{1+(y')^2}} - \frac{y}{\sqrt{1+(y')^2}} = p$$

$$\Rightarrow \quad (xy' - y)^2 = p^2(1+(y')^2)$$

The correct option is therefore (B). As expected, this is a first order DE. ■

Example 2

Which of the following is the solution to the DE $xy^2 \dfrac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$?

(A) $y + 2\tan^{-1} y = \ln x + \dfrac{x^2}{3} + C$ (B) $y - \dfrac{1}{2}\tan^{-1} y = \ln x + \dfrac{x^2}{2} + C$

(C) $y - \tan^{-1} y = \ln x - \dfrac{x^2}{2} + C$ (D) None of these

Solution: This DE is of the variable separable form, as can be made evident by a slight rearrangement:

$$xy^2 \frac{dy}{dx} = (1-x^2)(1+y^2)$$

$$\Rightarrow \left(\frac{y^2}{1+y^2}\right) dy = \left(\frac{1-x^2}{x}\right) dx$$

$$\Rightarrow \left(1 - \frac{1}{1+y^2}\right) dy = \left(\frac{1}{x} - x\right) dx$$

Integrating both sides, we have

$$y - \tan^{-1} y = \ln x - \frac{x^2}{2} + C$$

This is the required general solution. The correct option is (C). ■

Example 3

What is the solution to the DE $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy = 0$?

(A) $e^{\frac{x}{y}} + \dfrac{x}{y} = \dfrac{C}{y}$ (C) $e^{\frac{x}{y}} + \dfrac{x}{y} = \dfrac{C}{x}$ (E) None of these

(B) $e^{\frac{x}{y}} - \dfrac{x}{y} = \dfrac{C}{y}$ (D) $e^{\frac{x}{y}} - \dfrac{x}{y} = \dfrac{C}{x}$

Solution: This DE can be rearranged to

$$\frac{dx}{dy} = \frac{e^{\frac{x}{y}}(\frac{x}{y} - 1)}{e^{\frac{x}{y}} + 1}$$

Using the substitution $x = vy$ (note : not $y = vx$) can reduce this DE to a VS form. We did not use $y = vx$ since that would've led to an expression involving complicated exponentials. We now have

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v + 1} \quad \Rightarrow \quad \frac{dy}{y} = -\frac{e^v + 1}{e^v + v}dv$$

Integrating both sides, we have

$$\ln y = -\ln\left|e^v + v\right| + \ln C \quad \Rightarrow \quad y(e^v + v) = C \quad \Rightarrow \quad e^{\frac{x}{y}} + \frac{x}{y} = \frac{C}{y}$$

The correct option is (A). ■

Example 4

What is the general solution to the DE $x\cos(\frac{y}{x})(ydx + xdy) = y\sin(\frac{y}{x})(xdy - ydx)$?

(A) $\dfrac{x}{y} = C\sec\left(\dfrac{y}{x}\right)$ (C) $\dfrac{x}{y} = C\sec\left(\dfrac{x}{y}\right)$ (E) None of these

(B) $xy = C\sec\left(\dfrac{y}{x}\right)$ (D) $xy = C\sec\left(\dfrac{x}{y}\right)$

Solution: Upon rearrangement, we have

$$ydx + xdy = \frac{y}{x}\tan\frac{y}{x}(xdy - ydx) = xy\tan\frac{y}{x}\left(\frac{xdy - ydx}{x^2}\right)$$

$$\Rightarrow \quad \frac{ydx + xdy}{xy} = \tan\frac{y}{x}\left(\frac{xdy - ydx}{x^2}\right)$$

This can now be written as

$$\frac{d(xy)}{xy} = \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

The solution is now obtained simply by integrating both sides:

$$\ln(xy) = \ln\left(\sec\left(\frac{y}{x}\right)\right) + \ln C \quad \Rightarrow \quad xy = C\sec\left(\frac{y}{x}\right)$$

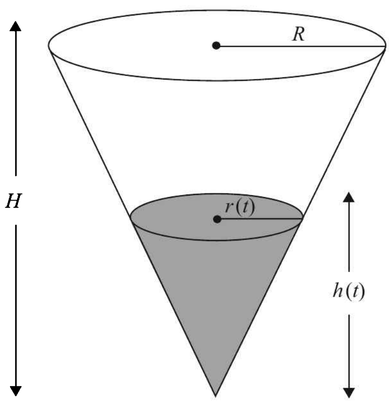
The correct option is (B). ■

Example 5

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = $k > 0$). What is the time after which the cone is empty?

(A) $\dfrac{H}{3k}$ (B) $\dfrac{H}{2k}$ (C) $\dfrac{H}{k}$ (D) $\dfrac{2H}{k}$ (E) None of these

Solution: We need to form a differential equation which describes the variation of the amount of water left in the cone with time.



Let us denote the height of the water remaining in the cup at time t by $h(t)$. Denote the volume at time t by $v(t)$

From the geometry described in the figure above,

$$\frac{r(t)}{h(t)} = \frac{R}{H} \quad \Rightarrow \quad h(t) = \frac{H}{R} r(t)$$

The volume $v(t)$ of the cone is

$$v(t) = \frac{1}{3}\pi r^2(t)h(t) \quad = \frac{\pi H}{3R} r^3(t) \tag{1}$$

Now, it is specified that the rate of evaporation (the rate of decrease of the water's volume) is proportional to the surface area in contact with air:

$$\frac{dv}{dt} = -k\pi r^2 \tag{2}$$

From (1) and (2), we have

$$\frac{H}{R} r^2 \frac{dr}{dt} = -kr^2 \quad \Rightarrow \quad \frac{dr}{dt} = -\frac{Rk}{H}$$

This is the DE representing the variation in the radius of the water surface with time. The inital radius is R and the final radius is 0. If the time taken for the entire water to evaporate is T, we have

$$\int_R^0 dr = -\frac{Rk}{H}\int_0^T dt \quad \Rightarrow \quad T = \frac{H}{k}$$

The correct option is (C). Note that the time taken is independent of the radius of the cone and depends only on its height. Thus, for example, two cones full of water, with the same height, but one of them having a radius say a 1000 times larger than the other, will become empty in the same amount of time! Can you see why? ■

Example 6

A and B are two separate reservoirs of water. The capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then water is released simultaneuously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water?

(A) $\dfrac{\ln 2}{\ln(\frac{4}{3})}$ hours (B) $\dfrac{\ln 2}{\ln(\frac{8}{3})}$ hours (C) $\dfrac{\ln 4}{\ln(\frac{4}{3})}$ hours (D) $\dfrac{\ln 4}{\ln(\frac{8}{3})}$ hours (E) None of these

Solution: Assume the intial volumes of water in A and B to be $2V$ and V. Denote the volume of water in A and B by v_1 and v_2 respectively. We have,

$$\frac{dv_1}{dt} = -k_A v_1, \quad \frac{dv_2}{dt} = -k_B v_2$$

where k_A and k_B are constants of proportionality (not given). These two DEs are in VS form and the solution can be obtained by simple integration.

$$\int_{2V}^{v_1(t)} \frac{dv_1}{v_1} = \int_0^t -k_A dt, \quad \int_V^{v_2(t)} \frac{dv_2}{v_2} = \int_0^t -k_B\, dt$$

$$\Rightarrow \quad \ln\frac{v_1(t)}{2V} = -k_A t, \quad \ln\frac{v_2(t)}{V} = -k_B t$$

$$\Rightarrow \quad v_1(t) = 2Ve^{-k_A t}, \quad v_2(t) = Ve^{-k_B t}$$

It is given that

$$v_1(t=1) = \frac{3}{2}v_2(t=1) \quad \Rightarrow \quad 2Ve^{-k_A} = \frac{3}{2}Ve^{k_B}$$

$$\Rightarrow \quad e^{(k_B - k_A)} = \frac{3}{4} \quad \Rightarrow \quad k_B - k_A = \ln\left(\frac{3}{4}\right)$$

Let T be the time at which the volumes in the two reservoirs become equal. We thus have,

$$v_1(t=T) = v_2(t=T) \quad \Rightarrow \quad 2Ve^{-k_A T} = Ve^{-k_B T}$$

$$\Rightarrow \quad e^{(k_A - k_B)T} = 2 \quad \Rightarrow \quad (k_A - k_B)T = \ln 2$$

$$\Rightarrow \quad T\ln\left(\frac{4}{3}\right) = \ln 2 \quad \Rightarrow \quad T = \frac{\ln 2}{\ln(\frac{4}{3})}\text{hours}$$

The correct option is (A). ■

SUBJECTIVE TYPE EXAMPLES

Example 7

Solve the DE $\dfrac{dy}{dx} = \dfrac{(x+y)+(x+y-1)\ln(x+y)}{\ln(x+y)}$.

Solution: The substitution $x + y = v$ will reduce this DE to the following VS form:

$$\frac{dv}{dx} - 1 = \frac{v + (v-1)\ln v}{\ln v} = (v-1) + \frac{v}{\ln v}$$

$$\Rightarrow \quad \frac{dv}{dx} = v + \frac{v}{\ln v} \quad \Rightarrow \quad \frac{\ln v}{v(1+\ln v)} dv = dx$$

Integrating, we have

$$\int \frac{\ln v}{v(1+\ln v)} dv = \int dx$$

To evaluate the integral on the LHS, we use the substitution $(1+\ln v) = t$, which gives $\frac{1}{v} dv = dt$. Thus,

$$\int \frac{t-1}{t} dt = \int dx \quad \Rightarrow \quad t - \ln t = x + C$$

$$\Rightarrow \quad (1+\ln v) - \ln(1+\ln v) = x + C$$

$$\Rightarrow \quad (1+\ln(x+y)) - \ln(1+\ln(x+y)) = x + C$$

■

Example 8

Solve the DE $\dfrac{xdy}{x^2+y^2} = \left(\dfrac{y}{x^2+y^2} - 1\right) dx$.

Solution: Upon rearrangement, this DE gives

$$\frac{xdy - ydx}{x^2+y^2} = -dx \tag{1}$$

We see that the left hand side of (1) is the exact differential $d(\tan^{-1} \frac{y}{x})$. Thus, our DE reduces to

$$d\left(\tan^{-1}\frac{y}{x}\right) + dx = 0$$

Integrating, we obtain the solution as

$$\tan^{-1}\frac{y}{x} + x = C$$

In most cases, it is very likely that we won't be able to make out just by inspection whether the DE is exact or not. If the DE is not exact, it can be rendered exact by multiplying it with an integrating factor IF. In the case of the first-order linear DE

$$\frac{dy}{dx} + Py = Q\,,$$

the IF $e^{\int Pdx}$ renders the DE exact:

$$\frac{d}{dx}\left(ye^{\int Pdx} \right) = Qe^{\int Pdx}$$

and the solution is now obtainable by integration. If fact, a systematic approach exists to determine the IF in a general case (if such an IF is possible at all.). However, we'll not be discussing that approach here since it is beyond our current scope of discussion. ■

Example 9

Solve the DE $\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$.

Solution: We substitute $x \to X + h$ and $y \to Y + k$ where h, k need to be determined:

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{(2Y - X) + (2k - h - 4)}{(Y - 3X) + (k - 3h + 3)}$$

h and k must be chosen so that

$$2k - h - 4 = 0$$
$$k - 3h + 3 = 0$$

This gives $h = 2$ and $k = 3$. Thus,

$$x = X + 2$$
$$y = Y + 3$$

Our DE now reduces to

$$\frac{dY}{dX} = \frac{2Y - X}{Y - 3X}$$

Using the substitution $Y = vX$, and simplifying, we have (verify),

$$\frac{v-3}{v^2 - 5v + 1}dv = \frac{-dX}{X}$$

We now integrate this DE which is VS; finally, we substitute $v = \frac{Y}{X}$ and

$$X = x - 2$$
$$Y = y - 3$$

to obtain the general solution. Completing the solution is left to the reader as an exercise. ■

Example 10

Solve the DE $\frac{dy}{dx} = x^3y^3 - xy$.

Solution: We have,

$$\frac{dy}{dx} + xy = x^3 y^3$$

Note that since the RHS contains the term y^3, this DE is not in the standard first order linear DE form. However, a little artifice can enable us to reduce this to the standard form. We divide both sides of the equation by y^3:

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3 \tag{1}$$

Substitute $\dfrac{1}{y^2} = v$:

$$\Rightarrow \quad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx} \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx} \tag{2}$$

Using (2) in (1), we have

$$\frac{-1}{2}\frac{dv}{dx} + xv = x^3 \quad \Rightarrow \quad \frac{dv}{dx} + (-2x)v = -2x^3 \tag{3}$$

This is now in the standard first-order linear DE form. The IF is

$$\text{IF} = e^{\int -2x dx} = e^{-x^2}$$

Thus, the solution to (3) is

$$v \times \text{IF} = \int Q(x) \times (\text{IF}) dx$$

$$\Rightarrow \quad ve^{-x^2} = -2\int x^3 e^{-x^2} dx$$

Performing the integration on the RHS by the substitution $t = -x^2$ and then using integration by parts, we obtain

$$ve^{-x^2} = e^{-x^2}(x^2+1) + C \quad \Rightarrow \quad \frac{1}{y^2}e^{-x^2} = e^{-x^2}(x^2+1) + C$$

This is the required general solution to the DE. ■

Example 11

Solve the following DEs

(a) $\dfrac{dy}{dx} = \dfrac{\cos x(2\cos y - \sin^2 x)}{\sin y}$ (b) $\left(y\dfrac{dy}{dx} + 2x\right)^2 = \left(1 + \left(\dfrac{dy}{dx}\right)^2\right)(y^2 + 2x^2)$

Solution: (a) We have,

$$\sin y\frac{dy}{dx} - 2\cos y\cos x = -\sin^2 \cos x \tag{1}$$

Observe that the substitution $-\cos y = z$ will reduce this DE to a standard linear DE:

$$-\cos y = z \quad \Rightarrow \quad \sin y \frac{dy}{dx} = \frac{dz}{dx} \tag{2}$$

Using (2) in (1), we have

$$\frac{dz}{dx} + (2\cos x)z = -\sin^2 x \cos x$$

The IF for this DE is $e^{\int 2\cos x dx} = e^{2\sin x}$. Thus, the solution will be given by

$$ze^{2\sin x} = -\int e^{2\sin x} \cos x \cdot \sin^2 x \, dx \tag{3}$$

To integrate the RHS, we use the substitution $\sin x = t \Rightarrow \cos x dx = dt$. Thus, the integral reduces to

$$\begin{aligned} -\int t^2 e^{2t}\, dt &= -\frac{t^2 e^{2t}}{2} - \frac{e^{2t}}{4} + \frac{te^{2t}}{2} + C' \text{ (Integration by parts)} \\ &= -\frac{\sin^2 x \cdot e^{2\sin x}}{2} - \frac{e^{2\sin x}}{4} + \frac{\sin x \cdot e^{2\sin x}}{2} + C' \end{aligned}$$

Finally, the solution to the DE is, from (3),

$$\begin{aligned} & z = -\cos y = -\frac{\sin^2 x}{2} - \frac{1}{4} + \frac{\sin x}{2} + C'e^{-2\sin x} \\ \Rightarrow \quad & 4\cos y = 2\sin^2 x - 2\sin x + 1 + Ce^{-2\sin x} \end{aligned}$$

(b) Let $\frac{dy}{dx} = p$. Thus, this DE is

$$\begin{aligned} & (py + 2x)^2 = (1 + p^2)(y^2 + 2x^2) \\ \Rightarrow \quad & 2x^2 p^2 - 4xyp + y^2 - 2x^2 = 0 \\ \Rightarrow \quad & p = \frac{4xy \pm \sqrt{16x^2y^2 - 8x^2(y^2 - 2x^2)}}{4x^2} = \frac{4xy \pm \sqrt{8x^2y^2 + 16x^4}}{4x^2} \\ & = \frac{4xy \pm 2\sqrt{2}\, x\sqrt{y^2 + 2x^2}}{4x^2} = \frac{y}{x} \pm \frac{\sqrt{y^2 + 2x^2}}{\sqrt{2}x} = \frac{y}{x} \pm \sqrt{\frac{1}{2}\left(\frac{y}{x}\right)^2 + 1} \end{aligned}$$

The substitution $y = vx$ reduces this DE to

$$\begin{aligned} & v + x\frac{dv}{dx} = v \pm \sqrt{\frac{v^2}{2} + 1} \\ \Rightarrow \quad & \frac{dv}{\sqrt{v^2 + 2}} = \pm \frac{1}{\sqrt{2}} \frac{dx}{x} \end{aligned}$$

Integrating both sides, we have

$$\ln|v+\sqrt{v^2+2}| = \pm\frac{1}{\sqrt{2}}\ln x + C'$$

$$\Rightarrow \quad \ln x \pm \sqrt{2}\ln|v+\sqrt{v^2+2}| = C$$

$$\Rightarrow \quad \ln x \pm \sqrt{2}\ln\left|\frac{y}{x}+\frac{\sqrt{y^2+2x^2}}{x}\right| = C \tag{4}$$

Thus, we obtain two different solutions to the DE, one corresponding to the ' + ' and one to the ' − ' sign in (4). ■

Example 12

Let $f:\mathbb{R}^+ \to \mathbb{R}$ be a differentiable function such that

$$f(x) = e-(x-1)\ln\frac{x}{e}+\int_1^x f(x)dx$$

Find a simple expression for $f(x)$.

Solution: Differentiating the given relation, we have

$$\frac{df}{dx} = -\frac{(x-1)}{x}-\ln\frac{x}{e}+f$$

$$\Rightarrow \quad \frac{df}{dx}-f = \frac{1}{x}-\ln x$$

This is evidently a first-order linear DE; the IF is $e^{\int -dx} = e^{-x}$. Multiplying it across both sides of the DE renders the DE exact and its solution is given by

$$e^{-x}\cdot f = \int e^{-x}\left(\frac{1}{x}-\ln x\right)dx = e^{-x}\ln x + C$$

$$\Rightarrow \quad f(x) = \ln x + Ce^x \tag{1}$$

From the relation specified in the equation, note that

$$f(1) = e-(1-1)\left(\ln\frac{1}{e}\right)+\int_1^1 f(x)dx = e$$

From (1), $f(1) = Ce$. This gives $C = 1$. Thus, the function $f(x)$ has the simple form

$$f(x) = \ln x + e^x$$

■

Example 13

A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the mid-point of AB. C passes through (1, 1). Determine its equation.

Solution: Let the curve be $y = f(x)$. The tangent at any point $P(x, y)$ has the equation

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets the axes in $A(x - y\frac{dx}{dy}, 0)$ and $B(0, y - x\frac{dy}{dx})$. Since P itself is the mid-point of AB, we have

$$x - y\frac{dx}{dy} = 2x, y - x\frac{dy}{dx} = 2y \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

This is in VS form and can be solved by straight forward integration:

$$\int\frac{dy}{y} = -\int\frac{dx}{x} \quad \Rightarrow \quad \ln y = -\ln x + \ln k \text{ (k is an arbitrary constant)}$$

$$\Rightarrow \quad xy = k$$

Since the curve C passes through (1, 1), we have $k = 1$. Thus, the equation of C is

$$xy = 1$$

■

Differential Equations

PART-C: Advanced Problems

OBJECTIVE TYPE EXAMPLES

P1. Let m be the order and n the degree of the differential equation

$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}}\right\} = 0, \quad p, q \in \mathbb{Z}$$

The value of $m + n$ is

(A) 4 (B) 5 (C) 6 (D) 7

P2. Consider a family of curves given by

$$y = A\cos(x + C) + B\sin(x + C)$$

where A, B, C are constants. The order of the differential equation describing this family is

(A) 1 (B) 2 (C) 3 (D) 4

P3. What is the area enclosed by the x-axis and the curve which passes through the point (0, 2) and satisfies the following differential equation?

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0$$

(A) $\frac{122}{12}$ (B) $\frac{125}{12}$ (C) $\frac{128}{12}$ (D) $\frac{133}{12}$ (E) None of these

P4. A vector given by $\vec{A} = f(t)\hat{i} + g(t)\hat{j} + \hat{k}$ moves such that it is always parallel to the vector $\vec{B} = -f''(t) + f'(t) + \hat{k}$. The magnitude of $\vec{A}$ is

(A) constant.
(B) a linear function of time.
(C) a quadratic function of time.
(D) a cubic function of time.
(E) none of these.

P5. A hemispherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet

and the height of water level above the outlet at time t and g is the acceleration due to gravity. What is the time it takes to empty the tank? Choose the closest answer:

(A) 1 hour 12 minutes (C) 1 hour 38 minutes (E) 1 hour 58 minutes
(B) 1 hour 26 minutes (D) 1 hour 42 minutes

P6. India's rate of growth of the economy (GDP) is 9%, and rate of growth of population is 1.5%. The respective figures for the USA are 3% and 0.5%. Suppose that today, the size of the USA economy is 10 times that of the Indian economy, while the population of India is 3 times that of the USA. After how many years will India overtake the USA in terms of the

(a) GDP?

(A) 38 years (B) 41 years (C) 44 years (D) 49 years

(b) GDP per capita?

(A) 60 years (B) 64 years (C) 68 years (D) 72 years

Choose the closest answers.

SUBJECTIVE TYPE EXAMPLES

P7. (a) Let c be a positive constant and a be a variable positive parameter, with $a > c$. Consider the family of ellipses represented by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

As a is varied, we get different ellipses of the family. What are the foci of these ellipses?

(b) If instead of $a > c$, we have $a < c$, what family of curves will the given equation represent?

(c) What is the relation between the two families of curves?

P8. (a) Let a family of curves be integral curves of a differential equation $y' = f(x, y)$. Let a second family have the property that at each point $P = (x, y)$, the angle from the curve of the first family through P to the curve of the second family through P is α. Find the differential equation describing the curves of the second family.

(b) Use this result to find the curves that form the angle $\frac{\pi}{4}$ with all circles $x^2 + y^2 = c^2$.

P9. A curve is such that the area bounded between the curve, the y-axis, the x-axis and the line parallel to the x-axis passing through $P(x, y)$ is equal to $x + y$. Find the equation of the curve.

P10. Evaluate the curve $f(x, y) = 0$ which passes through the point $(0, \frac{\pi}{2})$ and satisfies the differential equation $\ln(\cos x)dy + \ln(\sin y)dx = y \tan x dx - x \cot y\, dy$.

P11. Find the shape of a curved 2-dimensional mirror such that light from a source located at the origin will be reflected parallel to the x-axis.

P12 Solve the differential equation $y\frac{d^2y}{dx^2} - 2(\frac{dy}{dx})^2 = y^2$.

P13. The following are all variants of first order linear differential equations. Solve them by first reducing them to the standard form:

(a) $\dfrac{dy}{dx} = x^3y^3 - xy$ (b) $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^3 x$ (c) $\dfrac{dz}{dx} + \dfrac{z}{x}\log(z) = \dfrac{z}{x^2}(\log z)^2$

P14. Let $f(x)$ be a solution of the first order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Find $g(x)$ such that $h(x) = f(x)g(x)$ is also a solution to this equation.

P15. Let $P(x)$ be a known function of x. Solve the following differential equation:

$$\frac{dy}{dx} + y\frac{dP(x)}{dx} = P(x)\frac{dP(x)}{dx}$$

P16. Let $u(x)$ and $v(x)$ satisfy the differential equations $\frac{du}{dx} + p(x)u = f(x)$ and $\frac{dv}{dx} + p(x)v = g(x)$,where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equations $y = u(x)$ and $y = v(x)$.

P17. A linear first order differential equation has the general form

$$y' = p(x) + q(x)y$$

Extending this, we get what is know as the *Riccati Equation*:

$$y' = p(x) + q(x)y + r(x)y^2$$

The Riccati equation cannot be solved by elementary methods. However, if a particular solution $y_1(x)$ is known, then the general solution has the form

$$y(x) = y_1(x) + z(x)$$

where $z(x)$ is the general solution of

$$z' - (q + 2ry_1)z = rz^2$$

(a) Prove this.
(b) Find the general solution of the equation

$$y' = \frac{y}{x} + x^3y^2 - x^5$$

P18. A particle is dropped from height h under gravity; the force of air resistance is proportional to the kinetic energy of the particle (proportionality constant k). Find the time after which it will reach the ground.

P19. A point P is dragged along the xy-plane by a string PT of length a. If T starts at the origin and moves along the positive y-axis, and if P starts at $(a, 0)$, what is the path of P?

P20. A rabbit starts at the origin and runs up the y-axis with speed a. At the same time, a dog running with speed b starts at the point $(c, 0)$ and pursues the rabbit. What is the dog's path, given that $a < b$?

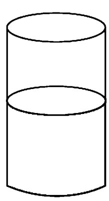
P21. The y-axis and the line $x = c$ are the banks of a river whose current has uniform speed a in the negative y-direction. A boat enters the river at $(c, 0)$ and heads directly towards the origin with speed b relative to the water. What is the boat's path?

P22. Find the shape assumed by a flexible chain suspended between two points and hanging under its own weight. Assume any constants you require.

P23. An inverted cone of height H and radius R is filled with a volatile liquid. The liquid evaporates with a rate proportional to the surface area of the liquid in contact with the air, and inversely proportional to the square of the time shift t from 12:00 Noon. Further, the liquid condenses with a rate proportional to V/t, where V is the volume of the liquid remaining at the time instant t. Let k_1, $3k_2$ be the two constants of proportionality, where k_1, k_2 are constants. If the cone is kept in the air at 1:00 PM, find the time it will take for the liquid to evaporate completely.

P24. (a) Find $f(x)$ such that the points $A(\vec{0})$, $B(\hat{i})$, $C(3\hat{i} + f(t)\hat{j} + \hat{k})$ and $D(2\hat{i} + f'(t)\hat{j} + 2\hat{k})$ are coplanar for all $t \in \mathbb{R}$.
(b) What will the answer be if B is the point $\hat{i} + \hat{j}$?

P25. Consider a massless cylinder initially filled with water up to a certain height. The initial mass of the system is A.



At $t = 0$, water starts leaking from the cylinder such that the mass of the system decreases with time in accordance with $\frac{dy}{dt} + y = 0$. At periodic intervals of period T, a person takes the following action. He takes out half of the water present in the cylinder at that instant for some experiment, and then puts in more water of mass A back into the cylinder. You can assume that this action is instantaneous.

(a) Find the mass of the system at $t = nT^+$ when $n \to \infty$.

(b) Sketch the variation of the mass of the system versus time.

P26. A rocket of structural mass m_1 contains fuel of initial mass m_2. It is fired straight up from the surface of the earth by burning fuel at a constant rate a (so that $dm/dt = -a$, where m is the variable total mass of the rocket) and expelling the exhaust products backward at a constant velocity b relative to the rocket. Neglecting all external forces except a gravitational force mg, where g is assumed constant, find the velocity and height attained at the moment when the fuel is exhausted (the *burnout velocity* and *burnout height*).

P27. Einstein's special theory of relativity states that the mass m of a particle moving with velocity v is given by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where c is the velocity of light and m_0 is the rest mass.

(a) If the particle starts from rest in empty space and moves for a long time under the influence of constant gravitational field, find v as a function of time, and show that $v \to c$ as $t \to \infty$.

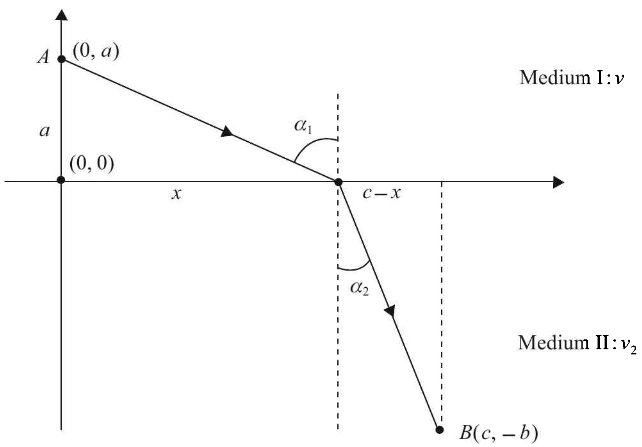
(b) Let $M = m - m_0$ be the increase in the mass of the particle. If the corresponding increase E in its energy is taken to be the work done on it by the force F acting on it, so that

$$E = \int_0^v F dx = \int_0^v \frac{d}{dt}(mv) dx = \int_0^v v d(mv),$$

verify that

$$E = Mc^2.$$

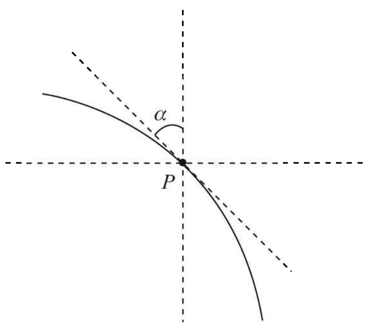
P28. (a) Light travels in medium I with velocity v_1 and a denser medium II with velocity v_2. It has to travel from the point $A(0, a)$ to the point $B(c, -b)$, as shown below:



Assuming that *Fermat's principle of least time*, which tells us that light travels from one path to another along the path requiring the shortest time, is true, find the relation between α_1 and α_2.

(b) Suppose that instead of two, there are three mediums, with velocities v_1, v_2 and v_3, and the corresponding angles the light ray makes with the vertical in the different materials being α_1, α_2 and α_3. Generalize the result of the previous part to this case.

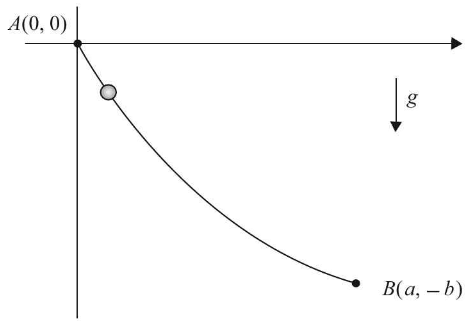
(c) Suppose that there is a medium with a continuously increasing optical density, so that light takes a 'smooth' path, as shown below:



At the point P, the speed of the ray is v and the angle it makes with the vertical is α.

Generalize the results of the previous two parts to this case.

(d) A wire in the form of a continuous curve joins the point $A(0, 0)$ to the point $B(a, -b)$. A bead travels on the wire under the influence of gravity:



What should be the shape of this wire so that the bead traverses the path $A \rightarrow B$ in the least amount of time? You can apply Fermat's principle of least time to this problem.

P29. (a) A spherical raindrop, starting from rest, falls under the influence of gravity. If it gathers in water vapor (assumed at rest) at a rate proportional to its surface, and if its initial radius is 0, show that it falls with constant acceleration, and find the value of this acceleration.

(b) If the initial radius of the raindrop in Part - a is r_0 and r is its radius at time t, find its acceleration at time t in terms of r_0 and r.

(c) A spherical raindrop, starting from rest, falls through a uniform mist. If it gathers in water droplets in its path (assumed at rest) as it moves, and if its initial radius is 0, show that it falls with constant acceleration, and find the value of this acceleration.

Differential Equations

PART-D: Solutions to Advanced Problems

S1. We have

$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}}\right\}=\frac{d}{dx}\left[\frac{d}{dx}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}}\right\}\right]=\frac{d}{dx}\left[-\frac{p}{q}\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}-1}\frac{d^3y}{dx^3}\right]$$

$$=-\frac{p}{q}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}-1}\frac{d^4y}{dx^4}-\left(1+\frac{p}{q}\right)\left(\frac{d^2y}{dx^2}\right)^{-\frac{p}{q}-2}\left(\frac{d^3y}{dx^3}\right)^2\right]$$

Clearly, the order is 4 and the degree is 1 (power of the highest order term). The value of $m + n$ is 5. The correct option is (B).

S2. It seems that the given equation has three arbitrary constants. However, we can express it in a simplified form as follows:

$$y=\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(x+C)+\frac{B}{\sqrt{A^2+B^2}}\sin(x+C)\right)$$

$$=\sqrt{A^2+B^2}\sin(x+C+\phi),\quad \tan\phi=\frac{A}{B}$$

$$=D\sin(x+E),$$

where $D=\sqrt{A^2+B^2}$ and $E=C+\phi$. Thus, to uniquely determine any curve in the given family, what we really require is only two constants: D and E. We have

$$y'=D\cos(x+E)\quad\Rightarrow\quad y''=-D\sin(x+E)\quad\Rightarrow\quad y''=-y$$

This differential equation describes the given family. The required order is thus 2. The correct option is (B).

S3. The given differential equation is a quadratic in $\frac{dy}{dx}$, and solving for it gives

$$\frac{dy}{dx}=\frac{y\pm\sqrt{x^2+y^2}}{x}=\frac{y}{x}\pm\sqrt{1+\left(\frac{y}{x}\right)^2}\tag{1}$$

Using $y=tx$, so that $\frac{dy}{dx}=t+x\frac{dt}{dx}$, (1) reduces to

$$t + x\frac{dt}{dx} = t \pm \sqrt{1+t^2} \quad \Rightarrow \quad \frac{dt}{\sqrt{1+t^2}} = \pm\frac{dx}{x}$$

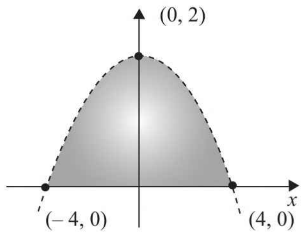
$$\Rightarrow \quad \ln|t + \sqrt{1+t^2}| = \pm \ln x + C'$$

$$\Rightarrow \quad t + \sqrt{1+t^2} = Cx \text{ or } \frac{C}{x}$$

$$\Rightarrow \quad y + \sqrt{x^2+y^2} = Cx^2 \text{ or } C$$

Only the second curve can pass through (0, 2), and the corresponding value of C is 4:

$$y + \sqrt{x^2+y^2} = 4 \quad \Rightarrow \quad y = 2 - \frac{x^2}{8}$$



The required area Δ is given by

$$\Delta = \int_{-4}^{4}\left(2 - \frac{x^2}{8}\right)dx.$$

$$= \frac{128}{12}$$

The correct option is (C).

S4. Since the coefficients of $\hat{k}$ in both $\vec{A}$ and $\vec{B}$ are the same, the only way that $\vec{A}$ and $\vec{B}$ can be parallel is that

$$f(t) = -f''(t), \quad f'(t) = g(t)$$

The first differential equation, $f''(t) = -f(t)$, tells us that $f(t)$ is a sinusoidal function:

$$f(t) = P\cos t + Q\sin t, \quad P, Q \in \mathbb{R}$$

$$= \sqrt{P^2+Q^2}\,\sin(t+\phi), \quad \text{where } \tan\phi = \frac{P}{Q}$$

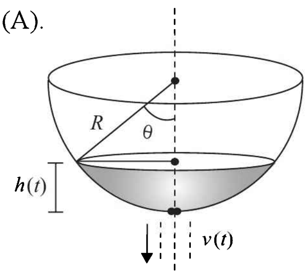
We note that $f_{\max}(t) = M = \sqrt{P^2+Q^2}$. Now,

$$|\vec{A}| = \sqrt{f^2(t) + g^2(t) + 1} = \sqrt{(P\cos t + Q\sin t)^2 + (-P\sin t + Q\cos t)^2 + 1}$$

$$= \sqrt{P^2+Q^2+1} = \sqrt{M^2+1}$$

We conclude that the magnitude of $\vec{A}$ is constant. The correct option is (A).

S5. We depict the situation at time t in the following diagram:



We note that the remaining volume of water can be written an a function of θ as follows:

$$V = \int_0^{\theta} \pi R^3 \sin\phi \, d\phi \qquad (1)$$

Now, the rate of decrease of the water's volume will be equal to the velocity of outflow $v(t)$ multiplied by the area A of the opening:

$$-\frac{dV}{dt} = Av(t) = A(0.6\sqrt{2gh(t)}) \tag{2}$$

We will now form a differential equation in θ and t. Noting that $h = R(1-\cos\theta) = 2R\sin^2\frac{\theta}{2}$, we have from (2):

$$-\frac{dV}{dt} = -\frac{dV}{d\theta}\cdot\frac{d\theta}{dt} = 1.2\ A\sqrt{gR}\sin\frac{\theta}{2} \tag{3}$$

Differentiating (1) with respect to θ, we have

$$\frac{dV}{d\theta} = \pi R^3\sin^3\theta$$

Using this in (3), we have

$$-\pi R^3\sin^3\theta\frac{d\theta}{dt} = 1.2A\ \sqrt{gR}\sin\frac{\theta}{2}$$

If T is the time taken to empty the tank, then we have

$$\int_{\pi/2}^{0} -\frac{\sin^3\theta}{\sin\frac{\theta}{2}}d\theta = \frac{1.2A\sqrt{gR}}{\pi R^3}\int_0^T dt$$

A little trigonometric manipulation can be used to solve the left hand integral I:

$$I = -\int_{\pi/2}^{0}\frac{\sin^3\theta}{\sin\frac{\theta}{2}}d\theta = -\int_{\pi/2}^{0}\left(\cos\frac{\theta}{2}-\frac{1}{2}\cos\frac{3\theta}{2}-\frac{1}{2}\cos\frac{5\theta}{2}\right)d\theta \qquad \text{(verify)}$$

$$= -\left(-2\sin\frac{\theta}{2}+\frac{1}{3}\sin\frac{3\theta}{2}+\frac{1}{5}\sin\frac{5\theta}{2}\right)\Bigg|_{\pi/2}^{0}$$

$$= \frac{28}{15\sqrt{2}}$$

Thus,

$$\frac{28}{15\sqrt{2}} = \frac{1.2A\sqrt{gR}}{\pi R^3}T \quad\Rightarrow\quad T = \frac{28\pi R^3}{18A\sqrt{2gR}}$$

Using $R = 2$ m, $A = 0.0012\text{ m}^2$ and $g = 9.8\text{ ms}^{-2}$, we have $T \approx 5202$ seconds $\approx$ 1 hour 26.7 minutes. The correct option is (B).

S6. **(a)** We express the given facts mathematically:

$$\frac{d(\text{GDP}_{India})}{dt} = 0.09\,\text{GDP}_{India},\quad \frac{d(\text{GDP}_{USA})}{dt} = 0.03\ \text{GDP}_{USA}$$

$$\Rightarrow\quad \frac{d\left(\frac{\text{GDP}_{India}}{\text{GDP}_{USA}}\right)}{dt} = 0.06\left(\frac{\text{GDP}_{India}}{\text{GDP}_{USA}}\right)$$

This last step is very important and you must carefully understand how we arrived at it. Thus, the ratio of the two GDPs grows as:

$$\frac{\text{GDP}_{India}}{\text{GDP}_{USA}} = 0.10e^{0.06t}$$

This will be greater than unity when $t \geq \frac{\ln 10}{0.06}$ years ≈ 38.4 years. The correct option is (A).

(b) Using a similar argument, we can show that

$$\frac{(\text{GDP per capita})_{India}}{(\text{GDP per capita})_{USA}} = \frac{1}{30}e^{0.05t}$$

which is greater than unity when $t \geq \frac{\ln 30}{0.05}$ years ≈ 68 years. The correct option is (C).

SUBJECTIVE TYPE EXAMPLES

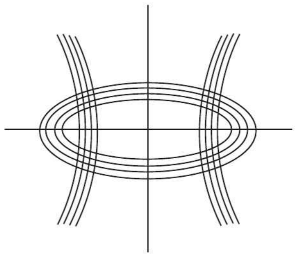
S7. **(a)** We have $b^2 = a^2 - c^2$, and

$$e^2 = 1 - \frac{b^2}{a^2} = \frac{c^2}{a^2} \quad \Rightarrow \quad e = \frac{c}{a}$$

The foci are at $(\pm ae, 0)$ or $(\pm c, 0)$.

(b) If $a < c$, we get a family of hyperbolas, with the foci of all the hyperbolas being $(\pm c, 0)$.

(c) To understand the relation between the two families, we plot approximate plots for a few curves from both families:



It seems as if there is a definite pattern to the intersection between any curve from one family and any curve from the other. To understand the pattern, we first find out the differential equation describing these families:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2yy'}{a^2 - c^2} = 0 \Rightarrow \quad a^2 = \frac{c^2 x}{x + yy'}$$

Substituting this back into the original equation, we have

$$\frac{x^2}{\frac{c^2 x}{x+yy'}} + \frac{y^2}{\frac{c^2 x}{x+yy'} - c^2} = 1$$

$$\Rightarrow \quad (x + yy')\left(x - \frac{y}{y'}\right) = c^2 \qquad (1)$$

We note that in (1), if we substitute $y' \to -\frac{1}{y'}$, it remains unchanged! What does this mean? It means that the same differential equation (1) describes *both* the first family (the family of ellipses) *and* the second family (the family of hyperbolas) which is orthogonal to the first family. In other words, (1) yields two sets of solutions, a set of ellipses ($a > c$), and a set of hyperbolas ($a < c$), and these are orthogonal to each other, because the orthogonal trajectory corresponding to (1) is (1) itself.

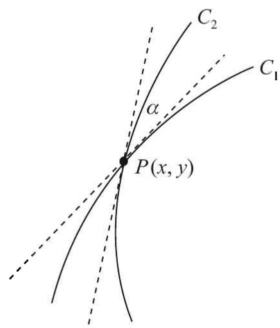
S8. **(a)** C_1 belongs to the first family, described by

$$y' = f(x, y),$$

and C_2 belongs to the second family. We note that the slope of C_1 at $P(x, y)$ is

$$m_{C_1} = y'_{C_1} = f(x, y)$$

If the slope of C_2 at $P(x, y)$ is m_{C_2}, then



$$\tan\alpha = \frac{m_{C_2} - m_{C_1}}{1 + m_{C_1} m_{C_1}}$$

$$\Rightarrow \quad m_{C_2} = \frac{m_{C_1} + \tan\alpha}{1 - m_{C_1} \tan\alpha}$$

Thus, the differential equation describing the second family of curves will be

$$y' = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$$

(b) Since $x^2 + y^2 = c^2$, we have

$$x + yy' = 0 \quad \Rightarrow \quad y' = -\frac{y}{x}$$

Thus, $f(x, y) = -\frac{y}{x}$, and the differential equation for the second family becomes

$$y' = \frac{-\frac{y}{x} + \tan\alpha}{1 + \frac{y}{x}\tan\alpha} = \frac{x\tan\alpha - y}{x + y\tan\alpha} = \frac{x - y}{x + y} \qquad (\tan\alpha = 1)$$

Using $y \to vx$ so that $y' = v + xv'$, we have

$$v + xv' = \frac{1 - v}{1 + v} \quad \Rightarrow \quad \frac{1 + v}{1 - 2v - v^2}\, dv = \frac{1}{x} dx$$

If $1 - 2v - v^2 \to u$, then $-2(1 + v)dv = du$, and the above differential becomes

$$-\frac{1}{2}\frac{du}{u} = \frac{dx}{x}$$

Integrating, we obtain

$$-\frac{1}{2}\ln u = \ln C_1 x.$$

$$\Rightarrow \quad ux^2 = C \quad \Rightarrow \quad (1 - 2v - v^2)x^2 = C$$

$$\Rightarrow \quad \left(1 - \frac{2y}{x} - \frac{y^2}{x^2}\right)x^2 = C \quad \Rightarrow \quad x^2 - 2xy - y^2 = C$$

S9. We can express the situation as follows:

$$\int_0^y x\, dy = x + y$$

Note that the curve passes through (0, 0). Differentiating this relation, we have

$$x = 1 + \frac{dx}{dy} \quad \Rightarrow \quad \frac{dx}{x - 1} = dy$$

$$\Rightarrow \quad x = 1 - e^y$$

The reader is urged to understand the differentiation step carefully.

S10. We rearrange the given differential equation as follows:

$$\{x \cot y\, dy + \ln(\sin y) dx\} + \{\ln(\cos x) dy - y \tan x\, dx\} = 0$$

We note that the two brackets contain exact differentials:

$$x \cot y\, dy + \ln(\sin y) dx = d(x \ln(\sin y))$$
$$\ln(\cos x) dy - y \tan x dx = d(y \ln(\cos x))$$

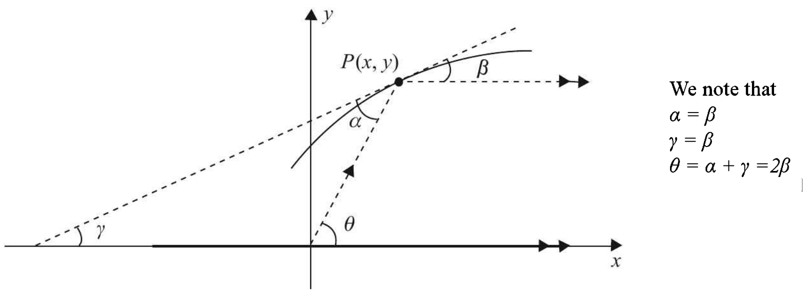
Thus, we have

$$d(x \ln(\sin y)) + d(y \ln(\cos x)) = 0$$
$$\Rightarrow \quad x \ln(\sin y) + y \ln(\cos x) = C_1$$
$$\Rightarrow \quad (\sin y)^x (\cos x)^y = C$$

Since the curve passes through $(0, \frac{\pi}{2})$, we'll obtain $C = 1$, and thus the required equation becomes

$$(\sin y)^x (\cos x)^y = 1$$

S11. The following figure represents the situation given:



Now,

$$\tan\theta = \frac{y}{x} = \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2(\frac{dy}{dx})}{1-(\frac{dy}{dx})^2} \tag{1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y} \quad \text{(Solving (1) as quadratic in } dy/dx)$$

$$\Rightarrow \quad \pm\frac{d(x^2+y^2)}{2\sqrt{x^2+y^2}} = dx$$

$$\Rightarrow \quad \pm\sqrt{x^2+y^2} = x + c$$

$$\Rightarrow \quad y^2 = 2cx + c^2$$

This represents the family of parabolas with the focus as the origin and the axis as the x-axis.

S12. Using $\frac{dy}{dx} = p$, and thus $\frac{dp}{dx} = \frac{d^2y}{dx^2} = p\frac{dp}{dy}$, we have

$$yp\frac{dp}{dy} - 2p^2 = y^2$$

Using $p^2 = z$, we have $p\frac{dp}{dy} = \frac{1}{2}\frac{dz}{dy}$

$$\Rightarrow \quad \frac{y}{2}\frac{dz}{dy} - 2z = y^2 \quad \Rightarrow \quad \frac{dz}{dy} - \frac{4}{y}z = 2y$$

This is a linear first order differential equation, whose solution is

$$z = c^2y^4 - y^2 \quad \Rightarrow \quad p = \pm y\sqrt{c^2y^2 - 1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm y\sqrt{c^2y^2 - 1}$$

$$\Rightarrow \quad y = A\sec(x + B),$$

where A and B are some constants (verify this last step).

S13. **(a)** Dividing by y^3 and rearranging, we have

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x^3$$

Substituting $\frac{1}{y^2} \to z$ so that $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$, we have

$$-\frac{1}{2}\frac{dz}{dx} + xz = x^3 \quad \Rightarrow \quad \frac{dz}{dx} - 2xz = -2x^3$$

This is in standard form. We have the IF as $e^{\int -2x dx} = e^{-x^2}$, and the solution is

$$ze^{-x^2} = \int -2x^3e^{-x^2}dx$$

$$= -\int te^{-t}dt \quad (x^2 \to t)$$

$$= -(-te^{-t} - \int(-e^{-t})dt) \text{ (Integration by parts)}$$

$$= te^{-t} + e^{-t} + C = e^{-x^2}(1 + x^2) + C$$

$$\Rightarrow \quad z = Ce^{x^2} + x^2 + 1 \quad \Rightarrow \quad y^2(Ce^{x^2} + x^2 + 1) = 1$$

(b) Dividing by $\cos y$ on both sides, we have

$$\tan y \sec y\frac{dy}{dx} + \sec y \tan x = \cos^3 x$$

Substituting $\sec y \to z$, so that $\tan y \sec y\frac{dy}{dx} = \frac{dz}{dx}$, we have

$$\frac{dz}{dx} + z\tan x = \cos^3 x$$

The IF $= e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$, and the solution is

$$z\sec x = \int \cos^2 x dx = \int \frac{1}{2}(1 + \cos 2x)dx$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C_1$$

$$\Rightarrow \quad 4\sec x \sec y = 2x + \sin 2x + C$$

(c) Dividing by z, we have

$$\frac{1}{z}\frac{dz}{dx} + \frac{\log z}{x} = \frac{(\log z)^2}{x^2}$$

Substituting $\log z \to \frac{1}{t}$, so that $\frac{1}{z}\frac{dz}{dx} = -\frac{1}{t^2}\frac{dt}{dx}$, we have

$$-\frac{1}{t^2}\frac{dt}{dx} + \frac{1}{tx} = \frac{1}{t^2x^2} \quad \Rightarrow \quad \frac{dt}{dx} + \left(-\frac{1}{x}\right)t = -\frac{1}{x^2}$$

The IF $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$, and the solution is

$$\frac{t}{x} = -\int \frac{1}{x^3}\,dx = \frac{1}{2x^2} + C_1 \quad \Rightarrow \quad t = \frac{1}{\log z} = \frac{1}{2x} + C_1 x = \frac{1+Cx^2}{2x}$$

$$\Rightarrow \quad z = e^{\left(\frac{2x}{1+Cx^2}\right)}$$

S14. $f' + Pf = Q$, since $f(x)$ is a solution to the given differential equation. If $f(x)\,g(x)$ has to be a solution, then

$$f'g + g'f + Pfg = Q$$

$$\Rightarrow \quad g(f' + Pf) + g'f = Q \quad \Rightarrow \quad g'f = Q(1-g)$$

$$\Rightarrow \quad \frac{g'}{1-g} = \frac{Q}{f} \quad \Rightarrow \quad \ln(1-g) = -\int \frac{Q}{f}\,dx + C'$$

$$\Rightarrow \quad g(x) = 1 - Ce^{-\int \frac{Q(x)}{f(x)}dx}$$

S15. The given differential equation can be written as $y' + yP' = PP'$, so that the integrating factor is $e^{\int P'dx} = e^P$. Thus, the solution to this equation in

$$ye^P = \int PP'e^P dx = \int te^t dt \quad \text{(Using } t = P(x)\text{)}$$

$$= C + e^t(t-1)$$

$$= C + e^P(P-1)$$

$$\Rightarrow \quad y(x) = Ce^{-P(x)} + P(x) - 1$$

S16. If $\psi(x) = e^{\int p(x)\,dx}$ represents the integrating factor for the two differential equations, then

$$\frac{d(u\psi)}{dx} = f\psi, \quad \frac{d(v\psi)}{dx} = g\psi$$

$$\Rightarrow \quad \frac{d\big((u-v)\psi\big)}{dx} = (f-g)\psi \qquad (1)$$

For every $x > x_1$, the RHS of (1) is positive, which implies that $(u-v)\psi$ is increasing on (x_1,∞). Also since $\psi > 0$, and $u(x_1) > v(x_1)$, we immediately conclude that $u(x) > v(x)$ for all $x > x_1$. Thus, for $x > x_1$, no point (x, y) can satisfy $y = u(x)$ and $y = v(x)$ simultaneously.

S17. **(a)** Substituting $y = y_1 + z$ in the Riccati equation, we get

$$y_1' + z' = p + q(y_1 + z) + r(y_1 + z)^2$$

$$\Rightarrow \quad y_1' + z' = (p + qy_1 + ry_1^2) + qz + 2ry_1z + rz^2 \tag{1}$$

Since y_1 is a particular solution, we have

$$y_1' = p + qy_1 + ry_1^2,$$

and thus (1) reduces to

$$z' = qz + 2ry_1z + rz^2$$

$$\Rightarrow \quad z' - (q + 2ry_1)z = rz^2 \tag{2}$$

(b) It is obvious that $y = x$ is a particular solution to the given Riccati equation. Also,

$$p(x) = -x^5,\ q(x) = \frac{1}{x},\ r(x) = x^3,\ y_1(x) = x$$

The reduced equation in (2), for this case, takes the form

$$z' - \left(\frac{1}{x} + 2x^4\right)z = x^3z^2$$

$$\Rightarrow \quad \frac{1}{z^2}\frac{dz}{dx} - \left(\frac{1}{x} + 2x^4\right)\frac{1}{z} = x^3$$

Substituting $\frac{1}{z} \to p$, so that $\frac{1}{z^2}\frac{dz}{dx} = -\frac{dp}{dx}$, we have

$$\frac{dp}{dx} + \left(\frac{1}{x} + 2x^4\right)p = -x^3 \tag{3}$$

This is a first order linear differential equation, and the IF is

$$\text{IF} = e^{\int\left(\frac{1}{x}+2x^4\right)dx} = e^{\ln x + \frac{2}{5}x^5} = xe^{2x^5/5}$$

The solution to (3) is

$$pxe^{2x^5/5} = \int -x^3(xe^{2x^5/5})\,dx + C_1 = -\int x^4e^{2x^5/5}dx + C_1$$

Putting $\frac{2x^5}{5} \to t$ so that $x^4dx = \frac{dt}{2}$, we have

$$\int x^4e^{2x^5/5}dx = \frac{1}{2}\int e^t dt = \frac{e^t}{2} = \frac{1}{2}e^{2x^5/5}$$

The solution to (3) becomes

$$pxe^{2x^5/5} = -\frac{1}{2}e^{2x^5/5} + C_1$$

$$\Rightarrow \quad p = \frac{C_1}{x}e^{-2x^5/5} - \frac{1}{2x} = \frac{C_2 e^{-2x^5/5} - 1}{2x}$$

$$\Rightarrow \quad z(x) = \frac{1}{p(x)} = \frac{2x}{C_2 e^{-2x^5/5} - 1}$$

The general solution to the given Riccati equation therefore is

$$y(x) = y_1(x) + z(x) = x + \frac{2x}{C_2 e^{-2x^5/5} - 1}$$

$$\Rightarrow \quad y(x) = x\left(1 + \frac{2}{C_2 e^{-2x^5/5} - 1}\right) = -x\left(\frac{1 + Ce^{2x^5/5}}{1 - Ce^{2x^5/5}}\right)$$

S18. Denoting by F the force of air resistance, and by v and m the velocity and mass of the particle respectively, we have

$$\frac{F}{m} = g - \frac{kv^2}{2} = \frac{dv}{dt}$$

Denoting by x the distance fallen by the particle, we have

$$\Rightarrow \quad v\frac{dv}{dx} = g - \frac{kv^2}{2} \quad \Rightarrow \quad \int \frac{v}{g - \frac{kv^2}{2}}\,dv = \int g\,dx$$

$$\Rightarrow \quad g - \frac{kv^2}{2} = Ce^{-kx}$$

Since $v = 0$ when $x = 0$, $C = g$. Thus,

$$v = \sqrt{\frac{2g}{k}(1 - e^{-kx})}$$

Writing v as $\frac{dx}{dt}$, we finally have a differential equation between x and t, from which we can find the time T taken by the particle to reach the ground:

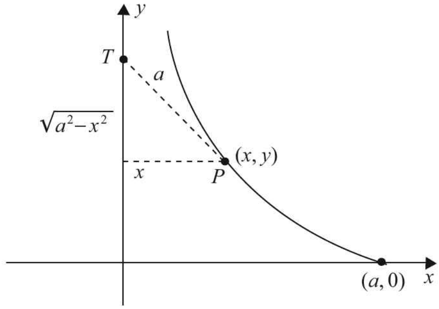
$$\int_0^h \frac{dx}{\sqrt{1 - e^{-kx}}} = \sqrt{\frac{2g}{k}}\int_0^T dt \quad \Rightarrow \quad T = \sqrt{\frac{k}{2g}}\int_0^h \frac{dx}{\sqrt{1 - e^{-kx}}}$$

To evaluate this integral, we make use of the substitution $1 - e^{-kx} = y^2$, and finally obtain the answer as

$$T = \sqrt{\frac{2}{kg}}\ln\left(\frac{1 + \sqrt{1 - e^{-kh}}}{1 - \sqrt{1 - e^{-kh}}}\right)$$

S19. The curve that P will move along is known as a *tractrix*. To determine the equation of the tractrix, consider the following figure:

From the figure, we observe that



$$\frac{dy}{dx} = \frac{-\sqrt{a^2-x^2}}{x}$$

$$\Rightarrow \quad dy = \frac{-\sqrt{a^2-x^2}}{x}\,dx$$

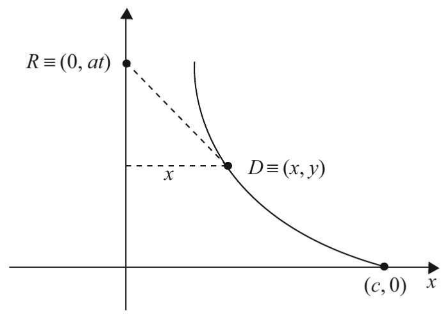
$$\Rightarrow \quad y(x) = \int_a^x \frac{-\sqrt{a^2-x^2}}{x}\,dx$$

To evaluate the integral, we use $x = \sin\theta$. Solving the integral is left to the reader as an exercise. The equation of the curve will be obtained as

$$y = a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) - \sqrt{a^2-x^2}$$

S20. We observe that at time t, the rabbit's position will be $(0, at)$. We assume that the dog's position is (x, y):

From the figure, we observe that



$$\frac{dy}{dx} = \frac{y-at}{x} \tag{1}$$

This is a differential equation in three variables: x, y, t. Writing $\frac{dy}{dx}$ as y', (1) can be written as

$$xy' - y = -at \tag{2}$$

We still have not made use of the fact that the dog runs with speed b. Assuming that the length of the arc the dog has traversed is s, we have $ds/dt = b$. Differentiating (2) with respect to x, we have

$$xy'' = -a\frac{dt}{dx} = -a\frac{dt}{ds}\frac{ds}{dx} = \frac{a}{b}\sqrt{1+(y')^2}$$

Substituting $y' \to z$, we have

$$xz' = \frac{a}{b}\sqrt{1+z^2} \quad \Rightarrow \quad \frac{z'}{\sqrt{1+z^2}} = \frac{a}{bx} \quad \Rightarrow \quad \frac{dz}{\sqrt{1+z^2}} = \frac{a}{b}\frac{dx}{x}$$

$$\Rightarrow \quad \ln(z+\sqrt{1+z^2}) = \ln\left(\frac{x}{c}\right)^{\frac{a}{b}} \tag{3}$$

Here, we have used the initial condition that $z = 0$ when $x = c$. Thus, solving for z from (3), we will have

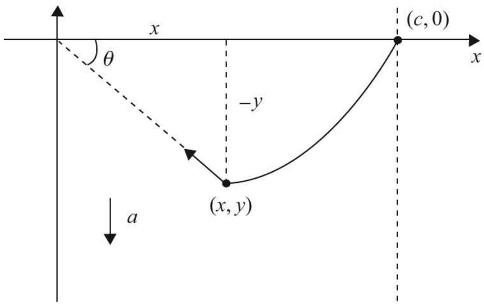
$$z = \frac{dy}{dx} = \frac{1}{2}\left\{\left(\frac{x}{c}\right)^{\frac{a}{b}} - \left(\frac{c}{x}\right)^{\frac{a}{b}}\right\} \tag{4}$$

Since $a < b$, *i.e.*, $a/b < 1$, we can now integrate (4) to obtain y as a function of x:

$$y(x) = \frac{1}{2}\left\{\frac{c}{(\frac{a}{b}+1)}\left(\frac{x}{c}\right)^{\frac{a}{b}+1} - \frac{c}{(1-\frac{a}{b})}\left(\frac{c}{x}\right)^{\frac{a}{b}-1}\right\} - \frac{abc}{a^2-b^2}$$

where, to evaluate the constant of integration, we have used the fact that $y = 0$ when $x = c$.

S21. The trajectory of the boat is approximately represented in the figure below:



We can write separate relations for the x and y components of the boat's velocity:

$$\frac{dx}{dt} = -b\cos\theta,\ \frac{dy}{dt} = -a + b\sin\theta$$

These yield

$$\frac{dy}{dx} = \frac{-a+b\sin\theta}{-b\cos\theta} = \frac{-a+b(\frac{-y}{\sqrt{x^2+y^2}})}{-b(\frac{x}{\sqrt{x^2+y^2}})} = \frac{a\sqrt{x^2+y^2}+by}{bx}$$

This is a homogeneous equation. Using the substitution $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we have

$$v + x\frac{dv}{dx} = \frac{a}{b}\sqrt{1+v^2} + v$$

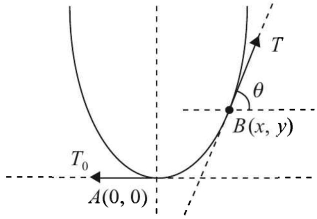
$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{a}{b}\frac{dx}{x}$$

$$\Rightarrow \quad \ln(v+\sqrt{1+v^2}) = \frac{a}{b}\ln x + C. \tag{1}$$

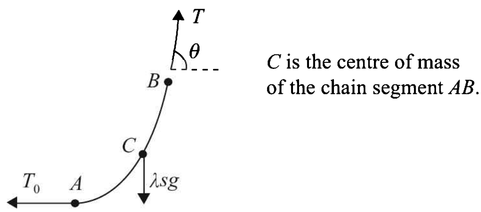
When $x = c$, $y = 0$, so $v = 0$. Thus, $C = -\frac{a}{b}\ln c$. Substituting $v = \frac{y}{x}$ in (1) and simplifying, the required equation of the trajectory will be obtained as

$$y + \sqrt{x^2+y^2} = \frac{x^{\frac{a}{b}+1}}{c^{\frac{a}{b}}}$$

S22. Consider the following diagram which represents the situation described in the problem:



Let $B \equiv (x, y)$ and let the arc length from A to B be s. The tension in the chain at the lowermost point is assumed to be T_0. We also assume that λ is the mass per unit length of the chain. Redrawing just the segment AB of the chain and the various forces acting on it, we have the following diagram:



We have $T_0 = T\cos\theta$ and $\lambda sg = T\sin\theta$. Thus,

$$\tan\theta = y' = \frac{\lambda sg}{T_0} \tag{1}$$

But since $\frac{ds}{dx} = \sqrt{1+(\frac{dy}{dx})^2}$, we can differentiate (1) with respect to x to obtain

$$y'' = \frac{\lambda g}{T_0}\sqrt{1+(y')^2}$$

Substituting $y' \to z$, we have

$$\frac{dz}{dx} = \frac{\lambda g}{T_0}\sqrt{1+z^2}$$

This is a straightforward differential equation; solving for z, we obtain

$$z = \frac{1}{2}\left(e^{\frac{\lambda g}{T_0}x} - e^{\frac{-\lambda g}{T_0}x}\right)$$

Finally, since $z = \frac{dy}{dx}$, we have

$$\int_0^y dy = \int_0^x \frac{1}{2}\left(e^{\frac{\lambda g}{T_0}x} - e^{\frac{-\lambda g}{T_0}x}\right)dx$$

$$\Rightarrow \quad y = \frac{T_0}{2\lambda g}\left(e^{\frac{\lambda g}{T_0}x} + e^{\frac{-\lambda g}{T_0}x}\right) - \frac{T_0}{\lambda g}$$

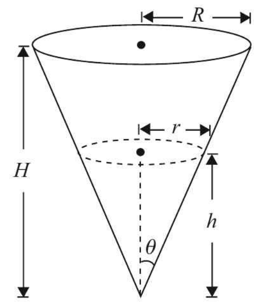
Taking $\frac{\lambda g}{T_0} = a$, and ignoring the constant of integration, the shape of the curve is given by

$$y = \frac{1}{2a}(e^{ax} + e^{-ax})$$

This curve is called a *catenary*, from the Latin word for chain: *catena*.

S23. Note that $h = r\cot\theta = \frac{rH}{R}$:

$$\Rightarrow \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \frac{H}{R}, \quad S = \pi r^2 \qquad (1)$$



According to the information provided,

$$\frac{dV}{dt} = \frac{3k_2 V}{(1+t)} - \frac{k_1 S}{(1+t)^2}$$

Using the expressions for V and S in (1) leads to a differential equation in r and t:

$$\frac{dr}{dt} - \frac{k_2 r}{(1+t)} = \frac{-Rk_1}{H(1+t)^2}$$

This is a first-order linear differential equation, with the integrating factor equal to

$$e^{-\int \frac{k_2}{(1+t)}dt} = \frac{1}{(1+t)^{k_2}}$$

The solution for r is

$$\frac{r}{(1+t)^{k_2}} = -\frac{k_1 R}{H}\int \frac{1}{(1+t)^{k_2+2}}\,dt$$

$$= \frac{k_1}{k_2+1}\cdot\frac{R}{H}\cdot\frac{1}{(1+t)^{k_2+1}} + C$$

At $t = 0$, $r = R$, which gives

$$C = R - \frac{k_1}{k_2+1}\cdot\frac{R}{H} \quad \Rightarrow \quad r(t) = \frac{k_1}{k_2+1}\cdot\frac{R}{H}\cdot\frac{1}{(1+t)} + R(1+t)^{k_2}\left(1 - \frac{k_2}{H(k_2+1)}\right)$$

The cone empties when $r(t) = 0$, *i.e.*,

$$t = \frac{1}{k_2+1}\ln\left(\frac{k_1}{k_1 - H(k_2+1)}\right) - 1$$

S24. (a) The four points A, B, C, D are coplanar if the three vectors $\overrightarrow{AB}(\hat{i}), \overrightarrow{AC}(3\hat{i} + f(t)\hat{j} + \hat{k})$ and $\overrightarrow{AD}(2\hat{i} + f'(t)\hat{j} + 2\hat{k})$ are coplanar or the STP of these three vectors is 0:

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 3 & f(t) & 1 \\ 2 & f'(t) & 2 \end{vmatrix} = 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad f'(t) = 2f(t) \quad \Rightarrow \quad f(t) = Ce^{2t}$$

The required function is $f(x) = Ce^{2x}$, $C \in \mathbb{R}$.

(b) The differential equation in this case will be

$$f'(t) - 2f(t) = -4$$

This is a linear first order differential equation, and the solution will be

$$f(x) = 2 + Ce^{2x}, C \in \mathbb{R}$$

S25. In the time interval [0, T], the mass y of the system decreases according to $y = Ae^{-t}$. At $t = T^+$,

$$y(T^+) = \frac{Ae^{-T}}{2} + A$$

Therefore, in $(T, 2T)$, the mass y of the system varies as

$$y = \left(A + \frac{Ae^{-T}}{2}\right)e^{-t}$$

Similarly,

$$y(2T^+) = \frac{1}{2}\left(Ae^{-T} + \frac{Ae^{-2T}}{2}\right) + A,$$

so that

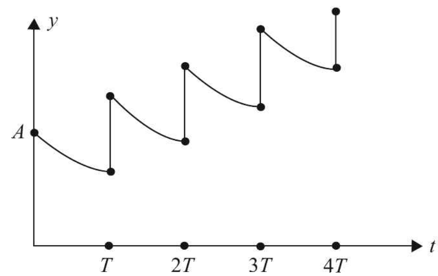
$$y = y(2T^+) \times e^{-t} \text{ in } t \in (2T, 3T)$$

Continuing this way,

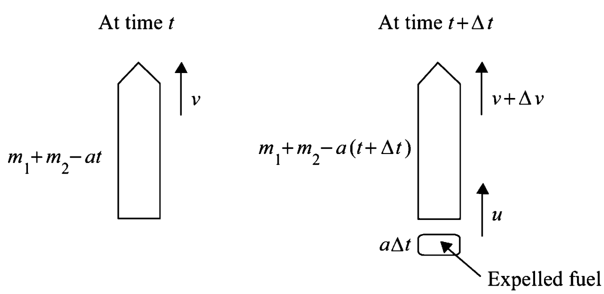
$$\lim_{n\to\infty} y\left(nT^+\right) = A + A\left(\frac{e^{-T}}{2}\right) + A\left(\frac{e^{-T}}{2}\right)^2 + \ldots\infty$$

$$= A\left(\frac{1}{1-\frac{e^{-T}}{2}}\right) = \frac{2A}{2-e^{-T}}$$

The approximate sketch of the mass variation is drawn below:



S26. We consider the rocket at two instants, t and $t + \Delta t$:



We assume that the expelled fuel in the time interval Δt has absolute velocity u. We note that

$$v + \Delta v - u = b$$

Now, we write the momentum conservation equation, including the effect of gravity:

$$\underbrace{(m_1 + m_2 - at)v}_{\text{Initial momentum of system}} \underbrace{-g(m_1 + m_2 - at)\Delta t}_{\text{Impulse of gravity}} = \underbrace{(m_1 + m_2 - a(t + \Delta t))(v + \Delta v) + a\Delta t u}_{\text{Final momentum of system}}$$

Simplifying this, we obtain

$$(m_1 + m_2 - at)(\Delta v + g\Delta t) = ab\Delta t$$

Dividing by Δt and rearranging, we have

$$\frac{\Delta v}{\Delta t} = \frac{ab}{m_1 + m_2 - at} - g$$

As $\Delta t \to 0$, this turns into a differential equation which can easily be solved:

$$\int_0^v dv = \int_0^t \left(\frac{ab}{m_1 + m_2 - at} - g \right) dt$$

$$\Rightarrow \quad v = b \ln\left(\frac{m_1 + m_2}{m_1 + m_2 - at} \right) - gt \tag{1}$$

If T is the time at which the fuel is exhausted, then $m_2 = aT$, and from (1), we have

$$v(T) = b \ln\left(\frac{m_1 + m_2}{m_1} \right) - gT = b \ln\left(1 + \frac{m_2}{m_1} \right) - \frac{m_2 g}{a}$$

This is the burnout velocity. To calculate the burnout height H, we write $v = \frac{dx}{dt}$ in (1), and solve for x:

$$\int_0^H dx = \int_0^T (b \ln(m_1 + m_2) - b \ln(m_1 + m_2 - at) - gt)\, dt$$

$$\Rightarrow \quad H = b \ln(m_1 + m_2)T - \frac{gT^2}{2} - b\left\{ \int_0^T \ln(m_1 + m_2 - at)\, dt \right\} \tag{2}$$

To evaluate the integral I in braces, we use integration by parts:

$$I = t \ln(m_1 + m_2 - at)\Big|_0^T - \int_0^T \frac{-at}{m_1 + m_2 - at}\, dt$$

$$= T \ln m_1 - \left\{ \int_0^T dt - (m_1 + m_2) \int_0^T \frac{1}{m_1 + m_2 - at}\, dt \right\}$$

$$= T \ln m_1 - T - \left(\frac{m_1 + m_2}{a} \right) \ln(m_1 + m_2 - at)\Big|_0^T$$

$$= T \ln m_1 - T + \left(\frac{m_1 + m_2}{a} \right) \ln\left(\frac{m_1 + m_2}{m_1} \right)$$

From (2),

$$H = bT\ln(m_1+m_2) - \frac{1}{2}gT^2 - bT\ln m_1 + bT - \frac{b}{a}(m_1+m_2)\ln\left(\frac{m_1+m_2}{m_1}\right)$$

$$= \frac{bm_2}{a}\ln\left(\frac{m_1+m_2}{m_1}\right) - \frac{1}{2}gT^2 + \frac{bm_2}{a} - \frac{b}{a}(m_1+m_2)\ln\left(\frac{m_1+m_2}{m_1}\right)$$

$$= -\frac{bm_1}{a}\ln\left(\frac{m_1+m_2}{m_1}\right) - \frac{1}{2}gT^2 + \frac{bm_2}{a}$$

$$= \frac{b}{a}\left(m_2 - m_1\ln\left(\frac{m_1+m_2}{m_1}\right)\right) - \frac{gm_2^2}{2a^2}$$

S27. (a) The differential equation describing the particle's motion will be

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{1}$$

Since $m = m_0(1-\frac{v^2}{c^2})^{\frac{-1}{2}}$, we have

$$\frac{dm}{dt} = -\frac{1}{2}m_0\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\frac{dv}{dt}$$

Using this in (1), we have

$$F = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{dv}{dt} + \frac{m_0v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{dv}{dt}$$

$$= m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{dv}{dt}\left\{1+\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1}\right\} = \frac{m_0c^3}{(c^2-v^2)^{\frac{3}{2}}}\frac{dv}{dt}$$

$$\Rightarrow \quad \int_0^v \frac{dv}{\left(c^2-v^2\right)^{3/2}} = \int_0^t \frac{Fdt}{m_0c^3}$$

Using the substitution $v \to c\sin\theta$, the first integral can easily be evaluated, and we'll obtain

$$\frac{v}{\sqrt{c^2-v^2}} = \frac{Ft}{m_0c}$$

$$\Rightarrow \quad v^2 = \frac{c^2}{\frac{m_0^2c^2}{F^2t^2}+1} \quad \Rightarrow \quad v = \frac{c}{\sqrt{1+\frac{m_0^2c^2}{F^2t^2}}}$$

It is obvious that as $t \to \infty$, $v \to c$.

(b) We have

$$E = \int_0^v vd\left(mv\right) = \int_0^v vd\left\{m_0v\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right\}$$

$$= \int_0^v \left\{ m_0 v \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + m_0 v^2 \left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right) \right\} dv$$

$$= \int_0^v m_0 v \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left\{1+\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1}\right\} dv$$

$$= \int_0^v m_0 v \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left(\frac{c^2}{c^2-v^2}\right) dv = m_0 c^3 \int_0^v \frac{v dv}{\left(c^2-v^2\right)^{3/2}}$$

Using $c^2 - v^2 \to y^2$, so that $vdv = -ydy$, with the new limits as c to $\sqrt{c^2-v^2}$, we have

$$E = -m_0 c^3 \int_c^{\sqrt{c^2-v^2}} \frac{ydy}{y^3} = m_0 c^3 \left(\frac{1}{y}\right)\Bigg|_c^{\sqrt{c^2-v^2}} = m_0 c^3 \left(\frac{1}{\sqrt{c^2-v^2}} - \frac{1}{c}\right)$$

$$= \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 = mc^2 - m_0 c^2 = (m-m_0)c^2 = Mc^2$$

S28. (a) The total time required for the journey is

$$T = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(c-x)^2}}{v_2}$$

We have to find that value of x for which T is minimum. Thus,

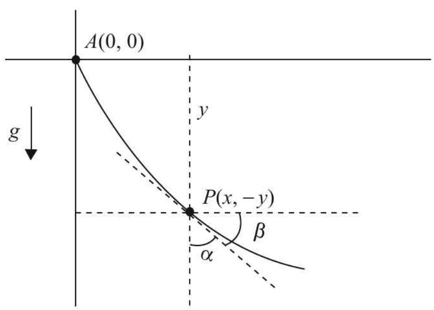
$$\frac{dT}{dx} = 0 \Rightarrow \frac{x}{v_1\sqrt{a^2+x^2}} = \frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}$$

$$\Rightarrow \frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$

The reader may recall that this is the *Snell's law of refraction.*

(b) $\dfrac{\sin\alpha_1}{v_1} = \dfrac{\sin\alpha_2}{v_2} = \dfrac{\sin\alpha_3}{v_3}$

(c) $\dfrac{\sin\alpha}{v} = \text{constant}$



(d) From conservation of energy, we have $v = \sqrt{2gy}$. Also, using Fermat's principle of least time, we can say that

$$\frac{\sin\alpha}{v} = \text{constant} = k\,(\text{say})$$

Additionally, $y' = \tan\beta$.

Now,

$$\sin\alpha = \cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}}$$

$$\Rightarrow \quad \frac{\sin\alpha}{v} = \frac{1}{\sqrt{2gy}\sqrt{1+(y')^2}} = k$$

$$\Rightarrow \quad y(1+(y')^2) = c \qquad \text{(another constant)}$$

This is the differential equation of the path we wish to determine. Separating variables, we have

$$dx = \sqrt{\frac{y}{c-y}}\, dy$$

To solve this differential equation, we use the substitution $\frac{y}{c-y} \to \tan^2\phi$. This gives

$$y = c\sin^2\phi,\ dy = 2c\sin\phi\cos\phi\ d\phi$$

$$\Rightarrow \quad dx = 2c\sin^2\phi\ d\phi = c(1-\cos 2\phi)\ d\phi$$

$$\Rightarrow \quad x = \frac{c}{2}(2\phi - \sin 2\phi) + \lambda$$

Since the curve passes through (0, 0), $\lambda = 0$. Thus,

$$x = \frac{c}{2}(2\phi - \sin 2\phi),\ y = c\sin^2\phi = \frac{c}{2}(1-\cos 2\phi)$$

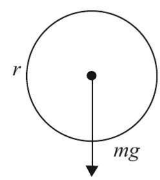
These relations can be written more simply by using $\frac{c}{2} \to p$ and $2\phi \to \theta$:

$$\boxed{x = p(\theta - \sin\theta),\ y = p(1-\cos\theta)}$$

The value of p can be obtained by using the fact that the curve passes through $B(a,-b)$. This is the equation of a *cycloid*, the curve generated by a point on the circumference of a rolling disc.

S29. (a) and (b) We will directly solve for the general case in (b), because the result in (a) is a special case of the result in (b).
If the density of water is ρ, then

$$V = \frac{4}{3}\pi r^3,\ m = \frac{4}{3}\rho\pi r^3 \qquad (1)$$



Also, it is given that

$$\frac{dV}{dt} = k(4\pi r^2) \qquad (2)$$

From (1) and (2), we can deduce that

$$\frac{dr}{dt} = k \quad \Rightarrow \quad r = r_0 + kt$$

Now, since $F = \frac{d(mv)}{dt}$, we have

$$F = mg = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Using (1) and simplifying, we obtain

$$g = \frac{dv}{dt} + \left(\frac{3k}{r}\right)v \tag{3}$$

This is a first order linear differential equation with the integrating factor IF being

$$\text{IF} = e^{\int \frac{3k}{r}dt} = e^{\int \frac{3k}{r_0 + kt}dt} = e^{3\ln(r_0 + kt)} = (r_0 + kt)^3$$

The solution to (3) is thus

$$v(r_0 + kt)^3 = \int g(r_0 + kt)^3 dt$$

$$\Rightarrow \quad v(r_0 + kt)^3 = \frac{g}{4k}(r_0 + kt)^4 + C$$

Since $r = r_0 + kt$, we can write this as

$$v = Cr^{-3} + \frac{gr}{4k} \tag{4}$$

At $t = 0$, $r = r_0$ and $v = 0$, so that $C = \frac{-gr_0^4}{4k}$. Thus, (4) becomes

$$v = \frac{g}{4k}\left(r - \frac{r_0^4}{r^3}\right) \tag{5}$$

To obtain the acceleration a, we differentiate (5) with respect to time, and use $\frac{dr}{dt} = k$:

$$a = \frac{g}{4}\left(1 + \frac{3r_0^4}{r^4}\right)$$

If $r_0 = 0$, then $a = \frac{g}{4}$.

(c) We will again solve for the general case where the initial radius is r_0. In this case, the difference from the previous two cases is that the faster the droplet is moving, the higher the rate at which it

will gather more water. So, instead of $\frac{dV}{dt} = k(4\pi r^2)$, the rate of change of volume in this case will be given by

$$\frac{dV}{dt} = k(4\pi r^2 v) \tag{6}$$

Convince yourself about this point. Since v is dx/dt, (6) can be written as

$$\frac{dV}{dx} = 4k\pi r^2 \tag{7}$$

Also, since $V = \frac{4}{3}\pi r^3$,

$$\frac{dV}{dx} = 4\pi r^2 \frac{dr}{dx} \tag{8}$$

From (7) and (8),

$$\frac{dr}{dx} = k \quad \Rightarrow \quad r = r_0 + kx$$

Now,

$$g = \frac{dv}{dt} + \frac{vdm}{mdt} = \frac{vdv}{dx} + \frac{v^2}{m}\frac{dm}{dx}$$

Also, since $m = \rho V$, $\frac{dm}{dx} = \rho \frac{dV}{dx}$. Thus,

$$g = v\frac{dv}{dx} + \frac{v^2}{(\frac{4}{3}\rho\pi r^3)}\left(\rho\, 4\pi r^2 \frac{dr}{dx}\right)$$

$$\Rightarrow \quad g = v\frac{dv}{dx} + 3\frac{kv^2}{r} \tag{9}$$

Substituting $v^2 \to u$, so that $v\frac{dv}{dx} = \frac{1}{2}\frac{du}{dx}$, we have

$$\frac{1}{2}\frac{du}{dx} + \frac{3ku}{r} = g$$

$$\Rightarrow \quad \frac{du}{dx} + \left(\frac{6k}{r}\right)u = 2g \tag{10}$$

(10) is a first order linear differential equation, and the IF is

$$\text{IF} = e^{\int \frac{6k}{r}dx} = e^{\int \frac{6k}{r_0 + kx}dx} = (r_0 + kx)^6 = r^6$$

The solution to (10) is

$$ur^6 = 2g\int r^6 dx = \frac{2g}{k}\left(\frac{r^7}{7}\right) + C$$

$$\Rightarrow \quad v^2 = \frac{C}{r^6} + \frac{2gr}{7k}$$

When $t = 0$, $x = 0$, $v = 0$ and $r = r_0$, so that

$$0 = \frac{C}{r_0^6} + \frac{2gr_0}{7k} \quad \Rightarrow \quad C = -\frac{2gr_0^7}{7k}$$

$$\Rightarrow \quad v^2 = \frac{2g}{7k}\left(r - \frac{r_0^7}{r^6}\right) = \frac{2gr}{7k}\left(1 - \frac{r_0^7}{r^7}\right)$$

If $r_0 = 0$, then

$$v^2 = \left(\frac{2g}{7k}\right)r \quad \Rightarrow \quad 2v\frac{dv}{dt} = \left(\frac{2g}{7k}\right)\frac{dr}{dt} = \left(\frac{2g}{7k}\right)\left(\frac{dr}{dx}\right)\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \quad 2v\frac{dv}{dt} = \left(\frac{2g}{7k}\right)(kv)$$

$$\Rightarrow \quad a = \frac{dv}{dt} = \frac{g}{7}$$